Appendix SA3.1 Basic Relationships in the Multiregional Input–Output Model

In standard input–output fashion, total demand for commodity i in region s is given by

$$\sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{i}^{s} \tag{A3.1.1}$$

The total supply of commodity i in region s is the total that is shipped in from other regions,

$$\sum_{r=1}^{p} z_{i}^{rs} \, (r \neq s)$$

plus the amount that is supplied from within the region, z_{i}^{ss}. This is just T_{i}^{s}, the sum of the elements in column s in Table 3.8, as defined in (3.18). Since shipments (supplies) occur only to satisfy needs (demands), we have, for each commodity i

$$T_{i}^{s} = \sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{i}^{s} \tag{A3.1.2}$$

Total production of i in region r is equivalent to the total amount of i shipped from r, including that kept within the region

$$x_{i}^{r} = \sum_{s=1}^{p} z_{i}^{rs} \tag{A3.1.3}$$

From the definition of the interregional proportions in section 3.4.2, $c_{i}^{rs} = z_{i}^{rs} T_{i}^{s}$, (A3.1.3) can be rewritten as

$$x_{i}^{r} = \sum_{s=1}^{p} c_{i}^{rs} T_{i}^{s} \tag{A3.1.4}$$

Putting T_{i}^{s} as defined in (A3.1.2), into (A3.1.4)

$$x_{i}^{r} = \sum_{s=1}^{p} c_{i}^{rs} \left(\sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{i}^{s} \right) \quad \left(i = 1, \ldots, n \right) \tag{A3.1.5}$$

Using familiar matrix notation, let

$$\mathbf{x}^r = \begin{bmatrix} x_1^r \\ \vdots \\ x_n^r \end{bmatrix}, \mathbf{x}^s = \begin{bmatrix} x_1^s \\ \vdots \\ x_n^s \end{bmatrix}, \mathbf{f}^s = \begin{bmatrix} f_1^s \\ \vdots \\ f_n^s \end{bmatrix}$$

$$\mathbf{A}^s = \begin{bmatrix} a_{11}^s & \cdots & a_{1n}^s \\ \vdots & & \vdots \\ a_{n1}^s & & a_{nn}^s \end{bmatrix}, \hat{\mathbf{c}}^{rs} = \begin{bmatrix} c_1^{rs} & 0 & \cdots & 0 \\ 0 & c_2^{rs} & & \\ \vdots & & & \\ 0 & & & c_n^{rs} \end{bmatrix}$$

The reader should be convinced that the entire set of n equations for outputs of goods in region r can be expressed as

$$x^r = \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}(\mathbf{A}^s\mathbf{x}^s + \mathbf{f}^s) = \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}\mathbf{A}^s\mathbf{x}^s + \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}\mathbf{f}^s \qquad \text{(A3.1.6)}$$

There will be p such matrix equations, one for each region r ($r = 1, \ldots, p$). Again using matrix notation, as in section 3.4, we can construct

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^s \\ \vdots \\ \mathbf{x}^p \end{bmatrix}, \mathbf{f} = \begin{bmatrix} \mathbf{f}^1 \\ \vdots \\ \mathbf{f}^s \\ \vdots \\ \mathbf{f}^p \end{bmatrix}, \mathbf{A} = \begin{bmatrix} \mathbf{A}^1 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & & \mathbf{A}^s & & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{A}^p \end{bmatrix} \text{ and}$$

$$\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{11} & \cdots & \hat{\mathbf{c}}^{1s} & \cdots & \hat{\mathbf{c}}^{1p} \\ \vdots & & \vdots & & \vdots \\ \hat{\mathbf{c}}^{r1} & \cdots & \hat{\mathbf{c}}^{rs} & \cdots & \hat{\mathbf{c}}^{rp} \\ \vdots & & \vdots & & \vdots \\ \hat{\mathbf{c}}^{p1} & \cdots & \hat{\mathbf{c}}^{ps} & \cdots & \hat{\mathbf{c}}^{pp} \end{bmatrix}$$

Then the p matrix equations in (A3.1.6) can be compactly expressed as

$$\mathbf{x} = \mathbf{C}(\mathbf{A}\mathbf{x} + \mathbf{f}) = \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{f}$$

from which

$$(\mathbf{I} - \mathbf{C}\mathbf{A})\mathbf{x} = \mathbf{C}\mathbf{f} \qquad \text{(A3.1.7)}$$

and

$$\mathbf{x} = (\mathbf{I} - \mathbf{C}\mathbf{A})^{-1}\mathbf{C}\mathbf{f} \qquad \text{(A3.1.8)}$$

as in (3.22) and (3.23) in the text.