Chapter 1: Real Numbers and Functions

Part A: Properties of Real Numbers

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Field Axioms

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2. Addition and multiplication are **associative**: $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for every $a, b, c \in \mathbb{R}$.

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The set of non-zero real numbers is denoted by $\mathbb{R}^*$.

We shall usually abbreviate $a \cdot b$ to ab.

Uniqueness of Identity and Inverse

Theorem

The field $\mathbb{R}$ has the following properties.

1. *0 is the only additive identity and 1 is the only multiplicative identity.*
2. *The additive inverse of any real number is unique.*
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Suppose $0'$ is also an additive identity. Then $0' = 0 + 0' = 0$. Next, suppose a has additive inverses b and c. Then,

$$b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c.$$

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You can similarly show the uniqueness of the multiplicative identity and inverses. □

Cancellation Laws

We denote the additive inverse of a by $-a$ and the multiplicative inverse by $1/a$ or a^{-1}.

Theorem

Let $a, b, c \in \mathbb{R}$. Then,

1. *If $a + b = a + c$ then $b = c$.*
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$$\begin{aligned} a + b = a + c &\implies (-a) + (a + b) = (-a) + (a + c) \\ &\implies ((-a) + a) + b = ((-a) + a) + c \\ &\implies 0 + b = 0 + c \implies b = c. \end{aligned}$$

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If $a \neq 0$ then it has a multiplicative inverse a^{-1} and we have

$$\begin{aligned} ab = ac \implies a^{-1}(ab) = a^{-1}bc \implies (a^{-1}a)b = (a^{-1}a)c \\ \implies 1 \cdot b = 1 \cdot c \implies b = c. \qquad \square \end{aligned}$$

Properties of Zero and Inverses

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Task

Verify that $-(a+b) = (-a) + (-b)$ *and* $(ab)^{-1} = a^{-1}b^{-1}$.

Subtraction and Division

The sum $a + (-b)$ is denoted by $a - b$ and is called the **difference** of a and b. The process of obtaining $a - b$ is called **subtraction**.

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Use the field axioms of $\mathbb{R}$ to prove the following:

1. $-\dfrac{a}{b} = \dfrac{-a}{b} = \dfrac{a}{-b}$ *if* $b \neq 0$,
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The **square** of a number x is defined by $x^2 = x \cdot x$.

Task

Show that $(-x)^2 = x^2$.

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These properties are called the **order axioms** of $\mathbb{R}$.

Combinations of Positive and Negative Numbers

Theorem

1. *If* $x, y \in \mathbb{R}^-$ *then* $x + y \in \mathbb{R}^-$.
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Combinations of Positive and Negative Numbers

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Ordering of Real Numbers

We say that a is **greater** than b, $a > b$, if $a - b \in \mathbb{R}^+$. In this case, we also say that b is **less** than a and denote that by $b < a$.

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1. $\mathbb{R}^+ = \{ x \in \mathbb{R} \mid x > 0 \}$ *and* $\mathbb{R}^- = \{ x \in \mathbb{R} \mid x < 0 \}$.
2. *(Trichotomy) Exactly one of the following holds:* $a = b$ *or* $a > b$ *or* $a < b$.
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2. First, $a = b \implies a - b = 0$ rules out $a > b$ and $a < b$. Next, let $a \neq b$. Then $a - b \neq 0$ and belongs to exactly one of $\mathbb{R}^+$ and $\mathbb{R}^-$. Apply the first part.
3. Hint: Consider $a - c = (a - b) + (b - c)$.

Ordering and Arithmetic

Theorem

Let $a, b, c \in \mathbb{R}$. Then the following hold.

1. *If $a > b$ then $a + c > b + c$.*
2. *Let $c > 0$. If $a > b$ then $ac > bc$.*
3. *Let $c < 0$. If $a > b$ then $ac < bc$.*
4. *If $a < b$ then $a < \dfrac{a+b}{2} < b$.*
5. *If $0 < a < b$ then $0 < 1/b < 1/a$.*
6. *Suppose $a, b > 0$. Then $a > b \iff a^2 > b^2$.*
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1. Hint: Consider $(a + c) - (b + c) = a - b$.
2. Hint: Consider $ac - bc = (a - b)c$.
3. As above.

Absolute Value

The **absolute value** of a real number x is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Theorem

Let $x, y \in \mathbb{R}$. Then we have the following.

1. $|x| \geq 0$.
2. $|x| = 0$ *if and only if* $x = 0$.
3. $|x^2| = |x|^2 = x^2$.
4. $|xy| = |x||y|$.
5. *(Triangle Inequality)* $|x + y| \leq |x| + |y|$.
6. $|x - y| \geq ||x| - |y||$.

Absolute Value (contd.)

The first two claims are obvious from the definition. To prove the others we use the earlier result that if $a, b \geq 0$ then $a = b \iff a^2 = b^2$.

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3. Since $x^2 \geq 0$, we have $|x^2| = x^2$. Further,

$$|x|^2 = \begin{cases} x^2 & \text{if } x \geq 0 \\ (-x)^2 & \text{if } x < 0 \end{cases} = x^2.$$

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6. $|x-y|^2 = x^2 + y^2 - 2xy \geq |x|^2 + |y|^2 - 2|x||y| = ||x|-|y||^2$.

$\square$

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$\square$

Task

For any $x, a \in \mathbb{R}$ with $a \geq 0$, prove that $|x| \leq a \iff -a \leq x \leq a$.

Distance

We call $|x - y|$ the **distance** between x and y.

Theorem

Let $x, y, z \in \mathbb{R}$. Then we have the following.

1. *(Positivity)* $|x - y| \geq 0$, *and* $|x - y| = 0$ *if and only if* $x = y$.
2. *(Symmetry)* $|x - y| = |y - x|$.
3. *(Triangle Inequality)* $|x - z| \leq |x - y| + |y - z|$.

The proofs are left as an exercise.

Some Types of Real Numbers

- By repeatedly adding 1 we generate the subset of **natural numbers**,

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- By dividing integers with each other we get the **rational numbers**,

$$\mathbb{Q} = \{\, a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0 \,\}.$$

Integer Powers

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\end{aligned}$$

Integer Powers

First define $x^0 = 1$ for any $x \in \mathbb{R}$. Then, for any $n \in \mathbb{N}$, define $x^n = x \cdot x^{n-1}$. If $x \neq 0$, we define $x^{-n} = (x^n)^{-1}$. We will use mathematical induction to prove the following:

$$\text{If } x \neq 0 \text{ and } n \in \mathbb{N} \text{ then } (x^{-1})^n = (x^n)^{-1}.$$

Let $P(n)$ be the statement $(x^{-1})^n = (x^n)^{-1}$. Then $P(1)$ is the statement $x^{-1} = x^{-1}$ which is certainly true. Now assume some $P(n)$ is true. We need to show that this forces $P(n+1)$ to be true:

$$\begin{aligned}
(x^{-1})^{n+1} &= x^{-1} \cdot (x^{-1})^n && \text{(definition of } a^{n+1}\text{)} \\
&= x^{-1}(x^n)^{-1} && \text{(induction hypothesis)} \\
&= (x \cdot x^n)^{-1} && \text{(because } (ab)^{-1} = a^{-1}b^{-1}\text{)} \\
&= (x^{n+1})^{-1} && \text{(definition of } a^{n+1}\text{).}
\end{aligned}$$

This shows the truth of $P(n+1)$. Therefore, by mathematical induction, $(x^{-1})^n = (x^n)^{-1}$ holds for every $n \in \mathbb{N}$.

Maximum and Minimum

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Example 3: $\mathbb{R}^+$ has neither a maximum nor a minimum.

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The Completeness Axiom lends itself to showing the existence of a number with a particular property by locating it between numbers which are too large or too small to have that property.

Existence of Square Roots

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Theorem

Let $x \in \mathbb{R}^+$. Then there is a unique $y \in \mathbb{R}^+$ such that $y^2 = x$. (We call y the **positive square root** *of x and denote it by $x^{1/2}$ or $\sqrt{x}$.)*

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If $y \in A$ then $y = \max(A)$, while if $y \in B$ then $y = \min(B)$. Therefore, if we show that A has no maximum and B has no minimum, we will have ruled out both $y^2 < x$ and $y^2 > x$, ensuring $y^2 = x$.

Existence of Square Roots (contd.)

To show that B has no least member, take any $m \in B$. We need to find an $m' \in B$ such that $m' < m$.

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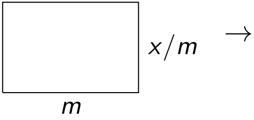

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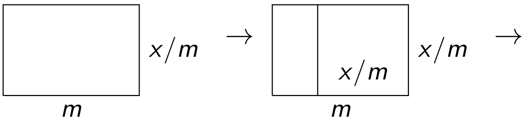

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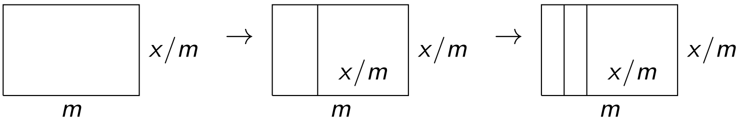

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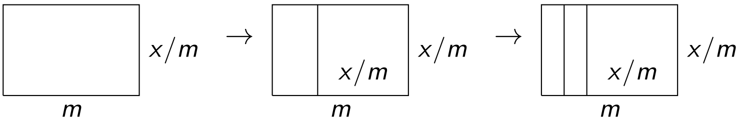

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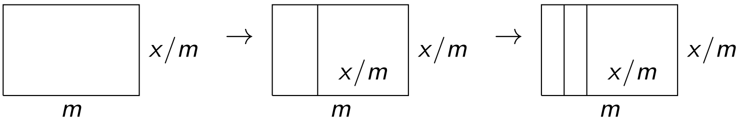

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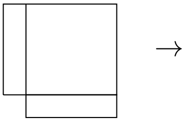

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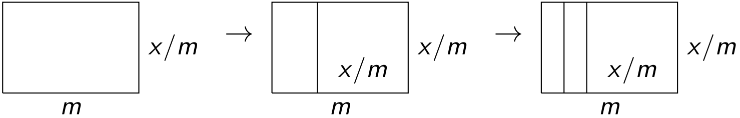

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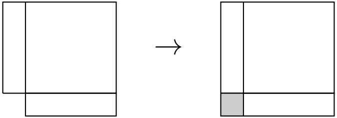

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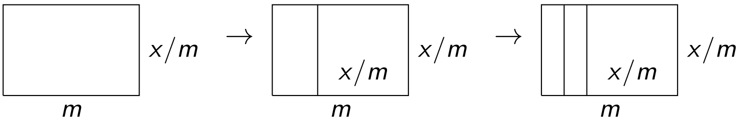

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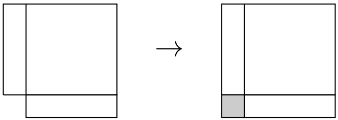

If the side of the final square is m' then it is clear that $m' < m$ while $m'^2 > x$.

Existence of Square Roots (contd.)

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The geometric argument given above leads to an algebraic one.

Define $m' = \frac{1}{2}(m + \frac{x}{m}) = \frac{x}{m} + \frac{1}{2}(m - \frac{x}{m})$.

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Hence A has no greatest element. This proves that $y^2 = x$. Uniqueness has been established earlier.

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Upper and Lower Bounds

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Let A be a subset of $\mathbb{R}$.

- $M \in \mathbb{R}$ is an **upper bound** of A if $a \leq M$ for every $a \in A$.
- $m \in \mathbb{R}$ is a **lower bound** of A if $m \leq a$ for every $a \in A$.

A is **bounded above** if it has an upper bound.
A is **bounded below** if it has a lower bound.
If A is bounded above *and* below, we say it is **bounded**.
If A is not bounded, we say it is **unbounded**.

Example 1: $(1, 2)$ has upper bound 2 and lower bound 1, hence is bounded.

Example 2: $\mathbb{R}^+$ has lower bound 0 but no upper bound, hence is unbounded.

Task

Is the empty set bounded as a subset of $\mathbb{R}$?

Archimedean Property

Ver. 1

The set $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

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Suppose $\mathbb{N}$ is bounded above.
Then the set B of all upper bounds of $\mathbb{N}$ is non-empty.

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Suppose $\mathbb{N}$ is bounded above.
Then the set B of all upper bounds of $\mathbb{N}$ is non-empty.
By definition of B, if $a \in \mathbb{N}$ and $b \in B$ then $a \leq b$.
Hence, by the Completeness Axiom, there is $\alpha \in \mathbb{R}$ such that $a \leq \alpha \leq b$ for every $a \in \mathbb{N}$, $b \in B$.

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Since $\alpha - 1 < \alpha$, we know $\alpha - 1 \notin B$.

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Hence there is an $N \in \mathbb{N}$ such that $N > \alpha - 1$.

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Hence there is an $N \in \mathbb{N}$ such that $N > \alpha - 1$.
But then $N + 1 \in \mathbb{N}$ and $N + 1 > \alpha$, a contradiction. □

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But then $N + 1 \in \mathbb{N}$ and $N + 1 > \alpha$, a contradiction. $\square$

Task

Show that $\mathbb{Z}$ has neither an upper nor a lower bound in $\mathbb{R}$.

Archimedean Property

Ver. 2

Let $x, y \in \mathbb{R}^+$. Then there exists $N \in \mathbb{N}$ such that $Nx > y$.

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By Ver. 1 there is $N \in \mathbb{N}$ such that $N > y/x$, hence $Nx > y$. $\square$

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Let $x, y \in \mathbb{R}^+$. Then there exists $N \in \mathbb{N}$ such that $0 < \frac{y}{N} < x$.

There is $N \in \mathbb{N}$ such that $Nx > y > 0$. Hence $0 < \frac{y}{N} < x$. $\square$

Archimedean Property Application

Theorem

Let $x, y \in \mathbb{R}$ *and* $M > 0$ *such that* $y - \frac{M}{n} \leq x \leq y + \frac{M}{n}$ *for every* $n \in \mathbb{N}$*. Then* $y = x$*.*

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Let $x, y \in \mathbb{R}$ and $M > 0$ such that $y - \frac{M}{n} \leq x \leq y + \frac{M}{n}$ for every $n \in \mathbb{N}$. Then $y = x$.

We apply trichotomy.

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Suppose $y > x$. Then $0 < y - x$.

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Suppose $y > x$. Then $0 < y - x$.
By Archimedean Property Ver. 3, there is $N \in \mathbb{N}$ such that $0 < M/N < y - x$.

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Let $x, y \in \mathbb{R}$ and $M > 0$ such that $y - \dfrac{M}{n} \leq x \leq y + \dfrac{M}{n}$ for every $n \in \mathbb{N}$. Then $y = x$.

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Hence $x < y - M/N$.

Archimedean Property Application

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This contradicts the given relationship between x, y, M.

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We similarly prove that $y < x$ is false.

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So $y > x$ is false.

We similarly prove that $y < x$ is false.

Therefore, by trichotomy, $y = x$. □

Greatest Integer for a Real Number

Theorem (Greatest Integer)

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For uniqueness, note that any integer with the given property would be the greatest element of A. □

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This m is called the **greatest integer** for x and is denoted by $[x]$.

Denseness of Rationals

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Theorem (Denseness of Rational Numbers)

Let $x, y \in \mathbb{R}$ with $x < y$. Then there is $r \in \mathbb{Q}$ such that $x < r < y$.

Denseness of Rationals

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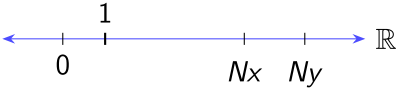
Let $x, y \in \mathbb{R}$ with $x < y$. Then there is $r \in \mathbb{Q}$ such that $x < r < y$.

By Archimedean Principle, Ver. 2, there is $N \in \mathbb{N}$ such that $1 < N(y - x) = Ny - Nx$.



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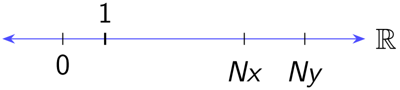

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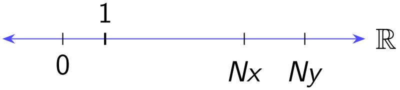

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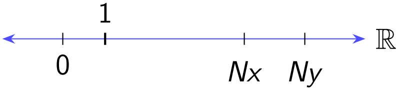

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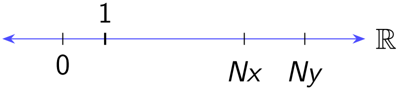

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Hence $Nx < M < Ny$ and $x < M/N < y$. □

Task

Show that there are infinitely many rational numbers between any two distinct real numbers.

Denseness of Irrationals

We know $\mathbb{Q} \neq \mathbb{R}$. For example, $\sqrt{2} \notin \mathbb{Q}$. A real number which is not rational is called **irrational**.

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We know $\mathbb{Q} \neq \mathbb{R}$. For example, $\sqrt{2} \notin \mathbb{Q}$. A real number which is not rational is called **irrational**.

Task

Let t be an irrational number. Show that:

1. *The numbers $-t$ and $1/t$ are irrational.*
2. *If r is rational then $r + t$ and $r - t$ are irrational.*
3. *If r is a non-zero rational then rt and r/t are irrational.*

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Theorem (Denseness of Irrational Numbers)

Let $x, y \in \mathbb{R}$ with $x < y$. Then there is an irrational number t such that $x < t < y$.

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Let $x, y \in \mathbb{R}$ with $x < y$. Then there is an irrational number t such that $x < t < y$.

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Theorem (Denseness of Irrational Numbers)

Let $x, y \in \mathbb{R}$ with $x < y$. Then there is an irrational number t such that $x < t < y$.

By the denseness of $\mathbb{Q}$, we have $r \in \mathbb{Q}$ such $\sqrt{2}x < r < \sqrt{2}y$. We can arrange for r to be non-zero. Then $t = r/\sqrt{2}$ is an irrational number such that $x < t < y$. □

Least Upper Bounds

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Let A be a non-empty subset of $\mathbb{R}$. $U \in \mathbb{R}$ is called the **least upper bound** (LUB) or **supremum** (sup) of A if

1. U is an upper bound of A, and
2. U is the least among all the upper bounds of A.

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The LUB of A need not belong to A. In particular, it may not be the maximum element of A.

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The LUB of A need not belong to A. In particular, it may not be the maximum element of A.

Example 3: $\mathbb{N}$ has no upper bounds, hence has no LUB.

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The condition $U \leq b$ for every $b \in B$ makes U the least among all the upper bounds of A.

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LUB of Union

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Task

Let $A, B \subseteq \mathbb{R}$ be non-empty and bounded above. Define $A + B = \{\, a + b \mid a \in A, b \in B \,\}$. Show that

$$\sup(A + B) = \sup(A) + \sup(B).$$

Greatest Lower Bound

Let A be a non-empty subset of $\mathbb{R}$. $L \in \mathbb{R}$ is called the **greatest lower bound** (GLB) or **infimum** (inf) of A if

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Let A be a non-empty subset of $\mathbb{R}$ which is bounded above. Define $-A = \{ x \in \mathbb{R} \mid -x \in A \}$. Show that

$$\inf(-A) = -\sup(A).$$