Chapter 14

```
In[1]:= Needs["Graphics`MultipleListPlot`"]
        Needs["Graphics`Legend`"]
        Needs["Graphics`PlotField`"]
```

à Question 1

In[4]:= $k = \frac{\text{Log[6] - Log[5.8]}}{10}$

Out[4]= 0.00339016

In[5]:= estp = 5.8 $E^{k\,(t-1701)}$

Out[5]= 5.8 $e^{0.00339016\,(-1701+t)}$

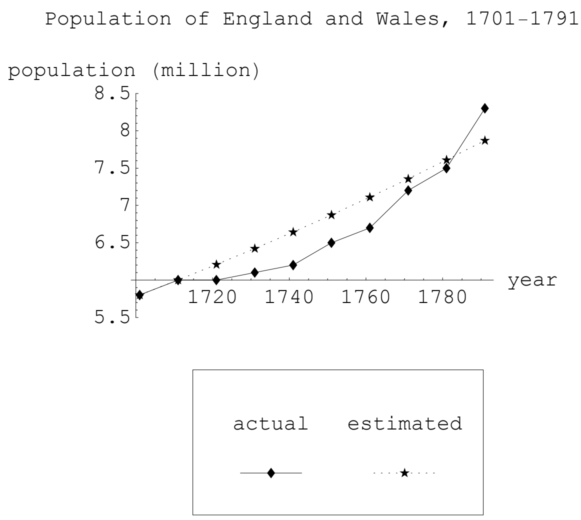
```
In[6]:= pointsestp = Table[{t, estp}, {t, 1701, 1791, 10}]

Out[6]= {{1701, 5.8}, {1711, 6.}, {1721, 6.2069}, {1731, 6.42093},
         {1741, 6.64234}, {1751, 6.87138}, {1761, 7.10833},
         {1771, 7.35344}, {1781, 7.60701}, {1791, 7.86932}}

In[7]:= pointsactp = {{1701, 5.8}, {1711, 6.0}, {1721, 6.0}, {1731, 6.1}, {1741, 6.2},
          {1751, 6.5}, {1761, 6.7}, {1771, 7.2}, {1781, 7.5}, {1791, 8.3}}

Out[7]= {{1701, 5.8}, {1711, 6.}, {1721, 6.}, {1731, 6.1}, {1741, 6.2},
         {1751, 6.5}, {1761, 6.7}, {1771, 7.2}, {1781, 7.5}, {1791, 8.3}}
```

```
In[8]:= MultipleListPlot[pointsactp, pointsestp, PlotJoined -> True,
          PlotRange -> {5.5, 8.5}, AxesLabel -> {"year", "population (million)"},
          PlotLabel -> "Population of England and Wales, 1701-1791\n\n",
          PlotLegend -> {"actual", "estimated"},
        LegendPosition -> {-0.35, -1.2},
        LegendOrientation -> Horizontal,
        LegendSize -> {1, .5}];
```



à Question 2

We have the following information for each country after the epidemic:

Country A: $b = 3\%$, $d = 2\%$ with $(b\text{-}d) = 1\%$ $\lambda = 0.01$

Country B: $b = 5\%$, $d = 3\%$ with $(b\text{-}d) = 2\%$ $\lambda = 0.02$

In[9]:= **Solve[2 p0 E$^{0.01\,t}$ == p0 E$^{0.02\,t}$, t]**

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[9]= {{t → 69.3147}, {t → -∞}}

à Question 3

We have

$$p(t) = \text{p0}\,E^{[(b-d)-m]\,t}$$

In[10]:= **FullSimplify[Solve[p0 E$^{((b-d)-m)\,t1}$ == $\frac{1}{2}$ p0 E$^{((b-d)-m)\,t0}$, t1]]**

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[10]= $\left\{\left\{t1 \to \frac{\text{Log}\left[\frac{1}{2}\, e^{(b-d-m)\,t0}\right]}{b-d-m}\right\}\right\}$

```
In[11]:= Apart[(Log[2] - Log[E^((b-d-m) t0)]) / (-b + d + m)]
```

$$\text{Out[11]}= -\frac{\text{Log}[2]}{b-d-m}+\frac{\text{Log}[e^{(b-d-m)\,t0}]}{b-d-m}$$

But

$$\text{Ln}(E^{(b-d-m)\,t0}) = (b-d-m)\,t0$$

Hence,

$$t1 = \frac{-\text{Log}[2]}{b-d-m} + t0$$

$$t1 - t0 = \frac{-\text{Log}[2]}{b-d-m}$$

To check, using the information in the previous question, let *b* = 5%, *d* = 3% and *m* = 3%, then *b*-*d*-*m* = -1%

```
In[12]:= -Log[2] / (-0.01)

Out[12]= 69.3147
```

à Question 4

Given

$$p(t) = p0\, E^{k\, t}$$

(i) Population trebbling in size

```
In[13]:= Solve[p0 E^(0.03 t1) == 3 p0 E^(0.03 t0), t1]

        Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

Out[13]= {{t1 → 33.3333 Log[3. e^(0.03 t0)]}}

In[14]:= Simplify[Solve[t1 == 33.3333 (0.03 t0 + Log[3]), t1]]

Out[14]= {{t1 → 36.6204 + 0.999999 t0}}
```

i.e., t1-t0 = 36.6204

(ii) General result

```
In[15]:= Clear[p0, λ, k]

In[16]:= Solve[p0 E^(k t1) == λ p0 E^(k t0), t1]

        Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$$\text{Out[16]}= \left\{\left\{t1 \to \frac{\text{Log}[e^{k\,t0}\,\lambda]}{k}\right\}\right\}$$

Which can be expressed

$$t1 - t0 = \frac{\text{Log}(\lambda)}{k}$$

à Question 5

To derive the Taylor expansion about the point *a/b*, we use the **Series** command

```
In[17]:= Series[p (a - b p), {p, a / b, 1}]
```

Out[17]= $-a\left(p-\frac{a}{b}\right)+O\left[p-\frac{a}{b}\right]^2$

Neglecting the error term, then

$$\dot{p} = -a(p - \tfrac{a}{b})$$

```
In[18]:= DSolve[{p'[t] == -a (p[t] - (a / b)), p[0] == p0}, p[t], t]
```

Out[18]= $\left\{\left\{p[t] \to \frac{e^{-a\,t}\left(a\,e^{a\,t}+b\left(-\frac{a}{b}+p0\right)\right)}{b}\right\}\right\}$

Which can be expressed

$$p(t) = \tfrac{a}{b} + (\mathrm{p0} - \tfrac{a}{b})\,E^{-a\,t}$$

Mathematica cannot take the limit of $E^{-a\,t}$ since it does not know whether a is positive or negative. Assuming a is positive, and given

```
In[19]:= Limit[E^-t, t -> ∞]
```

Out[19]= 0

then $p(t) \to a / b$, which implies that equilibrium is never achieved in a finite time period.

à Question 6

(i)

The fixed points are found from

```
In[20]:= Solve[p (a + c p) == 0, p]
```

Out[20]= $\left\{\{p \to 0\},\ \left\{p \to -\frac{a}{c}\right\}\right\}$

The turning point is

```
In[21]:= Solve[D[p (a + c p), p] == 0, p]
```

Out[21]= $\left\{\left\{p \to -\frac{a}{2\,c}\right\}\right\}$

which is negative since $a > 0$ and $c > 0$. Furthermore,

```
In[22]:= D[D[p (a + c p), p], p]
```

Out[22]= 2 c

which is positive, hence the turning point is a minimum.

(ii)

These observations can be verified by assuming for example $a = 0.2$ and $c = 0.004$.

```
In[23]:= Clear[a, c, p]
```

```
In[24]:= example = (a + c p) p /. {a -> 0.2, c -> 0.004}

Out[24]= (0.2 + 0.004 p) p

In[29]:= Solve[D[r (M - p) (p - m), p] == 0, p]

Out[29]= {{p -> (m + M)/2}}

In[25]:= Plot[example, {p, -100, 100}, AxesLabel -> {"p", "dp/dt"}];
In[30]:= D[D[r (M - p) (p - m), p], p] /. p -> (m + M)/2

Out[30]= -2 r
```

which is negative, hence the turning point is a maximum.

```
In[31]:= D[r (M - p) (p - m), p] /. p -> m

Out[31]= (-m + M) r
```

Hence, for positive population p0, the population grows indefinitely.

(iii)

```
In[26]:= DSolve[p'[t] == p[t] (a + c p[t]), p[t], t]
```

$$\text{Out[26]= } \left\{\left\{p[t] \to -\frac{a\, e^{a t}}{c\, e^{a t} - e^{C[1]}}\right\}\right\}$$

If we let $k = E^{C[1]}$, a constant, then this can be expressed

$$p(t) = \frac{-a}{c - k\, E^{-a\, t}}$$

But $p(t) = \infty$ if $c - k\, E^{-a\, t} = 0$.

```
In[27]:= Solve[k E^(-a t) == c, t]

   Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

$$\text{Out[27]= } \left\{\left\{t \to -\frac{\text{Log}[\frac{c}{k}]}{a}\right\}\right\}$$

Hence, an infinite population is reached in a finite time period

à Question 7

(i)

The two fixed points are

```
In[28]:= Solve[r (M - p) (p - m) == 0, p]

Out[28]= {{p -> m}, {p -> M}}
```

The turning point is

```
In[29]:= Solve[D[r (M - p) (p - m), p] == 0, p]
```

Out[29]= $\left\{\left\{p \to \frac{m+M}{2}\right\}\right\}$

```
In[30]:= D[D[r (M - p) (p - m), p], p] /. p -> (m + M)/2
```

```
Out[30]= -2 r
```

which is negative, hence the turning point is a maximum.

```
In[31]:= D[r (M - p) (p - m), p] /. p -> m
```

```
Out[31]= (-m + M) r
```

which is positive since $M > m$, so the fixed point m is locally unstable.

```
In[32]:= D[r (M - p) (p - m), p] /. p -> M
```

```
Out[32]= - (-m + M) r
```

which is negative since $M > m$, so the fixed point M is locally stable.

The parameter m is associated with such factors as:
(a) reproduction system
(b) density of species (i.e., the area over which it operates).

The parameter M is associated with such factors as:
(a) food availability
(b) predation
(c) catch

(ii)

In the case of (b) species can begin to die out well before $p = 0$, i.e., at $p = m$. It is important, therefore, to establish the value of m, e.g., the blue whale.

à Question 8

(i)

```
In[33]:= DSolve[{p'[t] == r p[t] (a - Log[p[t]]), p[0] == p0}, p[t], t]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[33]= $\left\{p[t] \to e^{e^{-r t} (-a + a e^{r t} + \text{Log}[p0])}\right\}$

In[34]:= PowerExpand$\left[E^{E^{-r t} (-a + a E^{r t} + \text{Log}[p0])}\right]$

Out[34]= $e^{e^{-r t} (-a + a e^{r t} + \text{Log}[p0])}$

In[35]:= FullSimplify$\left[E^{E^{-r t} (-a + a E^{r t} + \text{Log}[p0])}\right]$

Out[35]= $e^{a + e^{-r t} (-a + \text{Log}[p0])}$

which can be written

$$E^{-a(E^{-r t}-1)}\, E^{E^{-r t}\,\mathrm{Log[p0]}} = E^{-a(E^{-r t}-1)}\, \mathrm{p0}^{E^{-r t}}$$

since

```
In[36]:= E^Log[p0^E^(-r t)]

Out[36]= p0^e^(-r t)
```

(ii) and (iii)

The fixed points are,

```
In[37]:= Solve[r p (a - Log[p]) == 0, p]

    Solve::verif : Potential solution {p → 0} (possibly
      discarded by verifier) should be checked by hand. May require use of limits.

Out[37]= {{p → e^a}}

In[38]:= Limit[r p (a - Log[p]), p -> 0]

Out[38]= 0
```

Hence, there are two fixed points one at $p = 0$ and the other at E^a.

```
In[39]:= D[r p (a - Log[p]), p]

Out[39]= -r + r (a - Log[p])
```

The slope of $\dot{p}$ at the point $p = 0$ cannot be determined since this involves Log[0]. On the other hand,

```
In[40]:= Simplify[D[r p (a - Log[p]), p] /. p -> E^a]

Out[40]= r (-1 + a - Log[e^a])
```

which is equal to $-r$, since Log[E^a] = a Log[E] = a. Since r is positive, then the slope of $\dot{p}$ at the point E^a is negative, and so the point $p = E^a$ is locally stable.

Also $\dot{p}$ is at a maximum at the value

```
In[41]:= Solve[D[r p (a - Log[p]), p] == 0, p]

Out[41]= {{p → e^(-1+a)}}
```

(iv)

```
In[42]:= Limit[E^(-a (E^(-r t)-1)) p0^E^(-r t), t -> ∞]

Out[42]= Limit[e^(-a (-1+e^(-r t))) p0^e^(-r t), t → ∞]
```

So *Mathematica* cannot solve for the limit directly. But we note that the limit of $E^{-r t}$ as t -> ∞ is zero. Hence, the limit is E^a as t -> ∞.

à Question 9

```
In[43]:= DSolve[{x'[t] == -3 y[t], y'[t] == -9 x[t]}, {x[t], y[t]}, t]
```

Out[43]= $\left\{\left\{x[t] \to \frac{1}{6} e^{-3\sqrt{3}\,t}\left(3\,C[1] + 3\,e^{6\sqrt{3}\,t}\,C[1] + \sqrt{3}\,C[2] - \sqrt{3}\,e^{6\sqrt{3}\,t}\,C[2]\right),\; y[t] \to -\frac{1}{2} e^{-3\sqrt{3}\,t}\left(-\sqrt{3}\,C[1] + \sqrt{3}\,e^{6\sqrt{3}\,t}\,C[1] - C[2] - e^{6\sqrt{3}\,t}\,C[2]\right)\right\}\right\}$

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\dot{y}}{\dot{x}} = \frac{-9x}{3y} = \frac{3x}{y}$$

then we can integrate by parts for we have,

$$y\,dy = 3\,x\,dx$$

In[44]:= $\int y \, dy$

Out[44]= $\frac{y^2}{2}$

In[45]:= $\int 3\,x \, dx$

Out[45]= $\frac{3\,x^2}{2}$

In[46]:= Solve$\left[\frac{y^2}{2} == \frac{3\,x^2}{2} + \frac{c}{2},\, y\right]$

Out[46]= $\left\{\left\{y \to -\sqrt{2}\sqrt{\frac{c}{2} + \frac{3\,x^2}{2}}\right\}, \left\{y \to \sqrt{2}\sqrt{\frac{c}{2} + \frac{3\,x^2}{2}}\right\}\right\}$

where $c/2$ is the constant of integration.

Alternatively, we can solve directly

```
In[47]:= DSolve[y'[x] == 3 x / y[x], y[x], x]
```

Out[47]= $\left\{\left\{y[x] \to -\sqrt{3\,x^2 + C[1]}\right\}, \left\{y[x] \to \sqrt{3\,x^2 + C[1]}\right\}\right\}$

Which can be expressed,

$y(x) = -\sqrt{3\,x^2 + c}$ and $y(x) = \sqrt{3\,x^2 + c}$, where $c = 3\,C[1]$.

à Question 10

The $\dot{T} = 0$ and $\dot{B} = 0$ phase lines are

In[48]:= Solve$\left[a\left(1 - \frac{T}{k1}\right) T - b\,T\,B == 0,\, B\right]$

Out[48]= $\left\{\left\{B \to \frac{a\,(k1 - T)}{b\,k1}\right\}\right\}$

In[49]:= $\text{Apart}\left[\frac{a\ (k1 - T)}{b\ k1}\right]$

Out[49]= $\frac{a}{b} - \frac{a\ T}{b\ k1}$

In[50]:= $\text{Solve}\left[c\left(1 - \frac{B}{k2}\right) B - d\ T\ B == 0,\ B\right]$

Out[50]= $\left\{\{B \to 0\},\ \left\{B \to \frac{k2\ (c - d\ T)}{c}\right\}\right\}$

In[51]:= $\text{Apart}\left[\frac{k2\ (c - d\ T)}{c}\right]$

Out[51]= $k2 - \frac{d\ k2\ T}{c}$

The two phase lines are then,

$$B = \frac{a}{b} - \left(\frac{a}{b\,\text{k1}}\right) T \qquad \text{for } \dot{T} = 0$$

$$B = \text{k2} - \left(\frac{d\,\text{k2}}{c}\right) T \qquad \text{for } \dot{B} = 0$$

which intersect at the point

In[52]:= $\text{Simplify}\left[\text{Solve}\left[\left\{B == \frac{a}{b} - \frac{a\ T}{b\ k1},\ B == k2 - \frac{d\ k2\ T}{c}\right\},\ \{B,\ T\}\right]\right]$

Out[52]= $\left\{\left\{B \to \frac{a\ (c - d\ k1)\ k2}{a\ c - b\ d\ k1\ k2},\ T \to \frac{c\ k1\ (a - b\ k2)}{a\ c - b\ d\ k1\ k2}\right\}\right\}$

which is as far as we can take the general result with *Mathematica*.

à Question 11

(i)

Given

$$\frac{N(t)}{K} = \frac{\text{N0}/K}{(\text{N0}/K) + (1 - (\text{N0}/K)\, E^{-r\,t}}$$

then $N(2)/K$ is

In[53]:= $\frac{0.25}{0.25 + (1 - 0.25)\ E^{-(0.71)\ 2}}$

Out[53]= 0.579662

and hence $N(2)$ is

In[54]:= $(0.579662)\ (80.5\ 10^6)$

Out[54]= 4.66628×10^7

or 46.6628×10^6 Kg

(ii)

In solving for t, we write $N(t)$ as Nt.

In[55]:= $\text{Solve}\left[\frac{\text{Nt}}{\text{K}} == \frac{\text{N0}/\text{K}}{(\text{N0}/\text{K}) + (1 - (\text{N0}/\text{K}))\,\text{E}^{-r\,t}},\ t\right]$

```
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.
```

Out[55]= $\left\{\left\{t \to \frac{\text{Log}\left[\frac{(\text{K}-\text{N0})\,\text{Nt}}{\text{N0}\,(\text{K}-\text{Nt})}\right]}{r}\right\}\right\}$

which can be expressed

$$t = -\left(\frac{1}{r}\right)\text{Log}\left[\frac{(\text{N0}/K)\,(1-\text{Nt}/K)}{\text{Nt}/K(1-\text{N0}/K)}\right]$$

(a) t at which $N(t) = 0.5\,K$ or $N(t)/K = 0.5$.

In[56]:= $-\left(\frac{1}{0.71}\right)\text{Log}\left[\frac{0.25\,(1-0.5)}{0.5\,(1-0.25)}\right]$

Out[56]= 1.54734

(b) t at which $N(t) = 0.75\,K$ or $N(t)/K = 0.75$.

In[57]:= $-\left(\frac{1}{0.71}\right)\text{Log}\left[\frac{0.25\,(1-0.75)}{0.75\,(1-0.25)}\right]$

Out[57]= 3.09468