Solutions to exercises

Exercise 1.1 Using Equation 1.1, $L = 4\pi R^2 \sigma T_{\text{eff}}^4$, the luminosities corresponding to each combination of radius and temperature are as shown in Table S1.1. (Remember to first convert the radii from solar units to metres, in order to calculate the luminosity L in watts. Then divide by the solar luminosity $L_\odot$ to convert the answer into solar units.)

As an example, the first entry in Table S1.1 may be calculated as follows:

$$\begin{aligned} L &= 4\pi R^2 \sigma T_{\text{eff}}^4 \\ &= 4\pi \times (0.1 \times 6.96 \times 10^8\ \text{m})^2 \times (5.671 \times 10^{-8}\ \text{J m}^{-2}\ \text{K}^{-4}\ \text{s}^{-1}) \times (2000\ \text{K})^4 \\ &= 5.52 \times 10^{22}\ \text{J s}^{-1} = 5.52 \times 10^{22}\ \text{J s}^{-1} \times \frac{\text{L}_\odot}{3.83 \times 10^{26}\ \text{W}} \\ &= 1.44 \times 10^{-4}\ \text{L}_\odot. \end{aligned}$$

Since Figure 1.1 has logarithmic axes, in order to plot this value, note that $\log_{10}(1.44 \times 10^{-4}) = -3.84$. Hence one end of the line connecting points with $R = 0.1\ \text{R}_\odot$ lies at the point $\log_{10}(L/\text{L}_\odot) = -3.84$, $T_{\text{eff}} = 2000$ K. The set of lines of constant radii illustrating all the results in Table S1.1 is shown on Figure S1.1 overleaf.

Table S1.1 Luminosities for stars of a given temperature and radius, for use with Exercise 1.1.

T_{eff}	$R = 0.1\ \text{R}_\odot$	$R = 1\ \text{R}_\odot$
2000 K	$L = 1.44 \times 10^{-4}\ \text{L}_\odot$	$L = 1.44 \times 10^{-2}\ \text{L}_\odot$
4000 K	$L = 2.31 \times 10^{-3}\ \text{L}_\odot$	$L = 2.31 \times 10^{-1}\ \text{L}_\odot$
6000 K	$L = 1.17 \times 10^{-2}\ \text{L}_\odot$	$L = 1.17\ \text{L}_\odot$
10 000 K	$L = 9.01 \times 10^{-2}\ \text{L}_\odot$	$L = 9.01\ \text{L}_\odot$
20 000 K	$L = 1.44\ \text{L}_\odot$	$L = 144\ \text{L}_\odot$
40 000 K	$L = 23.1\ \text{L}_\odot$	$L = 2.31 \times 10^{3}\ \text{L}_\odot$
T_{eff}	$R = 10\ \text{R}_\odot$	$R = 100\ \text{R}_\odot$
2000 K	$L = 1.44\ \text{L}_\odot$	$L = 1.44 \times 10^{2}\ \text{L}_\odot$
4000 K	$L = 23.1\ \text{L}_\odot$	$L = 2.31 \times 10^{3}\ \text{L}_\odot$
6000 K	$L = 117\ \text{L}_\odot$	$L = 1.17 \times 10^{4}\ \text{L}_\odot$
10 000 K	$L = 901\ \text{L}_\odot$	$L = 9.01 \times 10^{4}\ \text{L}_\odot$
20 000 K	$L = 1.44 \times 10^{4}\ \text{L}_\odot$	$L = 1.44 \times 10^{6}\ \text{L}_\odot$
40 000 K	$L = 2.31 \times 10^{5}\ \text{L}_\odot$	$L = 2.31 \times 10^{7}\ \text{L}_\odot$

Exercise 1.2 Let us suppose the Sun contains N ions in total. Each hydrogen ion will be matched by one electron, whereas each helium ion will be matched by two electrons. The mean molecular mass $\mu_\odot$ is therefore

$$\mu_\odot = \frac{N_{\text{H}}(m_{\text{H}}/u) + N_{\text{He}}(m_{\text{He}}/u) + N_{\text{e}}(m_{\text{e}}/u)}{N_{\text{H}} + N_{\text{He}} + N_{\text{e}}}.$$

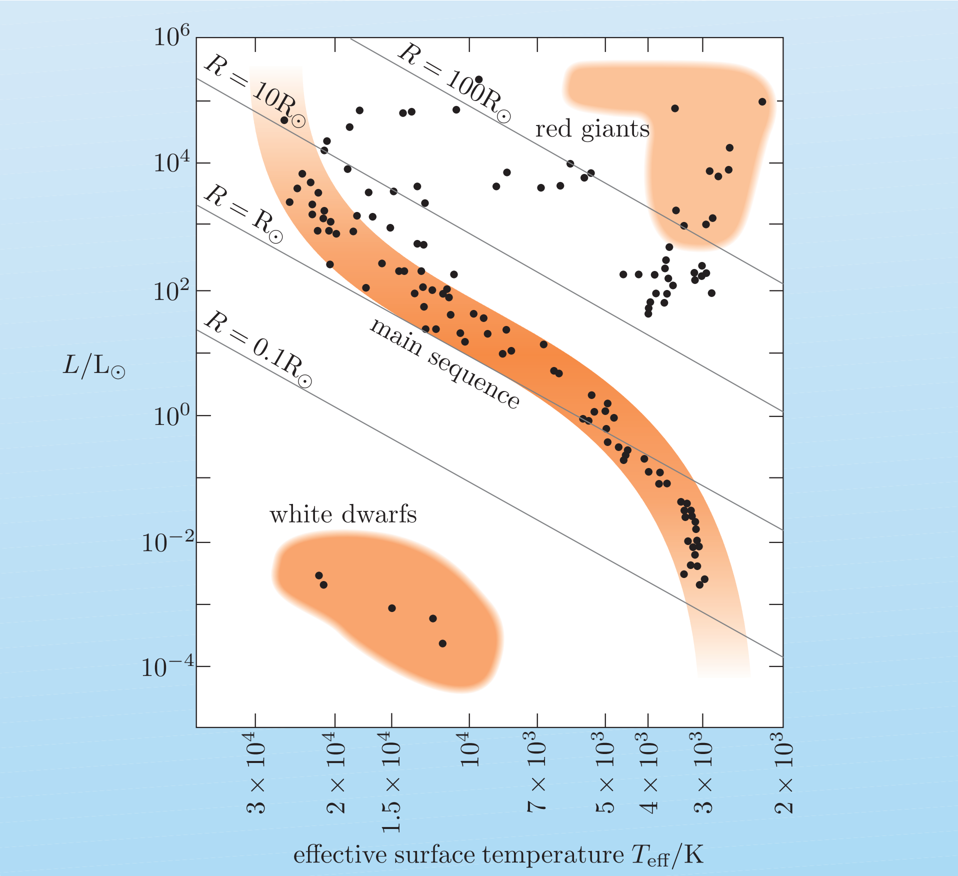

Figure S1.1 A schematic Hertzsprung–Russell diagram. The sloping lines indicate where stars would have radii $R = 0.1\,R_\odot$, $R_\odot$, $10\,R_\odot$ and $100\,R_\odot$.

Given the proportions in the question, $N_H = 0.927N$, $N_{He} = 0.073N$ and $N_e = 0.927N + (2 \times 0.073N) = 1.073N$. Assuming $m_H/u \approx 1$, $m_{He}/u \approx 4$ and $m_e/u \approx 0$, we can write

$$\mu_\odot \approx \frac{0.927N + (0.073N \times 4)}{(0.927 + 0.073 + 1.073)N} \approx 1.219/2.073 \approx 0.6.$$

So the mean molecular mass is $\mu_\odot \approx 0.6$ or $\overline{m}_\odot \approx 0.6u$.

Exercise 1.3 With a Kramers opacity, the opacity is given by $\kappa(r) \propto \rho(r)/T^{3.5}(r)$. Using the argument from the previous example, the mean opacity may be expressed as simply $\overline{\kappa} \propto \overline{\rho}/T_c^{3.5}$, where T_c is the star's central temperature and $\overline{\rho}$ is its mean density. Now, from Worked Example 1.1, we already have the relationships $\overline{\rho} \propto M/R^3$ and $T_c \propto M/R$, where M and R are the mass and radius of the star. So, the mean value of the Kramers opacity may be re-written as $\overline{\kappa} \propto R^{0.5}/M^{2.5}$.

We derive the same penultimate equation as in the previous worked example, namely $L \propto M^3/\overline{\kappa}$, where L is the star's surface luminosity. So, using the relationship above, this becomes $L \propto M^{5.5}R^{-0.5}$.

Exercise 1.4 The second branch of the proton–proton chain will include one electron–positron annihilation reaction as only one instance of the initial proton+proton reaction is involved. The overall reaction may be written as

$$2e^- + 4p \longrightarrow {}^4_2\text{He} + \nu_e(\text{pp}) + \nu_e(\text{Be}) + \gamma_{\text{pd}} + \gamma_{\text{He}} + 2\gamma_e,$$

where $\nu_e(\text{pp})$ is the electron neutrino released by the proton+proton reaction step, $\nu_e(\text{Be})$ is the electron neutrino released by the beryllium-7 electron capture reaction step, γ_{pd} is the gamma-ray released by the proton+deuterium reaction step, γ_{He} is the gamma-ray released by the helium-3 + helium-4 reaction step, and $2\gamma_e$ are the gamma-rays released by the electron–positron annihilation step.

Now, using the masses from earlier, i.e. $1.672\,623 \times 10^{-27}$ kg for the ${}^1_1$H *nucleus* and $6.644\,656 \times 10^{-27}$ kg for the ${}^4_2$He *nucleus*, the mass defect can be calculated as

$$\begin{aligned}\Delta m &= \text{initial mass} - \text{final mass}\\ &= 2m(e^-) + m(4p) - m({}^4_2\text{He}) - m(\nu_e(\text{pp})) - m(\nu_e(\text{Be})) - m(\gamma_{\text{pd}}) - m(\gamma_{\text{He}}) - m(2\gamma_e)\\ &= 1.8218 \times 10^{-30}\ \text{kg} + 6.690\,492 \times 10^{-27}\ \text{kg} - 6.644\,656 \times 10^{-27}\ \text{kg} - 0 - 0 - 0 - 0 - 0\\ &= 4.7658 \times 10^{-29}\ \text{kg}.\end{aligned}$$

Exactly as for branch ppI, for branch ppII the energy equivalent $E = (\Delta m)c^2$ is 4.2833×10^{-12} J or 26.74 MeV.

This includes the energy that goes into the γ-rays, which is then absorbed by the surrounding gas. As before, the two neutrinos escape the star without depositing their energy, which in this case removes 0.26 MeV for the $\nu_e(\text{pp})$ neutrino and $(0.9 \times 0.86) + (0.1 \times 0.38)$ MeV = 0.81 MeV for the $\nu_e(\text{Be})$ neutrino. This leaves $26.74\,\text{MeV} - 0.26\,\text{MeV} - 0.81\,\text{MeV} = 25.67\,\text{MeV}$ for the star.

Exercise 2.1 The free-fall time for the Sun is

$$\begin{aligned}t_{\text{ff}} &= \left(\frac{3\pi}{32G\rho}\right)^{1/2} = \left(\frac{3\pi}{32 \times (6.673 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}) \times (1.41 \times 10^3\ \text{kg m}^{-3})}\right)^{1/2}\\ &= 1770\ \text{s}.\end{aligned}$$

The free-fall time of the Sun is therefore about half an hour.

Exercise 2.2 The limiting case is when $f = 0.5$. So, the first term in Equation 2.19

$$\begin{aligned}\left(\frac{36}{\pi}\frac{3c}{4\sigma}\frac{(1-f)}{f^4}\right)^{1/2} &= \left(\frac{36}{\pi} \times \frac{3 \times 2.998 \times 10^8\ \text{m s}^{-1}}{4 \times 5.671 \times 10^{-8}\ \text{J m}^{-2}\ \text{K}^{-4}\ \text{s}^{-1}} \times \frac{1-0.5}{0.5^4}\right)^{1/2}\\ &= 6.029 \times 10^8\ \text{m}^{1/2}\ \text{s K}^2\ \text{kg}^{-1/2}.\end{aligned}$$

The second term in Equation 2.19 becomes

$$\left(\frac{k}{m}\right)^2 = \left(\frac{1.381 \times 10^{-23}\ \text{J K}^{-1}}{0.6 \times 1.661 \times 10^{-27}\ \text{kg}}\right)^2 = 1.920 \times 10^8\ \text{m}^4\ \text{s}^{-4}\ \text{K}^{-2}.$$

Finally, the third term in Equation 2.19 becomes

$$\left(\frac{1}{G}\right)^{3/2} = \left(\frac{1}{6.673 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2}}\right)^{3/2} = 1.835 \times 10^{15}\ \text{kg}^{3/2}\ \text{m}^{-9/2}\ \text{s}^3.$$

So the upper mass limit for a star is given by

$$\begin{aligned}M &\approx (6.029 \times 10^8\ \text{m}^{1/2}\ \text{s K}^2\ \text{kg}^{-1/2}) \times (1.920 \times 10^8\ \text{m}^4\ \text{s}^{-4}\ \text{K}^{-2}) \times (1.835 \times 10^{15}\ \text{kg}^{3/2}\ \text{m}^{-9/2}\ \text{s}^3)\\ &\approx 2.12 \times 10^{32}\ \text{kg} \approx 100\,\text{M}_\odot.\end{aligned}$$

Exercise 2.3 The Kelvin–Helmholtz timescale for the Sun is

$$\tau_{\mathrm{KH},\odot} = \frac{GM_\odot^2}{R_\odot L_\odot} = \frac{6.673 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}} \times (1.99 \times 10^{30}\ \mathrm{kg})^2}{6.96 \times 10^8\ \mathrm{m} \times 3.83 \times 10^{26}\ \mathrm{J\ s^{-1}}} = 9.90 \times 10^{14}\ \mathrm{s}.$$

This is equivalent to 9.90×10^{14} s / $(365.25 \times 24 \times 3600)$ s yr^{-1} $\approx 3 \times 10^7$ yr.

Exercise 2.4 For a $0.5\,\mathrm{M}_\odot$ star, the Kelvin–Helmholtz contraction time is $\tau_{\mathrm{KH},0.5} \approx 3 \times 10^7\ \mathrm{yr} \times 0.5^{-2.4} \approx 1.6 \times 10^8$ yr.

For a $2\,\mathrm{M}_\odot$ star, the Kelvin–Helmholtz contraction time is $\tau_{\mathrm{KH},2} \approx 3 \times 10^7\ \mathrm{yr} \times 2^{-2.4} \approx 5.7 \times 10^6$ yr.

For a $5\,\mathrm{M}_\odot$ star, the Kelvin–Helmholtz contraction time is $\tau_{\mathrm{KH},5} \approx 3 \times 10^7\ \mathrm{yr} \times 5^{-2.4} \approx 6.3 \times 10^5$ yr.

Exercise 3.1 The time-independent Schrödinger equation in one dimension, for a constant barrier potential V is

$$\left[-\frac{\hbar^2}{2m_\mathrm{r}}\frac{\partial^2}{\partial r^2} + V\right]\psi_\mathrm{s}(r) = E\psi_\mathrm{s}(r),$$

where m_r is the reduced mass. Equation 3.4 can also be written

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \frac{2m_\mathrm{r}}{\hbar^2}(V - E)\psi_\mathrm{s}(r)$$

or

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \chi^2\psi_\mathrm{s}(r) \quad \text{where} \quad \chi^2 = \frac{2m_\mathrm{r}}{\hbar^2}(V - E).$$

To verify that the wave function $\psi_\mathrm{s}(r) = \exp(\chi r)$ is a solution for a constant potential (i.e. when V and hence χ do not depend on r), substitute this into the left-hand side of the Schrödinger equation:

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \frac{\partial^2}{\partial r^2}\exp(\chi r)$$

Expand the second derivative

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \frac{\partial}{\partial r}\frac{\partial}{\partial r}\exp(\chi r).$$

Evaluate the first derivative

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \frac{\partial}{\partial r}\chi\exp(\chi r)$$

and then the second, but note that $\exp(\chi r) = \psi_\mathrm{s}(r)$

$$\frac{\partial^2}{\partial r^2}\psi_\mathrm{s}(r) = \chi^2\exp(\chi r) = \chi^2\psi_\mathrm{s}(r),$$

which equals the right-hand side of Schrödinger equation, as required. Note that if the barrier potential had not been constant, then χ would depend on r, and the differentiation would not be so straightforward.

Exercise 3.2 The Gamow energy is $E_\mathrm{G} = 2m_\mathrm{r}c^2(\pi\alpha Z_\mathrm{A}Z_\mathrm{B})^2$, where m_r is the reduced mass of the two-body system, given by $m_\mathrm{r} = m_\mathrm{A}m_\mathrm{B}/(m_\mathrm{A} + m_\mathrm{B})$.

(a) Begin by calculating the reduced mass:

$$m_{\mathrm{r}} = \frac{m_{\mathrm{p}} m_{\mathrm{p}}}{m_{\mathrm{p}} + m_{\mathrm{p}}} = \frac{m_{\mathrm{p}}^2}{2m_{\mathrm{p}}} = \frac{m_{\mathrm{p}}}{2} = \frac{1.673 \times 10^{-27}\ \mathrm{kg}}{2} = 8.365 \times 10^{-28}\ \mathrm{kg}.$$

Then

$$\begin{aligned} E_{\mathrm{G}} &= 2m_{\mathrm{r}}c^2(\pi\alpha Z_{\mathrm{p}} Z_{\mathrm{p}})^2 \\ &= 2 \times 8.365 \times 10^{-28}\ \mathrm{kg} \times (2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2 \times (\pi \times \frac{1}{137.0} \times 1 \times 1)^2 \\ &= 7.907 \times 10^{-14}\ \mathrm{kg\ m^2\ s^{-2}} = 7.907 \times 10^{-14}\ \mathrm{J}. \end{aligned}$$

Since 1 eV = 1.602 $\times 10^{-19}$ J, $E_{\mathrm{G}} = 7.907 \times 10^{-14}$ J/1.602 $\times 10^{-19}$ J eV^{-1} = 493.6 keV.

(b) In this case, the reduced mass is

$$m_{\mathrm{r}} = \frac{m_3 m_3}{m_3 + m_3} = \frac{m_3^2}{2m_3} = \frac{m_3}{2} = \frac{3m_{\mathrm{p}}}{2} = \frac{3 \times 1.673 \times 10^{-27}\ \mathrm{kg}}{2} = 2.510 \times 10^{-27}\ \mathrm{kg}.$$

Then

$$\begin{aligned} E_{\mathrm{G}} &= 2m_{\mathrm{r}}c^2(\pi\alpha Z_{3\mathrm{He}} Z_{3\mathrm{He}})^2 \\ &= 2 \times 2.510 \times 10^{-27}\ \mathrm{kg} \times (2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2 \times (\pi \times \frac{1}{137.0} \times 2 \times 2)^2 \\ &= 3.796 \times 10^{-12}\ \mathrm{kg\ m^2\ s^{-2}} = 3.796 \times 10^{-12}\ \mathrm{J}. \end{aligned}$$

Since 1 eV = 1.602 $\times 10^{-19}$ J, $E_{\mathrm{G}} = 3.796 \times 10^{-12}$ J/1.602 $\times 10^{-19}$ J eV^{-1} = 23.70 MeV.

Exercise 3.3 The probability of barrier penetration is

$$P_{\mathrm{pen}} \approx \exp\left[-\left(\frac{E_{\mathrm{G}}}{E}\right)^{1/2}\right] \approx \exp\left[-\left(\frac{E_{\mathrm{G}}}{kT_{\mathrm{c}}}\right)^{1/2}\right].$$

(a) proton–proton:

$$P_{\mathrm{pen}} \approx \exp\left[-\left(\frac{493.6\ \mathrm{keV}}{1.3\ \mathrm{keV}}\right)^{1/2}\right] = 3.4 \times 10^{-9}.$$

(b) ${}^3_2\mathrm{He}$–${}^3_2\mathrm{He}$:

$$P_{\mathrm{pen}} \approx \exp\left[-\left(\frac{23\,700\ \mathrm{keV}}{1.3\ \mathrm{keV}}\right)^{1/2}\right] = 2.3 \times 10^{-59}.$$

Note that the answers have been given to only 2 significant figures, rather than the 3 s.f. available, because the approximation that the energy is given by $E \approx kT_{\mathrm{c}}$ degrades the accuracy further.

Exercise 3.4 Since $S(E)$ is being treated as a constant, the integrand can be written

$$f(E) = S \exp\left[-\left(\frac{E_{\mathrm{G}}}{E}\right)^{1/2} - \frac{E}{kT}\right].$$

Differentiating gives

$$\frac{\mathrm{d}f(E)}{\mathrm{d}E} = S\frac{\mathrm{d}}{\mathrm{d}E}\exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2} - \frac{E}{kT}\right]$$

but using Hint 2 gives

$$\begin{aligned}\frac{\mathrm{d}f(E)}{\mathrm{d}E} &= S\exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2} - \frac{E}{kT}\right] \times \frac{\mathrm{d}}{\mathrm{d}E}\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2} - \frac{E}{kT}\right]\\ &= S\exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2} - \frac{E}{kT}\right] \times \left[-E_\mathrm{G}^{1/2}\left(-\frac{1}{2}E^{-3/2}\right) - \frac{1}{kT}\right]\\ &= S\exp\left[-\left(\frac{E_\mathrm{G}}{E}\right)^{1/2}\right]\exp\left(-\frac{E}{kT}\right) \times \left[\frac{E_\mathrm{G}^{1/2}}{2}(E^{-3/2}) - \frac{1}{kT}\right].\end{aligned}$$

The integrand $f(E)$ is either a minimum or maximum when $\mathrm{d}f(E)/\mathrm{d}E = 0$, which is when one of the following terms is zero:

(i) $S = 0$; this is a trivial, uninteresting case.

(ii) $\exp[-(E_\mathrm{G}/E)^{1/2}] \to 0$; this occurs when E becomes very small.

(iii) $\exp[-E/kT] \to 0$; this occurs when E becomes very large.

(iv)

$$\left[\frac{E_\mathrm{G}^{1/2}}{2}(E^{-3/2}) - \frac{1}{kT}\right] = 0.$$

We rearrange this to get an expression for E

$$\frac{E_\mathrm{G}^{1/2}}{2}(E^{-3/2}) = \frac{1}{kT}, \quad \text{and simplifying, we get} \quad \frac{E_\mathrm{G}^{1/2}}{E^{1/2}} = \frac{2E}{kT}.$$

Comment: This result will be useful later.

Collecting terms in E, we have

$$E^{3/2} = \frac{kT}{2}E_\mathrm{G}^{1/2},$$

then take the $(2/3)$-power of all terms

$$E = \left(\frac{kT}{2}\right)^{2/3}E_\mathrm{G}^{1/3} = \left(E_\mathrm{G}\left(\frac{kT}{2}\right)^2\right)^{1/3}.$$

This is the interesting case, and the energy $E_0 = [E_\mathrm{G}(kT/2)^2]^{1/3}$ is called the Gamow peak.

(Do not confuse this with the Gamow energy E_G.)

Exercise 3.5 (a) First calculate the energy ratio:

$$\frac{E_G}{4kT} = \frac{2m_p m_{13C}}{m_p + m_{13C}} \times \frac{c^2(\pi\alpha Z_p Z_{13C})^2}{4kT}$$

$$= \frac{2 \times 1u \times 13u}{1u + 13u} \times \frac{(2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2 \times (\pi \times \frac{1}{137} \times 1 \times 6)^2}{4 \times 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} \times 15.6 \times 10^6\ \mathrm{K}}$$

$$= 3.667 \times 10^{30} u \frac{(\mathrm{m\ s^{-1}})^2}{\mathrm{J}}$$

$$= 3.667 \times 10^{30} \times 1.661 \times 10^{-27}\ \mathrm{kg} \frac{\mathrm{m^2\ s^{-2}}}{\mathrm{kg\ m^2\ s^{-2}}} = 6091.$$

Next compute the fusion rate per unit mass fraction:

$$\frac{R_{p13C}}{X_{13C}} = 6.48 \times 10^{-24} \times \frac{(A_p + A_{13C})\rho_c^2 X_p}{(A_p A_{13C} u)^2 [\mathrm{m^{-6}}] Z_p Z_{13C}} \times \frac{S(E_0)}{[\mathrm{keV\ barns}]} \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]\ \mathrm{m^{-3}\ s^{-1}}$$

$$= 6.48 \times 10^{-24} \times \frac{(1 + 13) \times (1.48 \times 10^5\ \mathrm{kg\ m^{-3}})^2 \times 0.5}{(1 \times 13 \times 1.661 \times 10^{-27}\ \mathrm{kg})^2 \times [\mathrm{m^{-6}}] \times 1 \times 6}$$

$$\times \frac{5.5\ \mathrm{keV\ barns}}{[\mathrm{keV\ barns}]} \times (6091)^{2/3} \times \exp\left[-3 \times (6091)^{1/3}\right]\ \mathrm{m^{-3}\ s^{-1}}$$

$$= 1.0 \times 10^{18}\ \mathrm{m^{-3}\ s^{-1}}.$$

(b) First calculate the energy ratio:

$$\frac{E_G}{4kT} = \frac{2m_p m_{14N}}{m_p + m_{14N}} \times \frac{c^2(\pi\alpha Z_p Z_{14N})^2}{4kT}$$

$$= \frac{2 \times 1u \times 14u}{1u + 14u} \times \frac{(2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2 \times (\pi \times \frac{1}{137} \times 1 \times 7)^2}{4 \times 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} \times 15.6 \times 10^6\ \mathrm{K}}$$

$$= 5.017 \times 10^{30} u \frac{(\mathrm{m\ s^{-1}})^2}{\mathrm{J}}$$

$$= 5.017 \times 10^{30} \times 1.661 \times 10^{-27}\ \mathrm{kg} \frac{\mathrm{m^2\ s^{-2}}}{\mathrm{kg\ m^2\ s^{-2}}} = 8333.$$

Next compute the fusion rate per unit mass fraction:

$$\frac{R_{p14N}}{X_{14N}} = 6.48 \times 10^{-24} \times \frac{(A_p + A_{14N})\rho_c^2 X_p}{(A_p A_{14N} u)^2 [\mathrm{m^{-6}}] Z_p Z_{14N}} \times \frac{S(E_0)}{[\mathrm{keV\ barns}]} \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]\ \mathrm{m^{-3}\ s^{-1}}$$

$$= 6.48 \times 10^{-24} \times \frac{(1 + 14) \times (1.48 \times 10^5\ \mathrm{kg\ m^{-3}})^2 \times 0.5}{(1 \times 14 \times 1.661 \times 10^{-27}\ \mathrm{kg})^2 \times [\mathrm{m^{-6}}] \times 1 \times 7}$$

$$\times \frac{3.3\ \mathrm{keV\ barns}}{[\mathrm{keV\ barns}]} \times (8333)^{2/3} \times \exp\left[-3 \times (8333)^{1/3}\right]\ \mathrm{m^{-3}\ s^{-1}}$$

$$= 1.5 \times 10^{15}\ \mathrm{m^{-3}\ s^{-1}}.$$

(c) First calculate the energy ratio:

$$\frac{E_G}{4kT} = \frac{2m_p m_{15N}}{m_p + m_{15N}} \times \frac{c^2(\pi\alpha Z_p Z_{15N})^2}{4kT}$$

$$= \frac{2 \times 1u \times 15u}{1u + 15u} \times \frac{(2.998 \times 10^8 \text{ m s}^{-1})^2 \times (\pi \times \frac{1}{137} \times 1 \times 7)^2}{4 \times 1.381 \times 10^{-23}\,\text{J K}^{-1} \times 15.6 \times 10^6\,\text{K}}$$

$$= 5.039 \times 10^{30} u \frac{(\text{m s}^{-1})^2}{\text{J}}$$

$$= 5.039 \times 10^{30} \times 1.661 \times 10^{-27}\text{ kg}\frac{\text{m}^2\text{ s}^{-2}}{\text{kg m}^2\text{ s}^{-2}} = 8370.$$

Next compute the fusion rate per unit mass fraction:

$$\frac{R_{p15N}}{X_{15N}} = 6.48 \times 10^{-24} \times \frac{(A_p + A_{15N})\rho_c^2 X_p}{(A_p A_{15N} u)^2 [\text{m}^{-6}] Z_p Z_{15N}} \times \frac{S(E_0)}{[\text{keV barns}]} \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right] \text{ m}^{-3}\text{ s}^{-1}$$

$$= 6.48 \times 10^{-24} \times \frac{(1 + 15) \times (1.48 \times 10^5\text{ kg m}^{-3})^2 \times 0.5}{(1 \times 15 \times 1.661 \times 10^{-27}\text{ kg})^2 \times [\text{m}^{-6}] \times 1 \times 7}$$

$$\times \frac{78\text{ keV barns}}{[\text{keV barns}]} \times (8370)^{2/3} \times \exp\left[-3 \times (8370)^{1/3}\right] \text{ m}^{-3}\text{ s}^{-1}$$

$$= 3.0 \times 10^{16}\text{ m}^{-3}\text{ s}^{-1}.$$

Exercise 3.6 (a) In equilibrium $R_{p12C} = R_{p14N}$, so $3.5 \times 10^{17} X_{12C} = 0.015 \times 10^{17} X_{14N}$. This means

$X_{14N}/X_{12C} = 3.5/0.015 = 230$ and therefore ${}^{14}_{7}\text{N}/{}^{12}_{6}\text{C} = 12/14 \times 230 = 200.$

(b) In equilibrium $R_{p14N} = R_{p15N}$, so $0.015 \times 10^{17} X_{14N} = 0.30 \times 10^{17} X_{15N}$. This means

$X_{14N}/X_{15N} = 0.30/0.015 = 20$ and therefore ${}^{14}_{7}\text{N}/{}^{15}_{7}\text{N} = 15/14 \times 20 = 21.$

Exercise 3.7 (a) We begin with

$$R_{AB} = \frac{6.48 \times 10^{-24}}{A_r Z_A Z_B} \times \frac{n_A n_B}{[\text{m}^{-6}]} \times \frac{S(E_0)}{[\text{keV barns}]} \times \left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right] \text{ m}^{-3}\text{ s}^{-1}.$$

Following Hint 1, define

$$a = \frac{6.48 \times 10^{-24}}{A_r Z_A Z_B} \times \frac{n_A n_B}{[\text{m}^{-6}]} \times \frac{S(E_0)}{[\text{keV barns}]}$$

so

$$R_{AB} = a\left(\frac{E_G}{4kT}\right)^{2/3} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right] \text{ m}^{-3}\text{ s}^{-1}$$

and in preparation for using Hint 2, write

$$u = a\left(\frac{E_G}{4kT}\right)^{2/3} \quad \text{and} \quad v = \exp\left[-3\left(\frac{E_G}{4kT}\right)^{1/3}\right]$$

so $\quad R_{\rm AB} = uv \quad$ and then $\quad \dfrac{{\rm d}R_{\rm AB}}{{\rm d}T} = u\dfrac{{\rm d}v}{{\rm d}T} + v\dfrac{{\rm d}u}{{\rm d}T}.$

For the sake of clarity, calculate these two parts separately.

Step 1: Calculate ${\rm d}v/{\rm d}T$.

$$\frac{{\rm d}v}{{\rm d}T} = \frac{\rm d}{{\rm d}T}\exp\left[-3\left(\frac{E_{\rm G}}{4kT}\right)^{1/3}\right]$$

but using Hint 3

$$\frac{{\rm d}\exp(y)}{{\rm d}x} = \frac{{\rm d}\exp(y)}{{\rm d}y} \times \frac{{\rm d}y}{{\rm d}x} = \exp(y)\frac{{\rm d}y}{{\rm d}x}.$$

Hence

$$\frac{{\rm d}v}{{\rm d}T} = \exp\left[-3\left(\frac{E_{\rm G}}{4kT}\right)^{1/3}\right]\frac{\rm d}{{\rm d}T}\left[-3\left(\frac{E_{\rm G}}{4kT}\right)^{1/3}\right].$$

Note the the first exponential is just v again. Taking constants out of the differentiation gives

$$\frac{{\rm d}v}{{\rm d}T} = v\left[-3\left(\frac{E_{\rm G}}{4k}\right)^{1/3}\right]\frac{\rm d}{{\rm d}T}T^{-1/3}.$$

So, differentiating the $T^{-1/3}$ part

$$\frac{{\rm d}v}{{\rm d}T} = v\left[-3\left(\frac{E_{\rm G}}{4k}\right)^{1/3}\right]\left(-\frac{1}{3}\right)T^{-4/3} = v\left[\left(\frac{E_{\rm G}}{4kT}\right)^{1/3}\right]\frac{1}{T}.$$

Step 2: Calculate ${\rm d}u/{\rm d}T$.

$$\frac{{\rm d}u}{{\rm d}T} = \frac{\rm d}{{\rm d}T}a\left(\frac{E_{\rm G}}{4kT}\right)^{2/3}$$

taking the constants out of the differentiation gives

$$\frac{{\rm d}u}{{\rm d}T} = a\left(\frac{E_{\rm G}}{4k}\right)^{2/3}\frac{\rm d}{{\rm d}T}T^{-2/3} = a\left(\frac{E_{\rm G}}{4k}\right)^{2/3}\left(-\frac{2}{3}\right)T^{-5/3} = a\left(\frac{E_{\rm G}}{4kT}\right)^{2/3}\left(-\frac{2}{3}\right)\frac{1}{T}$$

but the first term on the right-hand side of the equation is just u again, so

$$\frac{{\rm d}u}{{\rm d}T} = u\left(-\frac{2}{3}\right)\frac{1}{T}.$$

Step 3: Calculate ${\rm d}R_{\rm AB}/{\rm d}T = u{\rm d}v/{\rm d}T + v{\rm d}u/{\rm d}T$.

Substitute the results from Steps 1 and 2:

$$\frac{{\rm d}R_{\rm AB}}{{\rm d}T} = uv\left[\left(\frac{E_{\rm G}}{4kT}\right)^{1/3}\right]\frac{1}{T} + vu\left(-\frac{2}{3}\right)\frac{1}{T}$$

take out the common factor uv/T

$$\frac{{\rm d}R_{\rm AB}}{{\rm d}T} = \frac{uv}{T}\left[\left(\frac{E_{\rm G}}{4kT}\right)^{1/3} - \frac{2}{3}\right]$$

and note that uv is simply R_{AB}

$$\frac{\mathrm{d}R_{\mathrm{AB}}}{\mathrm{d}T} = \frac{R_{\mathrm{AB}}}{T}\left[\left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3} - \frac{2}{3}\right].$$

(b) From the chain rule

$$\frac{\mathrm{d}\log_{\mathrm{e}} R_{\mathrm{AB}}}{\mathrm{d}\log_{\mathrm{e}} T} = \frac{\mathrm{d}\log_{\mathrm{e}} R_{\mathrm{AB}}}{\mathrm{d}R_{\mathrm{AB}}}\frac{\mathrm{d}R_{\mathrm{AB}}}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}\log_{\mathrm{e}} T} = \frac{1}{R_{\mathrm{AB}}}\frac{\mathrm{d}R_{\mathrm{AB}}}{\mathrm{d}T}\left(\frac{\mathrm{d}\log_{\mathrm{e}} T}{\mathrm{d}T}\right)^{-1} = \frac{T}{R_{\mathrm{AB}}}\frac{\mathrm{d}R_{\mathrm{AB}}}{\mathrm{d}T}.$$

So finally

$$\frac{\mathrm{d}\log_{\mathrm{e}} R_{\mathrm{AB}}}{\mathrm{d}\log_{\mathrm{e}} T} = \left[\left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3} - \frac{2}{3}\right].$$

Exercise 3.8 $R_{\mathrm{AB}} \propto T^{[(E_{\mathrm{G}}/4kT)^{1/3} - \frac{2}{3}]}$, so you need to evaluate the value

$$\nu = \left(\frac{E_{\mathrm{G}}}{4kT}\right)^{1/3} - \frac{2}{3}$$

for each reaction. Recall that $E_{\mathrm{G}} = 2m_{\mathrm{r}}c^2(\pi\alpha Z_{\mathrm{A}}Z_{\mathrm{B}})^2$ so

$$\nu = \left(\frac{2m_{\mathrm{r}}c^2(\pi\alpha Z_{\mathrm{A}}Z_{\mathrm{B}})^2}{4kT}\right)^{1/3} - \frac{2}{3}.$$

(a) For p + p, begin by calculating the reduced mass:

$$m_{\mathrm{r}} = \frac{m_{\mathrm{p}}m_{\mathrm{p}}}{m_{\mathrm{p}} + m_{\mathrm{p}}} = \frac{m_{\mathrm{p}}^2}{2m_{\mathrm{p}}} = \frac{m_{\mathrm{p}}}{2} = \frac{1.673 \times 10^{-27}\,\mathrm{kg}}{2} = 8.365 \times 10^{-28}\,\mathrm{kg}.$$

Then, using $T_{\odot,\mathrm{c}} = 15.6 \times 10^6$ K we obtain

$$\begin{aligned}\nu &= \left(\frac{2m_{\mathrm{r}}c^2(\pi\alpha Z_{\mathrm{p}}Z_{\mathrm{p}})^2}{4kT}\right)^{1/3} - \frac{2}{3}\\ &= \left(\frac{2 \times 8.365 \times 10^{-28}\,\mathrm{kg} \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})^2 \times (\pi \times \frac{1}{137.0} \times 1 \times 1)^2}{4 \times 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \times 15.6 \times 10^6\,\mathrm{K}}\right)^{1/3} - \frac{2}{3}\\ &= 3.84\end{aligned}$$

i.e. $R_{\mathrm{pp}} \propto T^{3.8}$.

(b) For p + $^{14}_{7}$N begin by calculating the reduced mass (where the reduced mass of $^{14}_{7}$N is given as m_{14}):

$$m_{\mathrm{r}} = \frac{m_{\mathrm{p}}m_{14}}{m_{\mathrm{p}} + m_{14}} = \frac{1u \times 14u}{1u + 14u} = \frac{14u^2}{15u} = \frac{14}{15}u = \frac{14}{15} \times 1.673 \times 10^{-27}\,\mathrm{kg} = 1.550 \times 10^{-27}\,\mathrm{kg}.$$

Then, using $T_{\odot,\mathrm{c}} = 15.6 \times 10^6$ K we obtain

$$\begin{aligned}\nu &= \left(\frac{2m_{\mathrm{r}}c^2(\pi\alpha Z_{\mathrm{p}}Z_{14\mathrm{N}})^2}{4kT}\right)^{1/3} - \frac{2}{3}\\ &= \left(\frac{2 \times 1.550 \times 10^{-27}\,\mathrm{kg} \times (2.998 \times 10^8\,\mathrm{m\,s^{-1}})^2 \times (\pi \times \frac{1}{137.0} \times 1 \times 7)^2}{4 \times 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \times 15.6 \times 10^6\,\mathrm{K}}\right)^{1/3} - \frac{2}{3}\\ &= 19.6. \quad \text{Therefore } R_{\mathrm{p14N}} \propto T^{19.6}.\end{aligned}$$

Exercise 3.9 Your amended Figure 3.6 should resemble Figure S3.1.

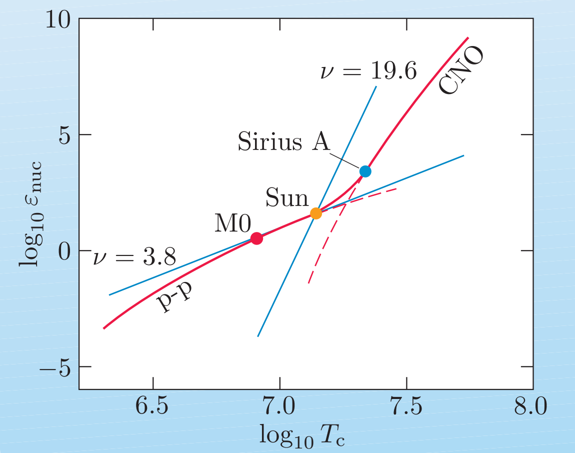

Figure S3.1 Copy of Figure 3.6, but with lines for $\varepsilon \propto T^\nu$ added, for $\nu = 3.8$ (p–p chain) and $\nu = 19.6$ (CNO cycle).

Exercise 4.1 The initial hydrogen content of the Sun is $0.70\,M_\odot$.

If it converted all of this into helium via the proton–proton chain, 0.0066 (i.e. $\approx 0.7\%$) of the hydrogen mass would be converted into energy. The total mass consumed would be

$$m = M_\odot \times 0.70 \times 0.0066 = 1.99 \times 10^{30}\,\text{kg} \times 0.70 \times 0.0066 = 9.2 \times 10^{27}\,\text{kg}.$$

This corresponds to an energy

$$E_{\text{fusion}} = mc^2 = 9.2 \times 10^{27}\,\text{kg} \times (2.998 \times 10^8\,\text{m s}^{-1})^2 = 8.3 \times 10^{44}\,\text{J}$$

over its lifetime.

The Sun's current luminosity is $L_\odot = 3.83 \times 10^{26}\,\text{J s}^{-1}$, so it could radiate at this rate for a lifetime given by

$$\tau_{\text{nuc}} = E_{\text{fusion}}/L_\odot = 8.3 \times 10^{44}\,\text{J}/3.83 \times 10^{26}\,\text{J s}^{-1} = 2.2 \times 10^{18}\,\text{s} \approx 70 \times 10^9\,\text{yr}$$

if it could indeed burn all of its hydrogen to helium.

Exercise 4.2 Lifetime $\propto M/L \propto M/M^{3.5} = 1/M^{2.5}$. If the solar lifetime is 10×10^9 yr, then the lifetime of a $0.5\,M_\odot$ star will be $1/0.5^{2.5} = 5.7$ times longer, i.e. 57×10^9 yr, and the lifetime of a $10\,M_\odot$ star will be $1/10^{2.5} = 0.0032$ times as long, i.e. 32×10^6 yr.

Exercise 4.3 For high-mass stars, the CNO cycle dominates energy production and the opacity is due entirely to electron scattering. For this case, $\nu \approx 17$. For stars of uniform chemical composition, μ is constant, so the μ-term can be absorbed into the unknown constant of proportionality.

(a) Equation 4.6 becomes $L \propto M^3\mu^4$, i.e. $L \propto M^3$.

(b) Equation 4.8 becomes

$$T_c \propto M^{4/(\nu+3)}\,\mu^{7/(\nu+3)} \propto M^{4/(17+3)} \propto M^{4/20}$$

i.e. $T_c \propto M^{0.2}$.

(c) Equation 4.7 becomes

$$R \propto M^{(\nu-1)/(\nu+3)}\,\mu^{(\nu-4)/(\nu+3)} \propto M^{(17-1)/(17+3)} \propto M^{16/20}$$

i.e. $R \propto M^{0.8}$.

That is, the luminosity increases strongly with mass, the core temperature increases weakly with mass, and the radius increases almost linearly with mass.

Exercise 4.4 For high-mass stars, the CNO cycle dominates energy production and the opacity is due entirely to electron scattering. For this case, $\nu \approx 17$. For stars of constant mass, The M-term can be absorbed into the unknown constant of proportionality.

(a) Equation 4.6 becomes $L \propto M^3\mu^4$, i.e. $L \propto \mu^4$.

(b) Equation 4.8 becomes

$$T_c \propto M^{4/(\nu+3)}\mu^{7/(\nu+3)} \propto \mu^{7/(17+3)} \propto \mu^{7/20}$$

i.e. $T_c \propto \mu^{0.4}$.

(c) Equation 4.7 becomes

$$R \propto M^{\frac{\nu-1}{\nu+3}}\mu^{\frac{\nu-4}{\nu+3}} \propto \mu^{\frac{17-4}{17+3}} \propto \mu^{\frac{13}{20}}$$

i.e. $R \propto \mu^{0.7}$.

That is, the luminosity increases steeply with mean molecular mass, the radius increases moderately and the core temperature increases very moderately with mean molecular mass.

Exercise 4.5 (a) (i) For fully ionized hydrogen and helium in Big Bang proportions, the mean molecular mass is

$$\mu = \frac{\sum_i n_i \frac{m_i}{u}}{\sum_i n_i} = \frac{N_{\rm H}m_{\rm p} + N_{\rm He}m_{\rm He} + N_{\rm e}m_{\rm e}}{u(N_{\rm H} + N_{\rm He} + N_{\rm e})}.$$

Let $N_{\rm nuc}$ be the (unknown) total number of nuclei, so $N_{\rm H} = 0.93N_{\rm nuc}$, $N_{\rm He} = 0.07N_{\rm nuc}$, $N_{\rm e} = N_{\rm H} + 2N_{\rm He} = 0.93N_{\rm nuc} + 2 \times 0.07N_{\rm nuc} = 1.07N_{\rm nuc}$, $m_{\rm He} \approx 4m_{\rm p}$, and $m_{\rm e}/m_{\rm p} = 9.109 \times 10^{-31}$ kg$/1.673 \times 10^{-27}$ kg $= 1/1837$, so $m_{\rm e} = m_{\rm p}/1837$. Note also that $m_{\rm p}/u = 1.673 \times 10^{-27}$ kg$/1.661 \times 10^{-27}$ kg $= 1.007$.

Substituting these into the expression for μ gives:

$$\mu = \frac{0.93N_{\rm nuc}m_{\rm p} + 0.07N_{\rm nuc}4m_{\rm p} + 1.07N_{\rm nuc}(m_{\rm p}/1837)}{u(0.93N_{\rm nuc} + 0.07N_{\rm nuc} + 1.07N_{\rm nuc})}$$

but the $N_{\rm nuc}$ terms cancel, and $m_{\rm p}$ is a common factor on the top line, so

$$\mu = \frac{(0.93 + 0.28 + 0.00058)}{(0.93 + 0.07 + 1.07)}\frac{m_{\rm p}}{u} = 0.58 \times 1.007 = 0.58.$$

Note that the electrons make a negligible contribution to the mass of the material (the numerator of the equation), but account for more than half of the *number* of particles, and hence greatly affect the denominator.

(ii) For fully ionized helium, the mean molecular mass is

$$\mu = \frac{\sum_i n_i \frac{m_i}{u}}{\sum_i n_i} = \frac{N_{\rm He} m_{\rm He} + N_{\rm e} m_{\rm e}}{u(N_{\rm He} + N_{\rm e})}.$$

Now, $N_{\rm e} = 2N_{\rm He}$, $m_{\rm He} = 4m_{\rm p}$ and $m_{\rm e} = m_{\rm p}/1837$, so

$$\mu = \frac{N_{\rm He} 4m_{\rm p} + 2N_{\rm He}(m_{\rm p}/1837)}{u(N_{\rm He} + 2N_{\rm He})} = \frac{(4 + 0.00109)}{1+2}\frac{m_{\rm p}}{u} = 1.33 \times 1.007 = 1.34.$$

The ratio of μ in case (ii) to μ in case (i) is $1.34/0.58 = 2.3$.

(b) (i) For stars burning hydrogen by the p–p chain: $R \propto \mu^0$, so $R_{\rm final}/R_{\rm initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^0 \propto 2.3^0 \approx 1$. So the star would be the same size.

$L \propto \mu^4$, so $L_{\rm final}/L_{\rm initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^4 \propto 2.3^4 \approx 28$. So the star would be a lot brighter!

$T_{\rm c} \propto \mu^1$, so $T_{\rm c,final}/T_{\rm c,initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^1 \propto 2.3^1 \approx 2.3$. So the star would be hotter.

(ii) For stars burning hydrogen by the CNO-cycle: $R \propto \mu^{0.7}$, so $R_{\rm final}/R_{\rm initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^{0.7} \propto 2.3^{0.7} \approx 1.8$. So the star would expand.

$L \propto \mu^4$, so $L_{\rm final}/L_{\rm initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^4 \propto 2.3^4 \approx 28$. So the star would be a lot brighter!

$T_{\rm c} \propto \mu^{0.4}$, so $T_{\rm c,final}/T_{\rm c,initial} \propto (\mu_{\rm final}/\mu_{\rm initial})^{0.4} \propto 2.3^{0.4} \approx 1.4$. So the star would be hotter.

Exercise 4.6 (a) For $s = 3$, $\gamma = (1 + s/2) \div (s/2) = (5/2) \div (3/2) = 5/3$, so the coefficient $(\gamma - 1)/\gamma = ((5/3) - 1) \div (5/3) = 2/5$.

(b) As $s \to \infty$, $\gamma \to (1/s + 1/2) \div (1/2) = (0 + 1) \div 1 = 1$, so the coefficient $(\gamma - 1)/\gamma = (1 - 1) \div 1 = 0$.

The critical temperature gradient for convection is

$$\frac{{\rm d}T}{{\rm d}r} < \frac{(\gamma - 1)}{\gamma}\frac{T}{P}\frac{{\rm d}P}{{\rm d}r}.$$

Since, in part (b), $\mathrm{d}T/\mathrm{d}r < 0$ and $(\gamma - 1)/\gamma = 0$, this material is *always* unstable to convection.

Exercise 5.1 (a) $\mu'({}^8_4\text{Be}) = \mu'({}^4_2\text{He}) + \mu'({}^4_2\text{He}) = 2\mu'({}^4_2\text{He})$.

(b) Since the chemical potential is $\mu' = mc^2 - kT \log_{\rm e}(g_s n_{\rm QNR}/n)$, in this case the result of part (a) and Equation 5.1 give

$$m_8 c^2 - kT \log_{\rm e}\left(\frac{g_8\, n_{\rm Q8}}{n_8}\right) = 2\left[m_4 c^2 - kT \log_{\rm e}\left(\frac{g_4\, n_{\rm Q4}}{n_4}\right)\right].$$

We need to find an expression for n_8/n_4, so rearrange the equation to work towards that goal. As a first step, collect the logarithms on one side and the mc^2 terms on the other:

$$m_8 c^2 - 2m_4 c^2 = kT\left(\log_{\rm e}\left(\frac{g_8\, n_{\rm Q8}}{n_8}\right) - 2\log_{\rm e}\left(\frac{g_4\, n_{\rm Q4}}{n_4}\right)\right)$$

$$\frac{(m_8 - 2m_4)c^2}{kT} = \log_{\rm e}\left[\frac{(g_8\, n_{\rm Q8}/n_8)}{(g_4\, n_{\rm Q4}/n_4)^2}\right]$$

$$\exp\left[\frac{(m_8 - 2m_4)c^2}{kT}\right] = \frac{(g_8\, n_{Q8}/n_8)}{(g_4\, n_{Q4}/n_4)^2}$$

$$\frac{n_8}{n_4^2} = \exp\left[\frac{-(m_8 - 2m_4)c^2}{kT}\right] \frac{g_8\, n_{Q8}}{(g_4\, n_{Q4})^2}. \qquad (5.9)$$

Since $n_{QA} = (2\pi m_A kT/h^2)^{3/2}$, the n_Q-term at the end of Equation 5.9 is

$$\frac{n_{Q8}}{(n_{Q4})^2} = \frac{(2\pi m_8 kT/h^2)^{3/2}}{\left[(2\pi m_4 kT/h^2)^{3/2}\right]^2} = \frac{(2\pi m_8 kT/h^2)^{3/2}}{(2\pi m_4 kT/h^2)^3} = \left(\frac{2\pi kT}{h^2}\right)^{-3/2} \left(\frac{m_8}{m_4^2}\right)^{3/2}.$$

Substituting this into Equation 5.9 gives

$$\frac{n_8}{n_4^2} = \exp\left[\frac{-(m_8 - 2m_4)c^2}{kT}\right] \frac{g_8}{g_4^2} \left(\frac{m_8}{m_4^2}\right)^{3/2} \left(\frac{h^2}{2\pi kT}\right)^{3/2}.$$

As we want to find the relative abundances of the nuclei, n_8/n_4, we must multiply both sides by n_4. Doing this, and using $n_4 = \rho X_4/m_4$, we obtain the final expression

$$\frac{n_8}{n_4} = \exp\left[\frac{-(m_8 - 2m_4)c^2}{kT}\right] \frac{g_8}{g_4^2} \rho X_4 \frac{m_8^{3/2}}{m_4^4} \left(\frac{h^2}{2\pi kT}\right)^{3/2}. \qquad (5.10)$$

We assume that the material is primarily $^4_2$He, so $X_4 = 1$. In evaluating the ratio $m_8^{3/2}/m_4^4$ we can use the approximation $m_4 = 4u$ and $m_8 = 8u$, but the term $m_8 - 2m_4$ involves the subtraction of nearly equal numbers, and for that we cannot use this approximation. However, we are given that $\Delta Q = (2m_4 - m_8)c^2 = -91.8\,\text{keV}$ so $(m_8 - 2m_4)c^2 = 91.8\,\text{keV}$. We calculated the polarizations g_4 and g_8 in the bulleted question at the end of Section 5.2.

So, evaluating Equation 5.10 at $T = 2 \times 10^8$ K and $\rho = 10^8$ kg m^{-3} gives:

$$\begin{aligned}\frac{n_8}{n_4} &= \exp\left[-\frac{91.8\text{ keV} \times 1.602 \times 10^{-16}\text{ J keV}^{-1}}{1.381 \times 10^{-23}\text{ J K}^{-1} \times 2 \times 10^8\text{ K}}\right] \times \frac{1}{1^2} \times 10^8\text{ kg m}^{-3} \times 1 \\ &\quad \times \frac{(8u)^{3/2}}{(4u)^4} \times \left(\frac{(6.626 \times 10^{-34}\text{ J s})^2}{2\pi \times 1.381 \times 10^{-23}\text{ J K}^{-1} \times 2 \times 10^8\text{ K}}\right)^{3/2} \\ &= 4.87 \times 10^{-3} \times 10^8\text{ kg m}^{-3} \times 0.0884 u^{-5/2} \times 1.27 \times 10^{-79}\text{ J}^{3/2}\text{ s}^3 \\ &= 5.47 \times 10^{-75} \times (1.661 \times 10^{-27})^{-5/2}\text{ kg}^{-3/2}\text{ m}^{-3}\text{ J}^{3/2}\text{ s}^3 \\ &= 4.86 \times 10^{-8}.\end{aligned}$$

Equivalently, $n_4/n_8 = 1/4.86 \times 10^{-8} = 2.1 \times 10^7$. That is, there is roughly one $^8_4$Be nucleus for every 21 million $^4_2$He nuclei!

Exercise 5.2 (a) The Gamow energy is $E_G = 2m_r c^2(\pi\alpha Z_A Z_B)^2$, where m_r is the reduced mass of the two-body system, given by $m_r = m_A m_B/(m_A + m_B)$.

Begin by calculating the reduced mass:

$$m_r = \frac{m_4 m_4}{m_4 + m_4} = \frac{m_4^2}{2m_4} = \frac{m_4}{2} = 2u = 2 \times 1.661 \times 10^{-27}\text{ kg} = 3.322 \times 10^{-27}\text{ kg}.$$

Then

$$E_G = 2m_r c^2 (\pi\alpha Z_4 Z_4)^2$$
$$= 2 \times 3.322 \times 10^{-27}\ \text{kg} \times (2.998 \times 10^8\ \text{m s}^{-1})^2 \times \left(\pi \times \frac{1}{137.0} \times 2 \times 2\right)^2$$
$$= 5.024 \times 10^{-12}\ \text{J}.$$

Since $1\,\text{eV} = 1.602 \times 10^{-19}$ J, the Gamow energy, $E_G = 5.024 \times 10^{-12}\ \text{J}/1.602 \times 10^{-19}\ \text{J eV}^{-1} = 31.4\,\text{MeV}$.

(b) The energy of the Gamow peak is $E_0 = (E_G(kT)^2/4)^{1/3}$. So, cubing both sides and multiplying by $4/E_G$ gives $(kT)^2 = 4E_0^3/E_G$. Taking the square root and dividing by k gives

$$T = \frac{2}{k}\sqrt{\frac{E_0^3}{E_G}}.$$

Now $\Delta Q = 91.8\,\text{keV}$ can be written in SI units as $\Delta Q = 91.8\ \text{keV} \times 1.602 \times 10^{-16}\ \text{J keV}^{-1} = 1.47 \times 10^{-14}$ J. So, if the energy of the Gamow peak coincides with this value of ΔQ, then

$$T = \frac{2}{k}\sqrt{\frac{E_0^3}{E_G}} = \frac{2}{1.381 \times 10^{-23}\ \text{J K}^{-1}}\sqrt{\frac{(1.47 \times 10^{-14}\ \text{J})^3}{5.024 \times 10^{-12}\ \text{J}}}$$
$$= 1.448 \times 10^{23}\text{J}^{-1}\ \text{K} \times 7.95 \times 10^{-16}\ \text{J} = 1.15 \times 10^8\ \text{K}.$$

Exercise 5.3 (a) The Gamow energy is $E_G = 2m_r c^2(\pi\alpha Z_A Z_B)^2$, where m_r is the reduced mass of the two-body system, given by $m_r = m_A m_B/(m_A + m_B)$.

Begin by calculating the reduced mass:

$$m_r = \frac{m_4 m_8}{m_4 + m_8} = \frac{4u \times 8u}{4u + 8u} = \frac{32u^2}{12u} = \frac{8u}{3} = 8 \times 1.661 \times 10^{-27}\ \text{kg}/3 = 4.429 \times 10^{-27}\ \text{kg}.$$

Then

$$E_G = 2m_r c^2 (\pi\alpha Z_4 Z_8)^2$$
$$= 2 \times 4.429 \times 10^{-27}\ \text{kg} \times (2.998 \times 10^8\ \text{m s}^{-1})^2 \times \left(\pi \times \frac{1}{137.0} \times 2 \times 4\right)^2$$
$$= 2.679 \times 10^{-11}\ \text{J}.$$

Since $1\,\text{eV} = 1.602 \times 10^{-19}$ J, the Gamow energy, $E_G = 2.679 \times 10^{-11}\ \text{J}/1.602 \times 10^{-19}\ \text{J eV}^{-1} = 167.2\,\text{MeV}$.

(b) The energy of the Gamow peak is $E_0 = (E_G(kT)^2/4)^{1/3}$. So, cubing both sides and multiplying by $4/E_G$ gives $(kT)^2 = 4E_0^3/E_G$. Taking the square root and dividing by k gives

$$T = \frac{2}{k}\sqrt{\frac{E_0^3}{E_G}}.$$

Now $\Delta Q = 287.7\,\text{keV}$ can be written in SI units as $\Delta Q = 287.7\ \text{keV} \times 1.602 \times 10^{-16}\ \text{J keV}^{-1} = 4.611 \times 10^{-14}$ J. So, if the energy of the Gamow peak coincides with this value of ΔQ, then

$$T = \frac{2}{k}\sqrt{\frac{E_0^3}{E_G}} = \frac{2}{1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}}}\sqrt{\frac{(4.611 \times 10^{-14}\ \mathrm{J})^3}{2.679 \times 10^{-11}\ \mathrm{J}}}$$
$$= 1.448 \times 10^{23}\mathrm{J^{-1}\ K} \times 1.913 \times 10^{-15}\ \mathrm{J} = 2.77 \times 10^8\ \mathrm{K}.$$

Exercise 5.4 (a) Ignoring the electronic parts, we find the mass defect is $m_{\text{initial}} - m_{\text{final}}$ = 3 × 4.002 60 amu − 12 amu = 0.007 800 amu.

(b) Using $E = \Delta m\, c^2$ we have $E = (0.007\,800 \times 1.661 \times 10^{-27}\ \mathrm{kg})$ $\times(2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2$ = 1.164×10^{-12} J. Since 1 eV = 1.602×10^{-19} J, this is equivalent to $E = 1.164 \times 10^{-12}$ J / 1.602×10^{-19} J eV^{-1} = 7.269×10^6 eV or about 7.27 MeV.

In the earlier subsections, Step 1 is said to require 91.8 keV and Step 2 is said to require 287.7 keV, whilst Step 3 releases 7.65 MeV. The net energy released is therefore $(7.65 - 0.2877 - 0.0918)$ MeV $= 7.27$ MeV, in agreement with the above.

(c) As a fraction of the initial mass this is 0.007 800 amu$/(3 \times 4.002\,60$ amu$) = 0.000\,65$. Recall that for hydrogen burning, the mass defect corresponds to 0.0066 of the initial mass, a factor of ten larger.

Exercise 5.5 (a) (i) For the proton

$$\lambda_{\mathrm{dB}}(\mathrm{p}) = h/(3m_{\mathrm{p}}kT)^{1/2}$$
$$= \frac{6.626 \times 10^{-34}\ \mathrm{J\ s}}{\sqrt{3 \times 1.673 \times 10^{-27}\ \mathrm{kg} \times 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}} \times 15.6 \times 10^6\ \mathrm{K}}}$$
$$= 6.372 \times 10^{-13}\ \mathrm{m}.$$

(ii) For the electron

$$\lambda_{\mathrm{dB}}(\mathrm{e}) = h/(3m_{\mathrm{e}}kT)^{1/2}$$
$$= \frac{6.626 \times 10^{-34}\ \mathrm{J\ s}}{\sqrt{3 \times 9.109 \times 10^{-31}\ \mathrm{kg} \times 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}} \times 15.6 \times 10^6\ \mathrm{K}}}$$
$$= 2.731 \times 10^{-11}\ \mathrm{m}.$$

(b) The ratio of de Broglie wavelengths is therefore

$$\frac{\lambda_{\mathrm{dB}}(\mathrm{e})}{\lambda_{\mathrm{dB}}(\mathrm{p})} = \frac{h}{(3m_{\mathrm{e}}kT)^{1/2}} \times \frac{(3m_{\mathrm{p}}kT)^{1/2}}{h} = \left(\frac{m_{\mathrm{p}}}{m_{\mathrm{e}}}\right)^{1/2} = \left(\frac{1.673 \times 10^{-27}\ \mathrm{kg}}{9.109 \times 10^{-31}\ \mathrm{kg}}\right)^{1/2} = 42.85.$$

The de Broglie wavelength of an electron is greater than that of a proton, by a factor of ≈ 43.

(c) From part (b), this ratio depends only on the mass of the particles, and hence is independent of the environment and hence of the temperature. The electron's wavelength is ≈ 43 times longer than the proton's in any star. Since a proton and a neutron have nearly the same mass, we can say that the de Broglie wavelength of an electron is about 40 times greater than that of a nucleon in any star.

Exercise 5.6 The second condition is $n \gg n_Q$, i.e. $n \gg (2\pi mkT/h^2)^{3/2}$.

Since $n = 1/l^3$, substituting for n gives $1/l^3 \gg (2\pi mkT/h^2)^{3/2}$.

Taking the (1/3)-power gives $1/l \gg (2\pi mkT/h^2)^{1/2}$, and multiplying through by $l/(2\pi mkT/h^2)^{1/2}$ gives

$$l \ll h/(2\pi mkT)^{1/2} = (3/2\pi)^{1/2} \times h/(3mkT)^{1/2} = (3/2\pi)^{1/2} \times \lambda_{\rm dB} < 0.7\lambda_{\rm dB}.$$

That is, from the second degeneracy condition $n \gg n_Q$ we obtain the first degeneracy condition $l \ll \lambda_{\rm dB}$.

Exercise 5.7 (a) $n_Q = (2\pi mkT/h^2)^{3/2}$, so the degeneracy condition $n \gg n_Q$ implies that $n \gg (2\pi mkT/h^2)^{3/2}$.

Taking the (2/3)-power and multiplying both sides by $h^2/2\pi m$ gives $n^{2/3}h^2/(2\pi m) \gg kT$, i.e. $kT \ll n^{2/3}h^2/(2\pi m)$.

The third equivalent condition for degeneracy is: the gas is degenerate if its temperature $T \ll n^{2/3}h^2/(2\pi mk)$.

(b) (i) $n_{\rm p} = \rho_{\odot,{\rm c}} X_{\rm H}/m_{\rm p} = 1.48 \times 10^5\ {\rm kg\ m^{-3}} \times 0.5/1.673 \times 10^{-27}\ {\rm kg} = 4.42 \times 10^{31}\ {\rm m^{-3}}$ (i.e. hydrogen nuclei per m^3).

Therefore

$$\frac{h^2 n_{\rm p}^{2/3}}{2\pi m_{\rm p} k} = \frac{(6.626 \times 10^{-34}\ {\rm J\ s})^2 \times (4.42 \times 10^{31}\ {\rm m^{-3}})^{2/3}}{(2 \times \pi \times 1.673 \times 10^{-27}\ {\rm kg} \times 1.381 \times 10^{-23}\ {\rm J\ K^{-1}})}$$
$$= 3.78 \times 10^3\ {\rm J\ s^2\ m^{-2}\ kg^{-1}\ K} \approx 3780\ {\rm K}.$$

(ii) In the solar core, all atoms are ionized. The electrons are provided by the hydrogen and helium which each account for 0.5 of the composition by mass.

$$n_{\rm e} = n_{\rm p} + 2n_{\rm He} = \frac{\rho_{\odot,{\rm c}} X_{\rm H}}{m_{\rm p}} + \frac{2\rho_{\odot,{\rm c}} X_{\rm He}}{m_{\rm He}} = \rho_{\odot,{\rm c}} \left(\frac{X_{\rm H}}{m_{\rm p}} + \frac{2X_{\rm He}}{4u} \right)$$
$$= 1.48 \times 10^5\ {\rm kg\ m^{-3}} \left(\frac{0.5}{1.673 \times 10^{-27}\ {\rm kg}} + \frac{2 \times 0.5}{4 \times 1.661 \times 10^{-27}\ {\rm kg}} \right)$$
$$= 6.65 \times 10^{31}\ {\rm m^{-3}}.$$

Therefore

$$\frac{h^2 n_{\rm e}^{2/3}}{2\pi m_{\rm e} k} = \frac{(6.626 \times 10^{-34}\ {\rm J\ s})^2 \times (6.65 \times 10^{31}\ {\rm m^{-3}})^{2/3}}{(2 \times \pi \times 9.109 \times 10^{-31}\ {\rm kg} \times 1.381 \times 10^{-23}\ {\rm J\ K^{-1}})}$$
$$= 9.12 \times 10^6\ {\rm J\ s^2\ m^{-2}\ kg^{-1}\ K} \approx 9.12 \times 10^6\ {\rm K}.$$

(c) The temperature condition for degeneracy is $T \ll n^{2/3}h^2/(2\pi mk)$. $T_{\odot,{\rm c}} = 15.6 \times 10^6$ K, so the temperature in the core of the Sun is (i) much too high for proton degeneracy to have set in, and (ii) marginally too high for electron degeneracy to have set in.

Exercise 6.1 (a) Equating the core pressure $P_{\rm c} = (\pi/36)^{1/3} G M^{2/3} \rho_{\rm c}^{4/3}$ to the pressure of non-relativistic degenerate electrons $P_{\rm NR} = K_{\rm NR}(\rho_{\rm c} Y_{\rm e}/m_{\rm H})^{5/3}$, where $K_{\rm NR} = (h^2/5m_{\rm e})\,(3/8\pi)^{2/3}$, we have $K_{\rm NR}(\rho_{\rm c} Y_{\rm e}/m_{\rm H})^{5/3} = (\pi/36)^{1/3} G M^{2/3} \rho_{\rm c}^{4/3}$.

Collecting terms in ρ_c on the left-hand side, and all others on the right, we get

$$\rho_c^{1/3} = \left(\frac{\pi}{36}\right)^{1/3} \frac{G}{K_{NR}} \frac{m_H^{5/3}}{Y_e^{5/3}} M^{2/3}$$

cubing this gives

$$\rho_c = \left(\frac{\pi}{36}\right) \left(\frac{G}{K_{NR}}\right)^3 \frac{m_H^5}{Y_e^5} M^2$$

and substituting for K_{NR} gives

$$\rho_c = \left(\frac{\pi}{36}\right) \left(\frac{5m_e}{h^2}\right)^3 \left(\frac{8\pi}{3}\right)^2 G^3 \frac{m_H^5}{Y_e^5} M^2.$$

Consolidating the numerical factors, we have

$$\rho_c = \left(\frac{16\pi^3}{81}\right) \left(\frac{5m_e}{h^2}\right)^3 G^3 \frac{m_H^5}{Y_e^5} M^2.$$

(b) The electron number density in the core of the star is $n_e = \rho_c Y_e / m_H$, so substituting the core density from above gives

$$n_e = \left(\frac{16\pi^3}{81}\right) \left(\frac{5m_e}{h^2}\right)^3 G^3 \frac{m_H^4}{Y_e^4} M^2.$$

Exercise 6.2 The Fermi energy is

$$E_F \approx \frac{25}{2} \left(\frac{2\pi^2}{27}\right)^{2/3} \left(\frac{G}{h}\right)^2 \left(\frac{m_H}{Y_e}\right)^{8/3} M^{4/3} m_e$$

which as a fraction of the electron rest-mass energy is

$$\frac{E_F}{m_e c^2} \approx \frac{25}{2} \left(\frac{2\pi^2}{27}\right)^{2/3} \left(\frac{G}{h}\right)^2 \left(\frac{m_H}{Y_e}\right)^{8/3} \frac{M^{4/3}}{c^2}.$$

So in this case for a $0.4\,M_\odot$ white dwarf, the Fermi energy as a fraction of the electron rest-mass energy is

$$\frac{E_F}{m_e c^2} \approx \frac{25}{2} \left(\frac{2\pi^2}{27}\right)^{2/3} \left(\frac{6.673 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}}{6.626 \times 10^{-34}\ \mathrm{J\ s}}\right)^2 \left(\frac{1.673 \times 10^{-27}\ \mathrm{kg}}{0.5}\right)^{8/3} \frac{(0.4 \times 1.99 \times 10^{30}\ \mathrm{kg})^{4/3}}{(2.998 \times 10^8\ \mathrm{m\ s^{-1}})^2}$$

$$\approx 0.228.$$

Exercise 6.3 (a) For the ultra-relativistic case, equating P_{UR} to P_c gives $P_{UR} = K_{UR}(\rho_c Y_e / m_H)^{4/3} = (\pi/36)^{1/3} G M^{2/3} \rho_c^{4/3}$ where $K_{UR} = (hc/4)(3/8\pi)^{1/3}$.

Note that in contrast to the non-relativistic case, the density term is the same on both sides ($\rho_c^{4/3}$), so cancels, leaving

$$K_{UR} \left(\frac{Y_e}{m_H}\right)^{4/3} = \left(\frac{\pi}{36}\right)^{1/3} G M^{2/3}.$$

Collecting M on one side and swapping left and right sides, gives

$$M^{2/3} = \left(\frac{36}{\pi}\right)^{1/3} \left(\frac{K_{UR}}{G}\right) \left(\frac{Y_e}{m_H}\right)^{4/3}.$$

Taking the (3/2)-power and substituting for K_{UR} gives

$$M = \left(\frac{36}{\pi}\right)^{1/2}\left(\frac{(hc/4)(3/8\pi)^{1/3}}{G}\right)^{3/2}\left(\frac{Y_{\mathrm{e}}}{m_{\mathrm{H}}}\right)^{2} = \left(\frac{3\times 36}{4^3\times 8\pi^2}\right)^{1/2}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Y_{\mathrm{e}}}{m_{\mathrm{H}}}\right)^{2}$$
$$= \frac{1}{\pi}\left(\frac{27}{128}\right)^{1/2}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Y_{\mathrm{e}}}{m_{\mathrm{H}}}\right)^{2}.$$

(b) Putting in the numbers we have

$$M = \frac{1}{\pi}\left(\frac{27}{128}\right)^{1/2}\left(\frac{hc}{G}\right)^{3/2}\left(\frac{Y_{\mathrm{e}}}{m_{\mathrm{H}}}\right)^{2}$$
$$= \frac{1}{\pi}\left(\frac{27}{128}\right)^{1/2}\left(\frac{6.626\times 10^{-34}\,\mathrm{J\,s}\times 2.998\times 10^{8}\,\mathrm{m\,s^{-1}}}{6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}}\right)^{3/2}\left(\frac{0.5}{1.673\times 10^{-27}\,\mathrm{kg}}\right)^{2}$$
$$= 2.121\times 10^{30}\ \mathrm{kg}.$$

Since $\mathrm{M}_\odot = 1.99\times 10^{30}\,\mathrm{kg}$, $M = 2.121\times 10^{30}\,\mathrm{kg}/1.99\times 10^{30}\,\mathrm{kg\,M_\odot^{-1}} = 1.07\,\mathrm{M}_\odot$.

Exercise 6.4 (a) The radius as a function of mass and average density is $R = (3M/4\pi\langle\rho\rangle)^{1/3}$.

(b) If the average density is 1/6 of the core density, then $R = (9M/2\pi\rho_{\mathrm{c}})^{1/3}$.

(c) Substituting the core density gives:

$$R_{\mathrm{WD}} = \left(\frac{9M}{2\pi}\right)^{1/3}\left(\frac{16\pi^3}{81}\right)^{-1/3}\left(\frac{5m_{\mathrm{e}}}{h^2}\right)^{-1} G^{-1}\frac{m_{\mathrm{H}}^{-5/3}}{Y_{\mathrm{e}}^{-5/3}}M^{-2/3}.$$

Rearranging the terms with negative powers

$$R_{\mathrm{WD}} = \left(\frac{9M}{2\pi}\right)^{1/3}\left(\frac{81}{16\pi^3}\right)^{1/3}\left(\frac{h^2}{5m_{\mathrm{e}}}\right)\frac{Y_{\mathrm{e}}^{5/3}}{Gm_{\mathrm{H}}^{5/3}}M^{-2/3}.$$

Now consolidate the mass terms and the numerical factors

$$R_{\mathrm{WD}} = \left(\frac{729}{32\pi^4}\right)^{1/3}\left(\frac{h^2}{5m_{\mathrm{e}}}\right)\frac{Y_{\mathrm{e}}^{5/3}}{Gm_{\mathrm{H}}^{5/3}}M^{-1/3}.$$

Now for convenience, express the mass in solar units

$$R_{\mathrm{WD}} = \left(\frac{729}{32\pi^4}\right)^{1/3}\left(\frac{h^2}{5m_{\mathrm{e}}}\right)\frac{Y_{\mathrm{e}}^{5/3}}{Gm_{\mathrm{H}}^{5/3}}\left(\frac{M}{\mathrm{M}_\odot}\right)^{-1/3}\times(1.99\times 10^{30}\,\mathrm{kg\,M_\odot^{-1}})^{-1/3}.$$

Then putting in the numbers

$$R_{\mathrm{WD}} = \left(\frac{729}{32\pi^4}\right)^{1/3}\frac{(6.626\times 10^{-34}\,\mathrm{J\,s})^2}{(5\times 9.109\times 10^{-31}\,\mathrm{kg})}\,\frac{(0.5)^{5/3}\times(1.99\times 10^{30}\,\mathrm{kg\,M_\odot^{-1}})^{-1/3}}{(6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})\times(1.673\times 10^{-27}\,\mathrm{kg})^{5/3}}\left(\frac{M}{\mathrm{M}_\odot}\right)^{-1/3}.$$

And so

$$R_{\mathrm{WD}} = 9.45\times 10^{6}\left(\frac{M}{\mathrm{M}_\odot}\right)^{-1/3}\ \mathrm{m}.$$

Since $R_\odot = 6.96 \times 10^8$ m,

$$R_{\rm WD} = \left(\frac{9.45 \times 10^6\ \rm m}{6.96 \times 10^8\ \rm m\ R_\odot^{-1}}\right) \times \left(\frac{M}{\rm M_\odot}\right)^{-1/3}$$
$$= \frac{\rm R_\odot}{74} \times \left(\frac{M}{\rm M_\odot}\right)^{-1/3}.$$

(d) Since $\rm R_{Earth} = 6.4 \times 10^6$ m,

$$R_{\rm WD} = \left(\frac{9.45 \times 10^6\ \rm m}{6.4 \times 10^6\ \rm m\ R_{Earth}^{-1}}\right) \times \left(\frac{M}{\rm M_\odot}\right)^{-1/3}$$
$$= 1.5\,\rm R_{Earth} \times \left(\frac{M}{\rm M_\odot}\right)^{-1/3}.$$

Exercise 6.5

Table S6.1 Nucleosynthesis processes

Process	Major reactions	Products	Mass range of stars	Ignition temp/K	Timescale
Big Bang	(not studied)	$^1_1$H, $^2_1$H, $^3_2$He $^4_2$He, $^7_4$Li	(not applicable)	(not studied)	$\sim$ 15 minutes
H-burning	p–p chain (3 branches) CNO cycle	$^4_2$He $^4_2$He, $^{13}_6$C, $^{14}_7$N	$M_{\rm ms} \geq 0.08\,\rm M_\odot$	(2–10) $\times 10^6$	$\sim 10^7$ to 10^{10} yr
He-burning	triple-alpha process	$^{12}_6$C, $^{16}_8$O	$M_{\rm ms} \geq 0.5\,\rm M_\odot$	(1–2) $\times 10^8$	$\sim 10^6$ yr
C-burning	$^{12}_6$C + $^{12}_6$C $^{12}_6$C + $^{12}_6$C $^{12}_6$C + $^{12}_6$C	$^{20}_{10}$Ne + $^4_2$He $^{23}_{11}$Na + p $^{23}_{12}$Mg + n	$M_{\rm ms} \geq 8\,\rm M_\odot$	(5–9) $\times 10^8$	$\sim$ 500 yr
Ne-burning	$^{20}_{10}$Ne + γ $^{20}_{10}$Ne + $^4_2$He	$^{16}_8$O + $^4_2$He $^{24}_{12}$Mg	$M_{\rm ms} \geq 10\,\rm M_\odot$	(1–2) $\times 10^9$	$\sim$ 1 yr
O-burning	$^{16}_8$O + $^{16}_8$O	$^{28}_{14}$Si + $^4_2$He	$M_{\rm ms} \geq 10\,\rm M_\odot$	(2–3) $\times 10^9$	$\sim$ 6 months
Si-burning	$^{28}_{14}$Si + γ $^{28}_{14}$Si + $n\,^4_2$He (successive captures of α-particles)	$^{24}_{12}$Mg + $^4_2$He $^{32}_{16}$S, $^{36}_{18}$Ar, $^{40}_{20}$Ca, $^{44}_{20}$Ca, $^{48}_{22}$Ti, $^{52}_{24}$Cr, $^{56}_{26}$Fe	$M_{\rm ms} \geq 11\,\rm M_\odot$	(3–4) $\times 10^9$	$\sim$ 1 day
neutron-capture	^A_ZX + n	$^{A+1}_{\ \ Z}$X			
β-decay	$^{A+1}_{\ \ Z}$X	$^{A+1}_{Z+1}$(X + 1) + e$^-$ + $\overline{\nu}_{\rm e}$			
	s-process	Zr, Mo, Ba, Ce, Pb, Bi	$M_{\rm ms} \geq 1\,\rm M_\odot$		$\sim 10^4$ yr
	r-process	Kr, Sr, Te, Xe, Cs, Os, Pt, Au, Hg, Th, U	$M_{\rm ms} \geq 10\,\rm M_\odot$		$\sim$ 1 second

Exercise 7.1 We begin by balancing the chemical potentials:

$$\mu'_4 = 2\mu'_\mathrm{p} + 2\mu'_\mathrm{n}$$

and since $\mu' = mc^2 - kT\log_\mathrm{e}(g_\mathrm{s} n_\mathrm{QNR}/n)$ we have

$$m_4c^2 - kT\log_\mathrm{e}\left(\frac{g_4\, n_\mathrm{Q4}}{n_4}\right) = 2m_\mathrm{p}c^2 - 2kT\log_\mathrm{e}\left(\frac{g_\mathrm{p}\, n_\mathrm{Qp}}{n_\mathrm{p}}\right) + 2m_\mathrm{n}c^2 - 2kT\log_\mathrm{e}\left(\frac{g_\mathrm{n}\, n_\mathrm{Qn}}{n_\mathrm{n}}\right)$$

where m_4, m_p and m_n are the masses of a helium-4 nucleus, a proton and a neutron respectively; n_Q4, n_Qp and n_Qn are the non-relativistic quantum concentrations of a helium-4 nucleus, a proton and a neutron respectively; g_4, g_p and g_n are the number of polarizations of helium-4 nuclei, protons and neutrons respectively; and n_4, n_p and n_n are the number densities of helium-4 nuclei, protons and neutrons respectively. This may be rearranged as

$$2m_\mathrm{p}c^2 + 2m_\mathrm{n}c^2 - m_4c^2 = kT\log_\mathrm{e}\left[\left(\frac{g_\mathrm{p}\, n_\mathrm{Qp}}{n_\mathrm{p}}\right)^2 \left(\frac{g_\mathrm{n}\, n_\mathrm{Qn}}{n_\mathrm{n}}\right)^2 \left(\frac{g_4\, n_\mathrm{Q4}}{n_4}\right)^{-1}\right].$$

The left-hand side is simply ΔQ, so taking the exponential of both sides and rearranging slightly, we have:

$$\exp\left(\frac{\Delta Q}{kT}\right) = \frac{(g_\mathrm{p}\, n_\mathrm{Qp}/n_p)^2 (g_\mathrm{n}\, n_\mathrm{Qn}/n_\mathrm{n})^2}{g_4\, n_\mathrm{Q4}/n_4}.$$

Since we are interested in the proportion of helium-4 nuclei that are dissociated, we take this fraction onto the left-hand side to get:

$$\frac{n_\mathrm{p}^2\, n_\mathrm{n}^2}{n_4} = \frac{g_\mathrm{p}^2\, g_\mathrm{n}^2}{g_4} \frac{n_\mathrm{Qp}^2\, n_\mathrm{Qn}^2}{n_\mathrm{Q4}} \exp\left(-\frac{\Delta Q}{kT}\right).$$

Exercise 7.2 The number of particles is $N = M/m$. For a total core mass of $M = (1/2) \times 1.4\ \mathrm{M}_\odot$, the number of helium-4 nuclei is $N = (1/2) \times 1.4\ \mathrm{M}_\odot$ / $4u$ = $(0.7 \times 1.99 \times 10^{30}$ kg) / $(4 \times 1.661 \times 10^{-27}$ kg) = 2.10×10^{56} nuclei.

Each nucleus absorbs 28.3 MeV = 28.3×10^6 eV $\times\ 1.602 \times 10^{-19}$ J eV^{-1} = 4.534×10^{-12} J.

So the core absorbs $2.10 \times 10^{56} \times 4.534 \times 10^{-12}$ J = 9.5×10^{44} J by the photodisintegration of helium-4 nuclei.

Exercise 7.3

Table S7.1 Properties of white dwarfs and neutron stars

White dwarf	Neutron star
(a) Pressure of non-relativistic degenerate matter	
$P_{\mathrm{NR}} = \frac{h^2}{5m_\mathrm{e}} \left(\frac{3}{8\pi}\right)^{2/3} n_\mathrm{e}^{5/3}$	$P_{\mathrm{NR}} = \frac{h^2}{5m_\mathrm{n}} \left(\frac{3}{8\pi}\right)^{2/3} n_\mathrm{n}^{5/3}$
(b) In terms of mass density	
$n_\mathrm{e} = Y_\mathrm{e}\rho_\mathrm{c}/m_\mathrm{H}$	$n_\mathrm{n} = \rho_\mathrm{c}/m_\mathrm{n}$
$P_{\mathrm{NR}} = \frac{h^2}{5m_\mathrm{e}} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{Y_\mathrm{e}\rho_\mathrm{c}}{m_\mathrm{H}}\right)^{5/3}$	$P_{\mathrm{NR}} = \frac{h^2}{5m_\mathrm{n}} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{\rho_\mathrm{c}}{m_\mathrm{n}}\right)^{5/3}$
(c) Clayton model	
$P_\mathrm{c} = (\pi/36)^{1/3} G M^{2/3} \rho_\mathrm{c}^{4/3}$	$P_\mathrm{c} = (\pi/36)^{1/3} G M^{2/3} \rho_\mathrm{c}^{4/3}$
$\rho_\mathrm{c} = (36/\pi)^{1/4} G^{-3/4} M^{-1/2} P_\mathrm{c}^{3/4}$	$\rho_\mathrm{c} = (36/\pi)^{1/4} G^{-3/4} M^{-1/2} P_\mathrm{c}^{3/4}$
(d) Put (b) into (c)	
$\rho_\mathrm{c} = \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$	$\rho_\mathrm{c} = \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$
$\times \left[\frac{h^2}{5m_\mathrm{e}} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{Y_\mathrm{e}\rho_\mathrm{c}}{m_\mathrm{H}}\right)^{5/3}\right]^{3/4}$	$\times \left[\frac{h^2}{5m_\mathrm{n}} \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{\rho_\mathrm{c}}{m_\mathrm{n}}\right)^{5/3}\right]^{3/4}$
$= \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$	$= \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$
$\times \left(\frac{h^2}{5m_\mathrm{e}}\right)^{3/4} \left(\frac{3}{8\pi}\right)^{1/2} \left(\frac{Y_\mathrm{e}\rho_\mathrm{c}}{m_\mathrm{H}}\right)^{5/4}$	$\times \left(\frac{h^2}{5m_\mathrm{n}}\right)^{3/4} \left(\frac{3}{8\pi}\right)^{1/2} \left(\frac{\rho_\mathrm{c}}{m_\mathrm{n}}\right)^{5/4}$
Collect powers of ρ_c	
$\rho_\mathrm{c}^{-1/4} = \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$	$\rho_\mathrm{c}^{-1/4} = \left(\frac{36}{\pi}\right)^{1/4} G^{-3/4} M^{-1/2}$
$\times \left(\frac{h^2}{5m_\mathrm{e}}\right)^{3/4} \left(\frac{3}{8\pi}\right)^{1/2} \left(\frac{Y_\mathrm{e}}{m_\mathrm{H}}\right)^{5/4}$	$\times \left(\frac{h^2}{5m_\mathrm{n}}\right)^{3/4} \left(\frac{3}{8\pi}\right)^{1/2} \left(\frac{1}{m_\mathrm{n}}\right)^{5/4}$
Raise both sides to the power 4/3	
$\rho_\mathrm{c}^{-1/3} = \left(\frac{36}{\pi}\right)^{1/3} G^{-1} M^{-2/3}$	$\rho_\mathrm{c}^{-1/3} = \left(\frac{36}{\pi}\right)^{1/3} G^{-1} M^{-2/3}$
$\times \left(\frac{h^2}{5m_\mathrm{e}}\right) \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{Y_\mathrm{e}}{m_\mathrm{H}}\right)^{5/3}$	$\times \left(\frac{h^2}{5m_\mathrm{n}}\right) \left(\frac{3}{8\pi}\right)^{2/3} \left(\frac{1}{m_\mathrm{n}}\right)^{5/3}$
$= \frac{324^{1/3}}{4\pi G} \left(\frac{h^2}{5m_\mathrm{e}}\right) \left(\frac{Y_\mathrm{e}}{m_\mathrm{H}}\right)^{5/3} M^{-2/3}$	$= \frac{324^{1/3}}{4\pi G} \left(\frac{h^2}{5m_\mathrm{n}}\right) \left(\frac{1}{m_\mathrm{n}}\right)^{5/3} M^{-2/3}$
(e) Adopt $\langle\rho\rangle = \rho_\mathrm{c}/6$ in density expression	
$R = (3M/4\pi\langle\rho\rangle)^{1/3} = (9M/2\pi\rho_\mathrm{c})^{1/3}$	$R = (3M/4\pi\langle\rho\rangle)^{1/3} = (9M/2\pi\rho_\mathrm{c})^{1/3}$
$R = (9/2\pi)^{1/3} \rho_\mathrm{c}^{-1/3} M^{1/3}$	$R = (9/2\pi)^{1/3} \rho_\mathrm{c}^{-1/3} M^{1/3}$
(f) Substitute for ρ_c using (d)	
$R_{\mathrm{WD}} = \left(\frac{729}{32\pi^4}\right)^{1/3} \frac{1}{G} \left(\frac{h^2}{5m_\mathrm{e}}\right) \left(\frac{Y_\mathrm{e}}{m_\mathrm{H}}\right)^{5/3} M^{-1/3}$	$R_{\mathrm{NS}} = \left(\frac{729}{32\pi^4}\right)^{1/3} \frac{1}{G} \left(\frac{h^2}{5m_\mathrm{n}}\right) \left(\frac{1}{m_\mathrm{n}}\right)^{5/3} M^{-1/3}$

Exercise 7.4 (a) Equation 7.5 may be re-arranged as

$$R_{\max} = \left[GM\left(\frac{P}{2\pi}\right)^2\right]^{1/3}.$$

So in this case,

$$R_{\rm max} = \left[6.673 \times 10^{-11}\ {\rm N\,m^2\,kg^{-2}} \times 1.99 \times 10^{30}\ {\rm kg} \times \left(\frac{1\ {\rm s}}{2\pi}\right)^2\right]^{1/3}$$
$$= 1.5 \times 10^6\ {\rm m} = 1500\ {\rm km}.$$

So any object larger than 1500 km radius cannot rotate more quickly than once a second. This effectively rules out rotating white dwarfs as the origin of pulsars.

(b) Equation 7.5 is

$$P_{\rm min} = 2\pi \left(\frac{R^3}{GM}\right)^{1/2}.$$

So in this case

$$P_{\rm min} = 2\pi \left(\frac{(10^4\ {\rm m})^3}{6.673 \times 10^{-11}\ {\rm N\,m^2\,kg^{-2}} \times 1.99 \times 10^{30}\ {\rm kg}}\right)^{1/2}$$
$$= 0.55 \times 10^{-3}\ {\rm s} = 0.55\ {\rm ms}.$$

So a $1\ {\rm M_\odot}$ neutron star can rotate as fast as 2000 times per second. Rapidly rotating neutron stars clearly can provide an explanation for pulsars.

Exercise 7.5 The magnetic field strength is

$$B = \frac{\mu_0 m}{4\pi R^3}.$$

Substituting for the magnetic dipole moment,

$$m = \left[-\frac{\dot{E}_{\rm rot}}{\omega^4 \sin^2\theta}\,\frac{3c^3}{2}\,\frac{4\pi}{\mu_0}\right]^{1/2}$$

we have

$$B = \frac{\mu_0}{4\pi R^3}\left[-\frac{\dot{E}_{\rm rot}}{\omega^4 \sin^2\theta}\,\frac{3c^3}{2}\,\frac{4\pi}{\mu_0}\right]^{1/2}$$

and then substituting for the rate of loss of energy $\dot{E}_{\rm rot} = I\omega\dot{\omega} = 2MR^2\omega\dot{\omega}/5$ we have

$$B = \frac{\mu_0}{4\pi R^3}\left[-\frac{2MR^2\omega\dot{\omega}}{5\omega^4 \sin^2\theta}\,\frac{3c^3}{2}\,\frac{4\pi}{\mu_0}\right]^{1/2}.$$

Then collecting together some terms we get

$$B = \left(\frac{\mu_0}{4\pi}\right)^{1/2} \frac{1}{R^2} \left(\frac{2M}{5}\right)^{1/2} \left(\frac{3c^3}{2}\right)^{1/2} \left(\frac{-\dot{\omega}}{\omega^3}\right)^{1/2} \frac{1}{\sin\theta}.$$

Now we note that, since $P = 2\pi/\omega$, then $\dot{P} = \dot{\omega} \times (-2\pi/\omega^2)$, so $-\dot{\omega}/\omega^3 = \dot{P}P/4\pi^2$, hence

$$B = \left(\frac{\mu_0}{4\pi}\right)^{1/2} \frac{1}{R^2} \left(\frac{2M}{5}\right)^{1/2} \left(\frac{3c^3}{2}\right)^{1/2} \frac{(\dot{P}P)^{1/2}}{2\pi}\,\frac{1}{\sin\theta}.$$

As required, this is an expression for the magnetic field strength of a pulsar in terms of its rotation period, rate of change of rotation period, mass, radius and other physical constants.

Exercise 7.6 Equation 7.10 may be rearranged to give

$$1 + \frac{2\dot{\omega}\tau}{\omega} = \frac{\omega^2}{\omega_0^2}$$

or

$$\omega_0 = \left(\frac{\omega^3}{\omega + 2\dot{\omega}\tau}\right)^{1/2}.$$

So for the Crab pulsar we have an initial angular frequency of

$$\omega_0 = \left(\frac{(190\ \mathrm{s}^{-1})^3}{(190\ \mathrm{s}^{-1}) + (2 \times -2.4 \times 10^{-9}\ \mathrm{s}^{-2} \times 950 \times 365 \times 24 \times 3600\ \mathrm{s})}\right)^{1/2}$$

$$\omega_0 \sim 400\ \mathrm{s}^{-1}.$$

Exercise 8.1 (a) The Jeans mass is $M_\mathrm{J} = 3kTR/2G\overline{m}$. Rearranging this, setting the mass equal to that of the Sun and setting $\overline{m} = 2u$, gives

$$R = M_\mathrm{J}\frac{2G\overline{m}}{3kT} = \mathrm{M}_\odot \frac{2G\,2u}{3kT}.$$

Substituting in values gives

$$R = 1.99 \times 10^{30}\ \mathrm{kg} \times \frac{2 \times 6.673 \times^{-11}\ \mathrm{N\ m^2\ kg^{-2}} \times 2 \times 1.661 \times 10^{-27}\ \mathrm{kg}}{3 \times 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}} \times 20\ \mathrm{K}}$$

$$= 1.07 \times 10^{15}\ \mathrm{m}.$$

This is the Jeans length for a solar mass of molecular hydrogen at a temperature of 20 K.

Since 1 pc = 3.086 $\times 10^{16}$ m, 1 AU = 1.496 $\times 10^{11}$ m, and $\mathrm{R}_\odot = 6.96 \times 10^8$ m, we can also express this distance as

$$R = 1.07 \times 10^{15}\ \mathrm{m}/3.086 \times 10^{16}\ \mathrm{m\ pc^{-1}} = 0.0347\ \mathrm{pc} \quad (\approx 0.03\ \mathrm{pc})$$

$$R = 1.07 \times 10^{15}\ \mathrm{m}/1.496 \times 10^{11}\ \mathrm{m\ AU^{-1}} = 7.15 \times 10^3\ \mathrm{AU} \quad (\approx 7000\ \mathrm{AU})$$

$$R = 1.07 \times 10^{15}\ \mathrm{m}/6.96 \times 10^8\ \mathrm{m\ R_\odot^{-1}} = 1.54 \times 10^6\ \mathrm{R}_\odot \quad (\approx 1.5\ \text{million}\ \mathrm{R}_\odot).$$

So, a cloud of molecular hydrogen at 20 K with a mass equal to that of the Sun will collapse if its radius is less than about 7000 AU.

(b) The Jeans density is

$$\rho_\mathrm{J} = \frac{3}{4\pi M^2}\left(\frac{3kT}{2G\overline{m}}\right)^3$$

so in this case

$$\rho_\mathrm{J} = \frac{3}{4\pi(1.99 \times 10^{30}\ \mathrm{kg})^2}\left(\frac{3 \times 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}} \times 20\ \mathrm{K}}{2 \times 6.673 \times 10^{-11}\ \mathrm{N\ m^2\ kg^{-2}} \times 2 \times 1.661 \times 10^{-27}\ \mathrm{kg}}\right)^3$$

$$\sim 4 \times 10^{-16}\ \mathrm{kg\ m^{-3}}.$$

The Jeans density for a 1 $\mathrm{M}_\odot$ cloud of molecular hydrogen at 20 K is 4×10^{-16} kg m^{-3} or about 120 billion molecules per cubic metre.

Exercise 8.2 The density is

$$\rho = \rho_\mathrm{J} = \frac{3}{4\pi M^2}\left(\frac{3kT}{2G\overline{m}}\right)^3$$

so the free-fall time is found by substituting this in Equation 2.5:

$$\tau_{\rm ff} = \left(\frac{3\pi}{32G}\frac{1}{\rho}\right)^{1/2} = \left(\frac{3\pi}{32G}\frac{4\pi M^2}{3}\left(\frac{2G\overline{m}}{3kT}\right)^3\right)^{1/2} = \left((\pi G)^2 M^2 \left(\frac{\overline{m}}{3kT}\right)^3\right)^{1/2}.$$

The mass of the molecules of H_2 is $\overline{m} \approx 2u$, so putting $M = \mathrm{M}_\odot$ we have

$$\begin{aligned}\tau_{\rm ff} &= \left((\pi G)^2 \mathrm{M}_\odot^2 \left(\frac{2u}{3kT}\right)^3\right)^{1/2} = \pi G \mathrm{M}_\odot \left(\frac{2u}{3kT}\right)^{3/2}\\ &= \pi \times 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times 1.99 \times 10^{30}\ \mathrm{kg} \times \left(\frac{2 \times 1.661 \times 10^{-27}\ \mathrm{kg}}{3 \times 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} \times 20\ \mathrm{K}}\right)^{3/2}\\ &= 3.35 \times 10^{12}\ \mathrm{s} \approx 1.1 \times 10^5\ \mathrm{yr}.\end{aligned}$$

So, a $1\,\mathrm{M}_\odot$ cloud of molecular hydrogen at 20 K with the Jeans density will collapse (if unopposed by internal pressure) within about one hundred thousand years.

Exercise 8.3 The Jeans density is

$$\rho_{\rm J} = \frac{3}{4\pi M^2}\left(\frac{3kT}{2G\overline{m}}\right)^3$$

which can be rearranged to give

$$M_{\rm J} = \sqrt{\frac{3}{4\pi\rho_{\rm J}}\left(\frac{3kT}{2G\overline{m}}\right)^3}.$$

We now substitute into this $\rho_{\rm J} = n\overline{m}$ to give

$$M_{\rm J} = \left(\frac{3}{4\pi n\overline{m}}\right)^{1/2}\left(\frac{3kT}{2G\overline{m}}\right)^{3/2}.$$

We can now evaluate the Jeans masses.

(a) Using $\overline{m} \approx 1u$ (i.e. 1 amu) for neutral atomic hydrogen,

$$\begin{aligned}M_{\rm J} &= \left(\frac{3 \times 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} \times 100\ \mathrm{K}}{2 \times 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times 1.661 \times 10^{-27}\ \mathrm{kg}}\right)^{3/2} \times \left(\frac{3}{4\pi \times 10^6\ \mathrm{m^{-3}} \times 1.661 \times 10^{-27}\ \mathrm{kg}}\right)^{1/2}\\ &= 3.06 \times 10^{34}\ \mathrm{kg}.\end{aligned}$$

Since $1\,\mathrm{M}_\odot = 1.99 \times 10^{30}$ kg, we have $M_{\rm J} = 3.06 \times 10^{34}\ \mathrm{kg}/1.99 \times 10^{30}\ \mathrm{kg\,M_\odot^{-1}}$ = $1.54 \times 10^4\,\mathrm{M}_\odot$.

(b) Using $\overline{m} \approx 2u$ (i.e. 2 amu) for molecular hydrogen,

$$\begin{aligned}M_{\rm J} &= \left(\frac{3 \times 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}} \times 10\ \mathrm{K}}{2 \times 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \times 2 \times 1.661 \times 10^{-27}\ \mathrm{kg}}\right)^{3/2} \left(\frac{3}{4\pi \times 10^9\ \mathrm{m^{-3}} \times 2 \times 1.661 \times 10^{-27}\ \mathrm{kg}}\right)^{1/2}\\ &= 7.66 \times 10^{30}\ \mathrm{kg}.\end{aligned}$$

Since $1\,\mathrm{M}_\odot = 1.99 \times 10^{30}$ kg, we have $M_{\rm J} = 7.66 \times 10^{30}\ \mathrm{kg}/1.99 \times 10^{30}\ \mathrm{kg\,M_\odot^{-1}}$ $= 3.85\,\mathrm{M}_\odot$.

Exercise 8.4 (a) Beginning with the Jeans mass (written in terms of density and temperature)

$$M_J = \sqrt{\frac{3}{4\pi\rho_J}\left(\frac{3kT}{2G\overline{m}}\right)^3}$$

we re-arrange this to get

$$\left(\frac{4\pi\rho_J}{3}\right)^{1/2}\left(\frac{2G\overline{m}}{3k}\right)^{3/2} M_J = T^{3/2}.$$

Raising each side to the (2/3)-power, and re-ordering the terms gives

$$T = \frac{2G\overline{m}}{3k}\left(\frac{4\pi}{3}\right)^{1/3} M_J^{2/3}\rho_J^{1/3}.$$

(b) Since $\log_{10} AB = \log_{10} A + \log_{10} B$, and $\log_{10} A^k = k\log_{10} A$, we get

$$\log_{10} T = \log_{10}\left(\frac{2G\overline{m}}{3k}\left(\frac{4\pi}{3}\right)^{1/3}\right) + \frac{2}{3}\log_{10} M_J + \frac{1}{3}\log_{10}\rho_J.$$

The first term on the right-hand side of the equation is merely a constant, whereas the second and third terms depend on mass and density. Drawn in the log temperature versus log density plane, the curve of T against ρ_J for a *given* protostellar mass M_J is a straight line of slope 1/3. The Jeans line for a protostellar mass 10 times higher (or lower) is also a straight line of slope 1/3, but offset vertically from the first by +2/3 (or –2/3) logarithmic units.

Exercise 8.5 (a) We have two expressions for pressure, $P \propto \rho^\gamma$ for an adiabatic process, and $P = \rho kT/\overline{m}$ for an ideal gas.

Substituting for P in the first gives $\rho kT/\overline{m} \propto \rho^\gamma$.

Dividing both sides by $\rho k/\overline{m}$ gives $T \propto (\overline{m}/k)\rho^{\gamma-1}$, but $\overline{m}$ and k are both constants, so we can write $T \propto \rho^{\gamma-1}$.

(b) Since $T \propto \rho^{\gamma-1}$, we can also write $T = \text{constant} \times \rho^{\gamma-1}$.
Taking logarithms of both sides of the equation gives
$\log_{10} T = \log_{10}(\text{constant}) + (\gamma - 1)\log_{10}\rho$.
Therefore the adiabats are straight lines of slope $\gamma - 1$.

(c) Since

$$\gamma = \frac{1 + (s/2)}{(s/2)},$$

so for $s = 3$, then $\gamma = 5/3$, and the slope $\gamma - 1$ of the adiabat of an ideal gas with three degrees of freedom is 5/3 – 3/3 = 2/3.

(d) s takes values from 3 to ∞, so γ takes values from $5/3$ to 1, and the slope is in the range from $2/3$ to 0.

(e) If $\gamma = 4/3$, then the adiabat has slope $\gamma - 1 = 1/3$.

Exercise 8.6 (a) Using Equation 8.5,

$$E_{DI} = \frac{M}{2m_H}E_D + \frac{M}{m_H}E_I = \frac{1.99\times 10^{30}\ \text{kg}}{1.673\times 10^{-27}\ \text{kg}}\left(\frac{4.5\ \text{eV}}{2} + 13.6\ \text{eV}\right) = 1.885\times 10^{58}\ \text{eV}.$$

Since 1 eV = 1.602×10^{-19} J, this is equivalent to $(1.885 \times 10^{58}\ \text{eV}) \times (1.602 \times 10^{-19}\ \text{J eV}^{-1}) \sim 3 \times 10^{39}$ J.

(b) Equating this energy to the change in gravitational potential energy, we have

$$\frac{GM^2}{R_2} - \frac{GM^2}{R_1} \sim 3 \times 10^{39}\ \text{J}$$

so

$$\begin{aligned} R_2 &= GM^2 \left(\frac{GM^2}{R_1} + 3 \times 10^{39}\ \text{J} \right)^{-1} \\ &= 6.673 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2} \times (1.99 \times 10^{30}\ \text{kg})^2 \\ &\qquad \times \left(\frac{6.673 \times 10^{-11}\ \text{N m}^2\ \text{kg}^{-2} \times (1.99 \times 10^{30}\ \text{kg})^2}{10^{15}\ \text{m}} + 3 \times 10^{39}\ \text{J} \right)^{-1} \\ &= 8.8 \times 10^{10}\ \text{m}. \end{aligned}$$

Hence the cloud collapses to a radius of about 10^{11} m or $\approx 150\,\text{R}_\odot$.

Exercise 8.7 Equation 8.8 can be re-written as

$$\frac{N(M_1)}{N(M_2)} = \left(\frac{M_1}{M_2} \right)^{-2.35}$$

where M_1 and M_2 are two particular masses. So the number of stars with each mass are:

$$N(50) = \left(\frac{50}{100} \right)^{-2.35} \sim 5 \quad \text{and} \quad N(10) = \left(\frac{10}{100} \right)^{-2.35} \sim 200,$$

$$N(5) = \left(\frac{5}{100} \right)^{-2.35} \sim 1000 \quad \text{and} \quad N(1) = \left(\frac{1}{100} \right)^{-2.35} \sim 50\,000,$$

$$N(0.5) = \left(\frac{0.5}{100} \right)^{-2.35} \sim 250\,000 \quad \text{and} \quad N(0.1) = \left(\frac{0.1}{100} \right)^{-2.35} \sim 11\ \text{million}.$$