

## Exercise: concepts from chapter 8

Reading: Fundamentals of Structural Geology, Ch 8

1) The following exercises explore elementary concepts associated with a linear elastic material that is isotropic and homogeneous with respect to elastic properties.

- a) It is commonly understood that longitudinal deformation, say shortening, implies compressive normal stress acting in the direction of this strain. Use the three dimensional form of Hooke's Law for an isotropic body with Young's modulus,  $E$ , and Poisson's ratio,  $\nu$ , as the two elastic moduli to demonstrate that this could be a misconception under some states of stress. As an illustrative example consider the state of uniaxial tension,  $\sigma_{xx} > 0$ ,  $\sigma_{yy} = 0 = \sigma_{zz}$ . Describe how your result depends upon the elastic moduli and define the full range of these quantities that you are considering.
- b) Now consider a three-dimensional state of stress that could develop in Earth's crust. The stresses are given by Anderson's standard state (an isotropic compression). We ignore strains that are associated with the development of this stress state. Suppose the rock body is subject to a tectonic stress state  $\sigma_{xx} > 0$ ,  $\sigma_{yy} = \sigma = \sigma_{zz}$ , and  $\sigma > 0$ . In other words a tectonic tension is applied in the  $x$ -direction and a tension of magnitude  $\sigma$  is applied in the  $y$ - and  $z$ -direction. Use Hooke's Law for an isotropic body with Young's modulus,  $E$ , and Poisson's ratio,  $\nu$ , as the two elastic moduli to determine those conditions under which the tectonic strains in  $y$  and  $z$  are a shortening even though the tectonic stress is tensile in those directions.
- c) It is commonly understood that shearing deformation implies shear stresses acting on planes associated with this strain. Use the three dimensional form of Hooke's Law for an isotropic body with Young's modulus and Poisson's ratio as the two elastic moduli to demonstrate that this is an accurate conception. Describe how your result depends upon the elastic moduli and define the full range of these quantities that you are considering.
- d) Many types of rubber have values of Poisson's ratio approaching the upper limit of  $1/2$ , whereas many varieties of cork have values approaching the lower limit of  $0$ . Both of these materials are used as stoppers for bottles containing liquids. What mechanical reason can you offer for the predominant usage of cork instead of rubber for wine bottle stoppers? Assume that the stopper would be a solid cylindrical shape whether rubber or cork. On the other hand, rubber is the choice for most stoppers in a chemistry lab, presumably because of its resistance to chemical reaction. Can you suggest why most of these rubber stoppers are tapered and not cylindrical in shape?

- e) Suppose a rock mass has a Young's modulus,  $E = 25$  GPa, and Poisson's ratio,  $\nu = 0.15$ . Determine values for Lamé's constants,  $G$  and  $\lambda$ , and the bulk modulus,  $K$ , and write down the equations you have used. Suppose you know values for the bulk modulus,  $K$ , and the shear modulus,  $G$ . Using algebra, derive equations for Young's modulus,  $E$ , and Poisson's ratio,  $\nu$ .

- 2) Consider the block of rock shown in Figure 1 to be linear elastic and isotropic and homogeneous with respect to elastic properties.

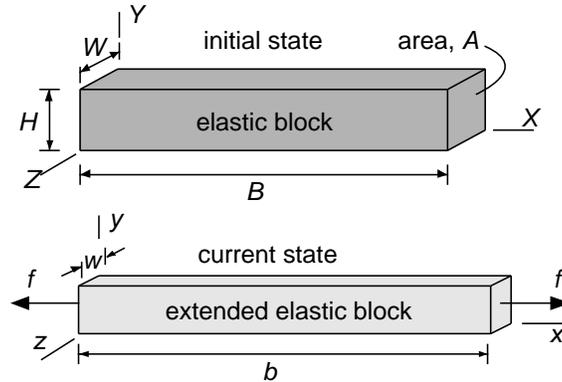


Figure 1. Idealized block of rock that is a linear elastic material. a) Initial unloaded state. b) Current loaded state.

Suppose the elastic properties of this block are Young's modulus,  $E = 50$  GPa, and Poisson's ratio,  $\nu = 0.20$ . Also suppose the side lengths  $B = 1000$  m,  $H = 150$  m, and  $W = 125$  m.

- a) Compute the three infinitesimal longitudinal strain components ( $\epsilon_{xx}$ ,  $\epsilon_{yy}$ ,  $\epsilon_{zz}$ ) in the coordinate directions within this block for the following state of stress:

$$\begin{aligned} \sigma_{xx} &= -50 \text{ MPa}, \quad \sigma_{yy} = -35 \text{ MPa}, \quad \sigma_{zz} = -30 \text{ MPa} \\ \sigma_{xy} &= \sigma_{yz} = \sigma_{zx} = 0 \end{aligned} \quad (1)$$

Note that the normal components are principal stresses and all are compressive. Assess the magnitude of the strains and indicate if they are within the range for typical elastic behavior.

- b) Recall the general kinematic equations relating the infinitesimal strain components to the displacement components (5.118):

$$\epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2)$$

Compute the displacement components ( $u_x$ ,  $u_y$ ,  $u_z$ ) at the point  $x = 1000$  m,  $y = 150$  m, and  $z = 125$  m for the stress state given in (1). Assume the rock mass is fixed (zero displacement) at the origin of the coordinate system (Figure 1).

- c) The stretch,  $S$ , the extension (also called the infinitesimal strain),  $\varepsilon$ , and the strain (also called the Lagrangian strain),  $E$ , are related to the initial length,  $B$ , and final length,  $b$ , of the block as follows:

$$S = \frac{b}{B}; \quad \varepsilon = \frac{b-B}{B}; \quad E = \left(\frac{b-B}{B}\right) + \frac{1}{2}\left(\frac{b-B}{B}\right)^2 \quad (3)$$

Calculate the final length of the block,  $b$ , and use this to calculate all three measures of deformation in (3). Compare the extension and the strain to determine the error introduced when using the infinitesimal strain approximation. Assess whether you were justified in using the infinitesimal theory in parts a) and b) of this exercise.

- d) Show algebraically how the extension and strain are calculated as functions of the stretch. Use MATLAB to plot both the extension and the strain versus the stretch over the range  $0 \leq S \leq 3$ . Comment on their graphical relationship to one another.
- e) Determine the approximate range of  $S$  within which the error introduced by neglecting the higher order term in equation (3) for  $E$  is less than 10%. Use MATLAB to plot the error as a percentage versus the stretch.

3) The quasi-static linear elastic solution for the edge dislocation originally found application to problems of plasticity at the scale of defects in the crystal lattice. As Weertman and Weertman (1964) point out, the dislocation solution has found application as a modeling tool for many different geological structures (Figure 2).

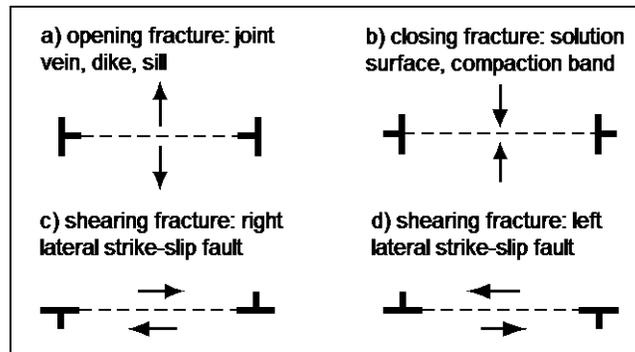


Figure 2. Pairs of edge dislocations used to model geological structures at length scales ranging from centimeters to kilometers.

- a) Describe in words and with a carefully labeled sketch what is meant by the following attributes of the edge dislocation: extra half plane of atoms, Burgers vector, dislocation line, tangent vector, glide plane, dislocation core.
- b) Consider equations (8.36) and (8.37) which give the displacement components that solve Navier's equations of motion for the edge dislocation. Put these equations in dimensionless form and plot each term as a function of the polar angle,  $\theta$ , for a circuit around the dislocation, keeping the radial coordinate,  $r$ , constant. Justify your choice of  $r$  based upon the size of the dislocation core. Identify which

term(s) contribute to the displacement discontinuity across the glide plane and show how this is related to the magnitude,  $b$ , of Burgers vector.

- c) Plot a contour map of each displacement component,  $u_x$  and  $u_y$ , around the edge dislocation centered in a region that is  $200b$  on a side. Choose elastic moduli such that  $G = \lambda = 3 \times 10^4$  MPa. Compare and contrast your contour plots to those of Hytch et al. (2003) from the frontispiece for chapter 8.
- d) Plot a contour map of the normal stress component  $\sigma_{xx}$  using the same elastic moduli and region as in part c). Describe the distribution of this stress component, pointing out any symmetry. Provide a physical explanation why this normal stress is tensile (positive) for  $y > 0$  and compressive (negative) for  $y < 0$ .
- e) Plot a contour map of the shear stress component  $\sigma_{xy}$  using the same elastic moduli and region as in part c). Describe the distribution of this stress component, pointing out any symmetry. Provide a physical explanation why the shear stress is positive for  $x < 0$  and negative for  $x > 0$  along the  $x$ -axis. Indicate how this stress distribution promotes further dislocation glide.
- f) Use your results from parts d) and e) to explain why the arrangement of dislocation pairs in Figure 2c represents a right lateral strike-slip fault.

4) Consider two-dimensional, plane strain conditions defined using the following constraints on the displacement components:

$$u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = 0 \quad (4)$$

In other words the two displacement components in the  $(x, y)$ -plane are only functions of  $x$  and  $y$ , and the  $z$ -component of displacement is zero. In this context explore the equations relating stress, strain, and displacement components, as well as the governing equations for the elastic boundary value problem.

- a) Start with the three-dimensional form of Hooke's Law for the isotropic elastic material using Lamé's constants with stress components as dependent variables:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} \quad (5)$$

Using the constraints imposed by (4) and the kinematic equations (2), expand (5) for each of the six Cartesian stress components as functions of the strain components.

- b) Start with the three-dimensional form of Hooke's Law for the isotropic elastic material using Young's modulus,  $E$ , and Poisson's ratio,  $\nu$ , as the elastic constants with infinitesimal strain components as dependent variables:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \quad (6)$$

Using the constraints imposed by (4), and the results from part a) of this exercise, expand (6) for the six Cartesian strain components as functions of the stress components.

- c) Explain how St. Venant's six equations of compatibility (7.143) – (7.148) for the infinitesimal strain components reduce to one equation relating the in-plane strain components under conditions of plane strain. Transform this compatibility equation so that it is written in terms of the stress components.
- d) Consider the following Airy stress function:

$$\Phi(x, y) = \frac{1}{6} Cy^3 \quad (7)$$

Derive equations for the three in-plane stress components ignoring body forces. Take the region of interest as the rectangle drawn in Figure 3. Sketch and label the traction boundary conditions acting on this region.

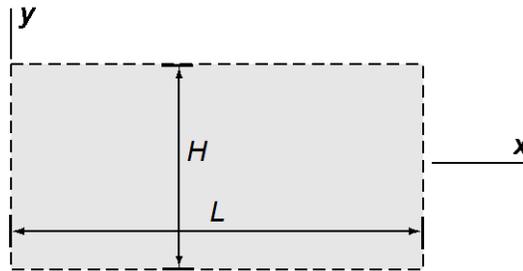


Figure 3. Rectangular region of interest for the traction boundary value problem with Airy stress function given in (7).

- e) Derive equations for the in-plane strain components associated with the stress distribution given in part d). Use the kinematic equations to derive the displacement components from the strains and cast these equations into dimensionless form. Plot and describe each normalized displacement component and the normalized displacement vector field for the region shown in Figure 3 where  $L = 2H$ . Plot the deformed shape of the originally rectangular region. Explain why this solution is called “pure bending”.

5) Cylindrical coordinates are the natural system for a number of important problems in structural geology. With zero displacement parallel to the cylindrical  $z$ -axis these constitute another important set of two-dimensional plane strain problems.

- a) The equilibrium equations for plane cylindrical problems are (8.73) and (8.74), and the equations relating the in-plane stress components to the Airy stress functions are (8.75) – (8.77). Show by substitution, while ignoring the body force terms, that these stress components satisfy the equilibrium conditions.
- b) Consider the Airy stress function for the stress perturbation due to a cylindrical valley that is 100m deep cut from an elastic half-space (Figure 8.21) with mass density,  $\rho$ , and uniform gravitational acceleration,  $g^*$ :

$$\Phi(r, \theta) = \frac{1}{2} \rho g^* R^2 r \theta \cos \theta \quad (8)$$

Derive the equation for the in-plane radial stress component,  $\sigma_{rr}$ , and write a MATLAB m-script to plot a contour map of the distribution. Describe the distribution and explain why this stress component is tensile everywhere.

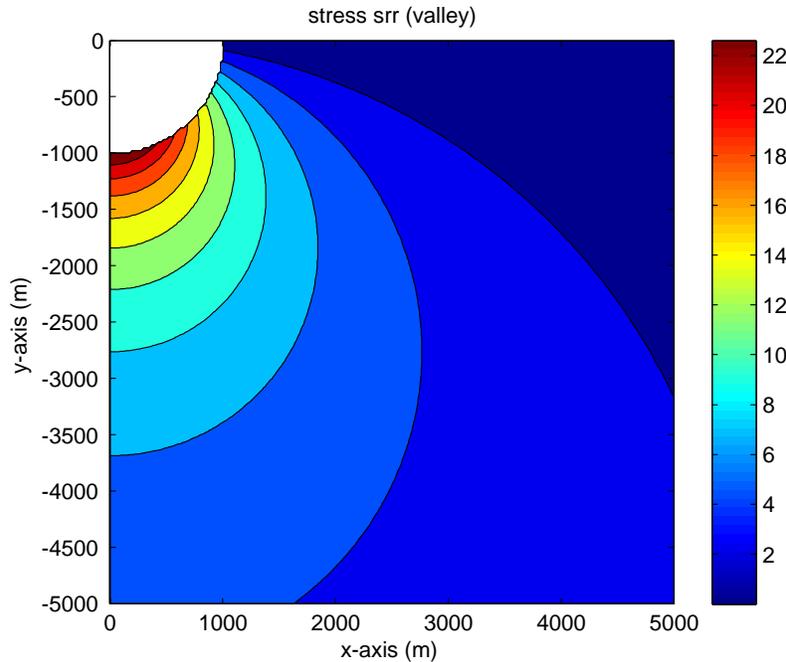


Figure 4. Radial component of normal stress,  $\sigma_{rr}$ , for the perturbation due to the incision of a cylindrical valley in an elastic half-space under gravitational loading. Valley is 1 km deep and the unit weight of the rock is  $\rho g^* = 25 \text{ MPa/km}$ .

- c) Add the stress state due to weight of the material in absence of the valley to the result from part b) and plot contour maps of the radial and circumferential stress components,  $\sigma_{rr}$  and  $\sigma_{\theta\theta}$ . Indicate where these components match the traction boundary conditions. Explain why the contours of the circumferential component are horizontal lines.
  - d) Consider the total state of stress due to the weight of the half-space and the perturbation of the valley and plot the stress trajectories. What simple relationship do the trajectories have to the cylindrical coordinate system? Why?
- 6) Uniaxial compression test results are given for two granites and for a particular limestone measured in two different parts of the same specimen. For both figures the axial stress,  $\sigma_a$ , is plotted versus the axial extension,  $e_a$ . Note that both the applied stress and the resulting extension are negative. That is, the stress is compressive and the extension is a shortening.
- a) Use the stress-extension graph for Georgia granite to estimate the apparent Young's modulus. Compare your value with the range of values given in Table 8.2 for granites and describe the modulus of Georgia granite relative to those. Would you

hear a “ring” or a “thud” if Georgia granite were struck with a geologist’s hammer?

- b) Stress-extension data from a uniaxial compression test on Colorado granite is found in the file colorado.txt with axial stress (MPa) in the first column and axial extension in the second column. Plot these data with stress (ordinate) as a function of extension (abscissa). Use a forward finite difference method to calculate the tangent elastic modulus for each value of the axial extension, omitting that at the origin. Plot the tangent elastic modulus versus axial extension. Describe the apparent non-linear behavior of this rock and provide possible explanations?

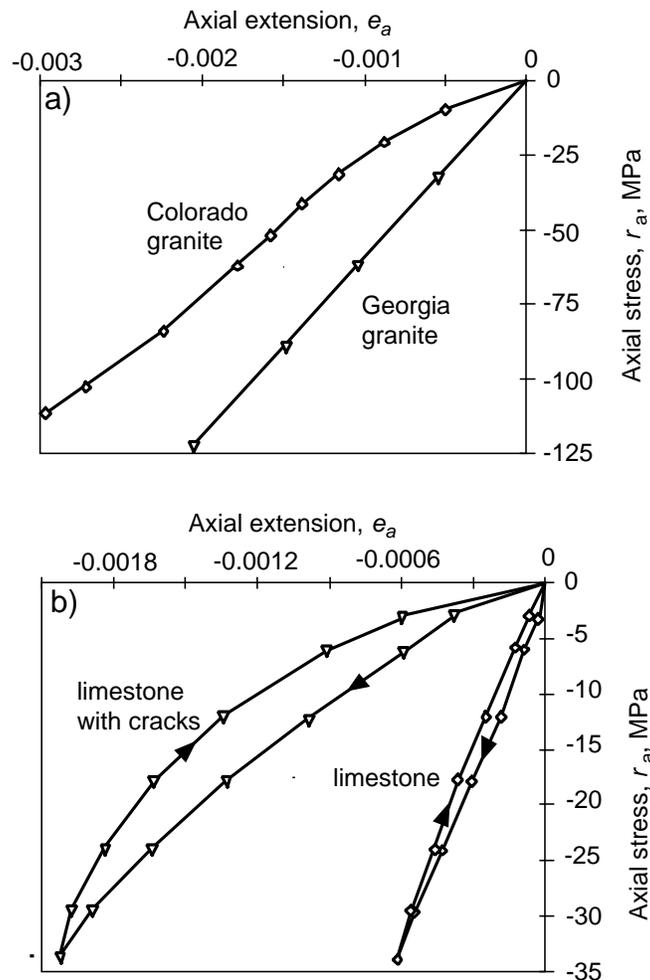


Figure 5. Uniaxial compression test results (Obert & Duvall, 1967). a) Colorado granite and Georgia granite. b) Two different areas on the same limestone sample.

- c) Data in the files limestone.txt and lmst\_cracks.txt were taken using samples from the same limestone formation. The axial stress is in the first column and the axial extension is in the second column of the data files. Construct a plot of axial stress versus axial extension. Describe three differences between mechanical responses of the two limestones. Determine how the tangent modulus differs for each test

and plot the tangent modulus versus the axial extension. Suggest what micro-mechanical mechanisms might explain the differences in the tangent moduli.

7) In this exercise the two-dimensional solution for the elastic boundary-value problem of a cylindrical inclusion (Figure 6) is used to study possible states of stress both within and near material heterogeneities in rock that could serve to concentrate or diminish a remotely applied stress. The radius of the inclusion is  $R$ , the only length dimension in this problem. The four elastic moduli are the shear modulus,  $G$ , and Poisson's ratio,  $\nu$ , for the inclusion (subscript i) and the surrounding material (subscript s). The remote boundary conditions are defined in terms of the principal stress components,  $\sigma_1^r$  and  $\sigma_2^r$ , acting in the  $x$ - and  $y$ -coordinate directions respectively.

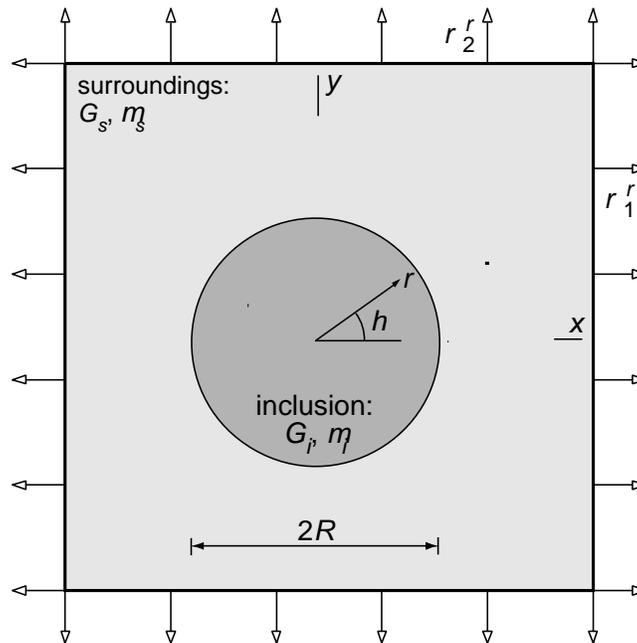


Figure 6. The plane strain elastic boundary value problem for a cylindrical inclusion loaded by remote biaxial principal stresses.

- a) Investigate the stress state within the inclusion as a function of the shear moduli inside and outside the inclusion. Apply a uniaxial remote stress of unit magnitude in the  $x$ -direction. Set Poisson's ratio inside and outside the inclusion to 0.25 and vary the shear moduli to consider the range from an open cavity to a rigid inclusion. Plot your results and describe how the stress state varies with the ratio of shear moduli. What conclusions can you draw from this study about stress concentration and stress diminution within the inclusion?
- b) Study the variation of stress within the inclusion for the same conditions as in part a) but set Poisson's ratio inside and outside the inclusion to 0.5 (incompressible) and then to 0 (perfectly compressible). Plot your results as a function of the shear moduli ratio and describe how the stress state varies for the two different values of Poisson's ratio. What conclusions can you draw from this study about stress concentration and diminution?

- c) The boundary conditions at the contact of the inclusion with the surroundings specify matching displacements, as though the two materials were tightly bonded together. What does this imply about the tractions acting on the surfaces of the two bodies in contact? What can you deduce about the stress states adjacent to these surfaces? Illustrate your answer with a sketch of the boundary and small volume elements with the appropriate cylindrical components of stress. If there is a discontinuity in any of the components, how does this vary with position on the interface? Illustrate your answers with a plot of the three cylindrical stress components just inside and just outside of the contact as a function of position,  $\theta$ , using the following parameters:

$$G_i = 10 \text{ GPa}, G_s = 30 \text{ GPa}, \nu_i = 0.1, \nu_s = 0.3, \sigma_1^r = 1, \sigma_2^r = 0 \quad (9)$$

- Use this plot to demonstrate that your code returns the correct boundary conditions at the interface. Explain why the apparent variation of stress inside the inclusion actually represents a homogeneous state of stress?
- d) Investigate the spatial variation of the stress components inside and just outside the inclusion,  $r = R^+$ , given the parameters in (9) except vary the inclusion shear modulus to consider three cases: an open cavity; a homogeneous body; and a much stiffer inclusion. Keep track of the position, orientation, and magnitude of the greatest tensile stress and use this to describe where and with what orientation opening cracks would be predicted to form if this tension equals the tensile strength. Now change the applied stress to a unit compression acting in the  $x$ -direction and address the same questions about opening cracks.
- e) Investigate the *radius of influence* of the inclusion on the stress field in the surrounding material. Because the remotely applied stresses are referred to the Cartesian coordinate axes use the Cartesian stress components. Consider the spatial variation of stress components along radial lines extending from the edge of the inclusion,  $r/R = 1$ , to a distance of six times the inclusion radius. Begin by considering the case of an open cavity under uniaxial stress of unit magnitude in the  $x$ -direction and use 10% of this stress as the threshold for identifying a significant perturbation. Determine whether the radius of influence changes for different stress components and for different orientations of the radial line. Determine whether the radius of influence is significantly different for the very stiff inclusion relative to the surroundings.

8) Use the two-dimensional solution for the elastic boundary-value problem of a cylindrical hole in an orthotropic material to study possible states of stress near holes in anisotropic rock that would serve to concentrate stress. For plane strain conditions there are two orthogonal axes of elastic symmetry in the  $(x, y)$ -plane.  $E_1$  and  $\nu_{12}$  are Young's modulus and Poisson's ratio in the  $x$ -coordinate direction and  $E_2$  and  $\nu_{21}$  are the respective moduli in the  $y$ -coordinate direction. The self-consistent shear modulus in the  $(x, y)$ -plane is  $G$ .

- a) Consider the first oil shale listed in Table 8.6 to be orthotropic and take the  $x$ -axis parallel to bedding and the  $y$ -axis perpendicular to bedding. Suppose the other two independent moduli are:

$$G = 6.0 \text{ GPa}, \nu_{12} = 0.2 \quad (10)$$

Calculate the value of the second Poisson's ratio,  $\nu_{21}$ , and write down all five elastic moduli.

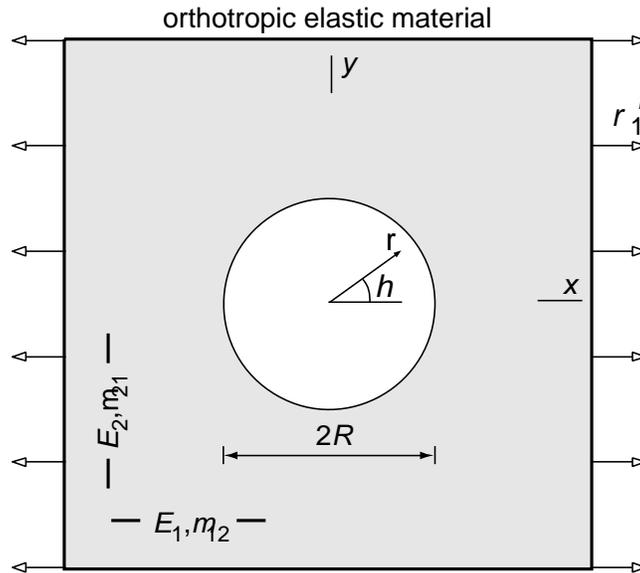


Figure 7. The plane strain elastic boundary value problem for a cylindrical hole of radius  $R$  in an orthotropic material loaded by a remote uniaxial stress.

- b) Write down the linear strain-stress equations using compliances and using the more familiar laboratory constants (Young's modulus and Poisson's ratio) for the orthotropic material. Use these equations to derive equations for the constants,  $C_1$  and  $C_2$ , employed in solutions to the orthotropic elastic boundary value problem:

$$C_1 = \frac{s_{11}}{s_{22}}, \quad C_2 = \frac{(s_{66} + 2s_{12})}{s_{22}} \quad (11)$$

- c) The elastic moduli for the oil shale from part a) must be related to elastic compliances that are real numbers. Test these values to determine if this condition holds starting with the following equations for the constants  $\alpha_1$  and  $\alpha_2$  which appear in the governing compatibility equation for orthotropic elastic boundary value problems:

$$C_1 = \alpha_1 \alpha_2, \quad C_2 = \alpha_1 + \alpha_2 \quad (12)$$

Place an upper bound on the shear modulus assuming the measured values of the two Young's moduli and the given value for the Poisson's ratio are correct.

- d) Use the elastic moduli found in part a) and calculate the circumferential stress around the circular hole in the orthotropic material for a unit remote normal stress and plot this distribution. Describe the concentration and diminution of stress

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around the hole. Compare your result to that for an isotropic rock using the Kirsh solution and plot this distribution on the same graph. Note that the Kirsh solution is independent of the elastic moduli. Evaluate the errors introduced in calculations of the stress state if you were to assume the oil shale is isotropic.