Correction of the Exercises

Exercise 1.1. The electrostatic field of the diffue layer. 1. As we are dealing with a binary symmetric electrolyte $q_{(\pm)} = \pm e$ and in the free electrolyte (the one not influence by the electrical diffuse layer), we have $C_{(\pm)}(\infty) = C_f$ where C_f is the salinity of the electrolyte. Far from the wall, the electrostatic potential created by the charge on the wall vanishes. So, $\mu_{(\pm)}(x \to \infty) = \mu^0_{(\pm)} + k_b T \ln C_f$ while in the vicinity of the wall, at distance x, we have $\mu_{(\pm)}(x) = \mu^0_{(\pm)} + k_b T \ln C_{(\pm)}(x) + q_{(\pm)}\psi(x)$. Equilibrium is given by the equality of the electrochemical potentials, so,

$$\mu^0_{(\pm)} + k_b T \ln C_f = \mu^0_{(\pm)} + k_b T \ln C_{(\pm)}(x) + q_{(\pm)}\psi(x), \tag{9.1}$$

$$C_{(\pm)}(x) = C_f \exp\left[-\frac{(\pm e)\psi(x)}{k_b T}\right]. \tag{9.2}$$

Another totally equivalent way to show this result is to say that at equilibrium the flux of ions due to the electrostatic field (due to the charged surface) counterbalances the diffusive flux of ions. This yields,

$$-D_{(\pm)}\nabla C_{(\pm)}(x) = -(\pm)C_{(\pm)}(x)\beta_{(\pm)}\nabla\varphi(x), \tag{9.3}$$

$$-(k_b T\beta_{(\pm)}/e)\nabla C_{(\pm)}(x) = -(\pm)C_{(\pm)}(x)\beta_{(\pm)}\nabla\varphi(x), \tag{9.4}$$

$$(k_b T/e)\nabla C_{(\pm)}(x) = (\pm)C_{(\pm)}(x)\nabla\varphi(x), \tag{9.5}$$

$$C_{(\pm)}(x) = C_f \exp\left[-\frac{(\pm e)\varphi(x)}{k_b T}\right]. \tag{9.6}$$

2. Because of the symmetry of the problem, the electrical field **E** has not one component normal to the charged wall therefore we can write $\varepsilon_f \nabla \cdot \mathbf{E} = \rho$ as,

$$-\varepsilon_f \frac{d^2\psi}{dx^2} = \rho(x). \tag{9.7}$$

Now the local bulk charge density is given by:

$$\rho(x) = eC_{(+)} - eC_{(-)}(x) = eC_f\left(\exp\left[-\frac{e\psi(x)}{k_b T}\right] - \exp\left[+\frac{e\psi(x)}{k_b T}\right]\right), \tag{9.8}$$

with $2\sinh(x) = \exp(x) - \exp(-x)$. It follows,

$$\frac{d^2\psi}{dx^2} = \frac{2eC_f}{\varepsilon_f}\sinh\left[\frac{e\psi(x)}{k_b T}\right]. \tag{9.9}$$

3. Note first that $k_b T/e$ has the dimension of an electrostatic potential. At 25°C (298K), its value is close to 25 mV. If the electrostatic potential $\psi(x)$ is much smaller than $k_b T/e$, we can use a first or second order development of the exponential.

$$\sinh(\varepsilon) = \varepsilon +, \tag{9.10}$$

… and therefore,

$$\frac{d^2\psi}{dx^2} = \frac{2e^2 C_f}{\varepsilon_f k_b T}\psi(x) . \tag{9.11}$$

Such type of linear partial differential equation is very classical and you should think that the solution is an exponential (the second derivative is equal to the function !). This is also a seocnd order linear partial differential equation with constant coefficients. We can try a solution of the type,

$$\psi(x) = \psi_0 \exp(-x/x_d) , \tag{9.12}$$

$$d\psi(x)/dx = -(\psi_0/x_d)\exp(-x/x_d) , \tag{9.13}$$

$$d^2\psi(x)/dx^2 = (\psi_0/{x_d}^2)\exp(-x/x_d) , \tag{9.14}$$

$$\text{and therefore } d^2\psi(x)/dx^2 = (1/{x_d}^2)\psi(x)) , \tag{9.15}$$

with,

$$x_d = \sqrt{\frac{\varepsilon_f k_b T}{2e^2 C_f}} . \tag{9.16}$$

A numerical application shows that x_d is much smaller than the size of the pore.

Exercise 1.2. Self-potential map. The self-potential map should look like the one shown in Figure 9.1.

Exercise 1.3. Stern Layer Complexation and CEC. 1. The solution of the problem is,

$$\Gamma^0_{\mathrm{SH}} = \frac{\Gamma^0_{\mathrm{S}} - \Gamma^0_{\mathrm{X}}}{1 + \frac{K_{\mathrm{H}}}{10^{-\mathrm{pH}}}\left(1 + C_f K_{\mathrm{Na}}\right)} , \tag{9.17}$$

$$\Gamma^0_{\mathrm{S^-}} = \frac{(\Gamma^0_{\mathrm{S}} - \Gamma^0_{\mathrm{X}})K_{\mathrm{H}}/10^{-\mathrm{pH}}}{1 + \frac{K_{\mathrm{H}}}{10^{-\mathrm{pH}}}\left(1 + C_f K_{\mathrm{Na}}\right)} , \tag{9.18}$$

$$\Gamma^0_{\mathrm{SNa}} = \frac{(\Gamma^0_{\mathrm{S}} - \Gamma^0_{\mathrm{X}})C_f K_{\mathrm{Na}} K_{\mathrm{H}}/10^{-\mathrm{pH}}}{1 + \frac{K_{\mathrm{H}}}{10^{-\mathrm{pH}}}\left(1 + C_f K_{\mathrm{Na}}\right)} . \tag{9.19}$$

2. The solution is given by,

$$f = f_M \left\{ \frac{C_f K_{\mathrm{Na}}}{f_M(1 + C_f K_{\mathrm{Na}}) + (1 - f_M)\left[1 + C_f K_{\mathrm{Na}} + \frac{10^{-\mathrm{pH}}}{K_H}\right]} \right\} \tag{9.20}$$

where

$$f_M = 1 - \frac{\Gamma^0_{\mathrm{X}}}{\Gamma^0_S} . \tag{9.21}$$

At high pH values,

$$f \approx f_M \left(\frac{C_f K_{\mathrm{Na}}}{1 + C_f K_{\mathrm{Na}}} \right). \tag{9.22}$$

3. The solution is given by,

$$\mathrm{CEC}(\mathrm{C}_f) = \mathrm{CEC}_M \left(\frac{C_f K_{\mathrm{Na}}}{1 + C_f K_{\mathrm{Na}}} \right). \tag{9.23}$$

where $\mathrm{CEC}_M = e(\Gamma_{\mathrm{S}}^0 - \Gamma_X^0) S_{sp}$ corresponds to the maximum CEC at value at high pH values.

Exercise 2.1. Prigogine Theorem. We write the fluxes in the new Lagrangian framework associated with deformation of the grain skeleton ("superscript "s"),

$$D^m = D^s - (\mathbf{v}_s - \mathbf{v}) \left\lfloor \rho s \nabla T + \sum_{i=1}^{N} \rho_i \phi (\nabla \mu_i - \mathbf{F}_i) + \rho_w \phi (\nabla \mu_w - \mathbf{F}_w) + \rho_s (1 - \phi)(\nabla \cdot \overline{\overline{\mu}}_s - \mathbf{F}_s) - \nabla \cdot \overline{\overline{\sigma}}^d \right\rfloor \tag{9.24}$$

$$D^s \equiv - \mathbf{S} \nabla T - \sum_{i=1}^{N} (\nabla \mu_i - \mathbf{F}_i) \mathbf{J}_i - (\nabla \mu_w - \mathbf{F}_w) \mathbf{J}_w + \overline{\overline{\sigma}}^d : \nabla \mathbf{v}_s . \tag{9.25}$$

In this equation D^s is the total dissipation function written in the Lagrangian framework associated with deformation of the solid phase. In order to demonstrate the validity of the Prigogine's theorem, $D^m = D^s$ in Eq. (A2), I can use the force balance equation written under the assumption of mechanical equilibrium,

$$\sum_{i=1}^{N} \rho_i \phi \mathbf{F}_i + \rho_w \phi \mathbf{F}_w + \rho_s (1 - \phi) \mathbf{F}_s + \nabla \cdot \overline{\overline{\sigma}} = 0 , \tag{9.26}$$

and where $\overline{\overline{\sigma}} = \overline{\overline{\sigma}}^e + \overline{\overline{\sigma}}^d$. In addition the Gibbs-Duhem relationship of thermostatic yields,

$$\rho s \nabla T + \nabla \cdot \overline{\overline{\sigma}}^e + \phi \left(\sum_{i=1}^{N} \rho_i \nabla \mu_i + \rho_w \nabla \mu_w \right) + \rho_s (1 - \phi) \nabla \cdot \overline{\overline{\mu}}_s = 0. \tag{9.27}$$

The equations yield immediately to Prigogine's theorem $D^m = D^s$.

Exercise 2.2. Donnan model of Osmotic pressure. 1. Thermodynamic equilibrium between the fictitious reservoir and the pore space implies,

$$\overline{\overline{\mu}}_{(\pm)} = \tilde{\mu}_{(\pm)} . \tag{9.28}$$

$$\mu_{(\pm)}^0 + k_b T \ln C_{(\pm)} = \mu_{(\pm)}^0 + k_b T \ln \overline{C}_{(\pm)} \pm e \overline{\varphi} , \tag{9.29}$$

$$\ln \bar{C}_{(\pm)} - \ln C_{(\pm)} = -\left(\pm \frac{e\bar{\varphi}}{k_b T} \right), \tag{9.30}$$

$$\bar{C}_{(\pm)} = C_{(\pm)} \exp\left[-\left(\pm \frac{e\bar{\varphi}}{k_b T} \right) \right], \tag{9.31}$$

2. Solution.

$$\bar{C}_{(\pm)} = C_f \left(\sqrt{1+\Theta^2} \pm \Theta \right), \tag{9.32}$$

$$\bar{\varphi} = -\frac{k_b T}{2e} \ln\left[\frac{\sqrt{1+\Theta^2} + \Theta}{\sqrt{1+\Theta^2} - \Theta} \right], \tag{9.33}$$

$$\Theta \equiv \frac{\bar{Q}_V}{2eC_f}. \tag{9.34}$$

3. Solution.

$$\pi = -\frac{k_b T}{\Omega_w} \ln C_w = -\frac{k_b T}{\Omega_w} \ln\left(1 - 2\Omega_f C_f\right) \approx 2k_b T C_f, \tag{9.35}$$

$$\mu_w = \mu_w^0 + \Omega_w p + k_b T \ln C_w, \tag{9.36}$$

$$\nabla \mu_w = \Omega_w \nabla p^* \text{ where } p^* = p - \pi. \tag{9.37}$$

4. Solution,

$$\bar{\pi} = -\frac{k_b T}{\Omega_w} \ln \bar{C}_w, \tag{9.38}$$

$$\bar{\mu}_w = \mu_w^0 + \Omega_w \bar{p} + k_b T \ln \bar{C}_w, \tag{9.39}$$

$$\Omega_w C_w + (\Omega_{(+)} + \Omega_{(-)}) C_f = 1, \tag{9.40}$$

$$\Omega_w \bar{C}_w + \Omega_{(+)} \bar{C}_{(+)} + \Omega_{(-)} \bar{C}_{(-)} = 1, \tag{9.41}$$

$$\bar{\mu}_w = \mu_w. \tag{9.42}$$

$$\delta\pi = \pi - \bar{\pi} = -\frac{k_b T}{\Omega_w} \ln\left(\frac{C_w}{\bar{C}_w} \right), \tag{9.43}$$

$$\delta\pi = -\frac{k_b T}{\Omega_w} \ln\left(\frac{1 - 2\Omega_w C_f}{1 - \Omega_w \bar{C}_{(+)} - \Omega_w \bar{C}_{(-)}} \right), \tag{9.44}$$

$$\delta\pi \approx -k_b T \left(\bar{C}_{(+)} + \bar{C}_{(-)} - 2C_f \right), \tag{9.45}$$

We have used a first-order Taylor expansion of the argument of the logarithm

5. Solution:

$$\pi = k_b T \left(2C_f \right), \tag{9.46}$$

$$\bar{\pi} = k_b T \left(\bar{C}_{(+)} + \bar{C}_{(-)} \right). \tag{9.47}$$

6. Solution

$$\delta\pi = -2C_f k_b T\left(\cosh\left(\frac{e\bar{\varphi}}{k_b T}\right)-1\right) \tag{9.48}$$

$$\delta\pi = -2C_f k_b T\left(\sqrt{1+\Theta^2}-1\right) \tag{9.49}$$

$$\bar{\pi} = 2C_f k_b T\cosh\left(\frac{e\bar{\varphi}}{k_b T}\right). \tag{9.50}$$

Similarly, the osmotic pressure in the pore space can be written as,

$$\delta\pi = \int_0^{\bar{\varphi}} \bar{Q}_V d\bar{\varphi}', \tag{9.51}$$

$$\delta\pi = e\int_0^{\bar{\varphi}} (\bar{C}_{(+)} - \bar{C}_{(-)}) d\bar{\varphi}', \tag{9.52}$$

$$\delta\pi = -2C_f k_b T\left[\cosh\left(\frac{e\bar{\varphi}}{k_b T}\right)-1\right], \tag{9.53}$$

$$\delta\pi = \pi - \bar{\pi}, \tag{9.54}$$

7. Solution.

$$\mathbf{j}_{(\pm)} = \bar{C}_{(\pm)}\mathbf{v}_c + \bar{C}_{(\pm)}\mathbf{v}_e + \bar{C}_{(\pm)}\mathbf{v}_m \tag{9.55}$$

$$\mathbf{v}_c = -b_{(\pm)}\nabla\mu_{(\pm)} \tag{9.56}$$

$$\mathbf{v}_e = (\pm 1)\beta_{(\pm)}\mathbf{E}. \tag{9.57}$$

where $b_{(\pm)}$ (in $\text{m}^2\ \text{s}^{-1}\ \text{J}^{-1}$) and $\beta_{(\pm)}$ (in $\text{m}^2\ \text{s}^{-1}\ \text{V}^{-1}$) are two mobilities defining the drift velocity for diffusion and electromigration, respectively ($\beta_{(\pm)} = b_{(\pm)}e$), and $\mu_{(\pm)}$ are the chemical potentials of the cations and anions.

Exercise 2.3. Ionic fluxes in the Stern layer in a capillary. From the Nernst-Planck equation, the ionic flux densities of the cations and anions are given by,

$$\mathbf{j}_{(\pm)} = -b^S_{(\pm)}\Gamma_{(\pm)}\nabla_S\left((\pm e)\psi + k_b T\ln\Gamma_{(\pm)}\right), \tag{9.58}$$

$$\mathbf{j}_{(\pm)} = -b^S_{(\pm)}\Gamma_{(\pm)}(\pm e)\nabla_S\psi - k_b T b^S_{(\pm)}\Gamma_{(\pm)}\nabla_S\ln\Gamma_{(\pm)}, \tag{9.59}$$

$$\nabla_S\ln\Gamma_{(\pm)} = \nabla_S\ln C_{(\pm)} \tag{9.60}$$

(from Eqs. 2.282 and 2.283). Therefore, we have,

$$\mathbf{j}_{(\pm)} = -b^S_{(\pm)}\Gamma_{(\pm)}(\pm e)\nabla\psi - k_b T b^S_{(\pm)}\Gamma_{(\pm)}\nabla\ln C_{(\pm)}, \tag{9.61}$$

$$\mathbf{j}_{(\pm)} = -b^S_{(\pm)}\Gamma_{(\pm)}\nabla\tilde{\mu}_{(\pm)}. \tag{9.62}$$

Exercise 2.4. Electrical conductivity and effective charge density of a bundle of capillaries. Solution: We consider first a single capillary *i*. The current density J_i is given by the local Ohm's law,

$$J_i = \frac{I_i}{A_i} = \sigma_f \frac{\Delta\psi}{L} + \frac{2\pi R_i}{\pi R_i^2}\Sigma_S \frac{\Delta\psi}{L}, \tag{9.63}$$

where I_i and A_i are the current and surface area of a cross-section of the capillary, respectively, $\Delta\psi/L$ is the electrical field, R_i the radius of the capillary, and L its length. The porosity is given by $\phi = n\pi\Pi_2 / A$. For a collection of n capillaries, the total current I is defined as the sum of all the currents through all the capillaries (fluxes are cumulative):

$$I = \sum_{i=1}^{n} I_i = \sum_{i=1}^{n}\left(\sigma_f A_i + 2R_i g_i \Sigma_S\right)\frac{\Delta\psi}{L}. \tag{9.64}$$

where g_i denotes the dicretized version of the probability density of the pore size distribution. The total (macroscopic) current density of the porous material is therefore given by:

$$J = \frac{I}{A} = \frac{1}{A}\sum_{i=1}^{n}\left(\sigma_f A_i + 2R_i g_i \Sigma_S\right)\frac{\Delta\psi}{L},$$

$$J = \left(\sigma_f \phi + \frac{1}{A}\left(\sum_{i=1}^{n} 2R_i g_i\right)\Sigma_S\right)\frac{\Delta\psi}{L},$$

$$J = \phi\left(\sigma_f + 2\left(\frac{\sum_{i=1}^{n} R_i g_i}{\sum_{i=1}^{n} R_i^2 g_i}\right)\Sigma_S\right)\frac{\Delta\psi}{L},$$

$$J = \phi\left(\sigma_f + 2\frac{\Pi_1}{\Pi_2}\Sigma_S\right)\frac{\Delta\psi}{L}.$$

As J is also given by a macroscopic Ohm's law and assuming that the capillaries are characterized by a certain tortuosity $F\ \phi$, the electrical conductivity of the porous material is given by,

$$\sigma = \frac{1}{F}\left(\sigma_f + 2\frac{\Pi_1}{\Pi_2}\Sigma_S\right).$$

We focus now on the scaling between the volumetric excess of charge and the raw moments of the pore size distribution. For a single capillary i, the local volumetric charge density $\bar{q}_V^i$ (averaged over the pore space of the capillary) is related to the effective surface charge density Q_S on the surface of the walls of the capillary by,

$$\bar{q}_V^i = -2Q_S \frac{1}{R_i}.$$

In this eqaution, the factor $2/R_i$ corresponds to the surface per pore volume ratio. If N_i represents the number of charges in excess to neutrality in the capillary i, we have also,

$$\bar{q}_V^i = \frac{N_i}{V_p^i},$$

where V_p^i is the volume of the capillary i. For a collection of n capillaries, the volumetric charge density of the porous material is defined by,

$$\bar{Q}_V = \frac{1}{V_p}\sum_{i=1}^{n} N_i = \frac{-2\pi L Q_S \sum_{i=1}^{n} R_i g_i}{L\pi \sum_{i=1}^{n} R_i^2 g_i},$$

where V_p is the total pore volume (therefore the total number of excess charge is cumulative). This yields,

$$\bar{Q}_V = -2Q_S \frac{\Pi_1}{\Pi_2}.$$

We observe therefore that surface conductivity and the volumetric charge density share the same scaling law.

2. We look now at the streaming current density. For a single capillary, we have,

$$J_i = \frac{I_i}{A_i} = \bar{q}_V \frac{R_i^2}{8\eta}\frac{\Delta p}{L}.$$

For a collection of n capillaries and following the chain rules as previously, we find,

$$J = \hat{Q}_V U,$$

$$\hat{Q}_V = \bar{Q}_V \frac{\Pi_2 \Pi_3}{\Pi_4 \Pi_1} \le \bar{Q}_V.$$

Exercise 3.1. Electrical conductivity tensor of a pile of *N* planar sedimentary layers. In the x and y directions, the total flux (total current for the conductivity) is the sum of the fluxes (current) of each layer (resistors in parallel). In the vertical direction, the all the fluxes in each layer are the same and the potential drop sum up to the total difference of potential through the pile (resistors in series).

Exercise 3.2. Complex conductivity of clayey materials. The conductivity data occurs over more than an order of magnitude. Using a linear fit would favor the high values while using a log log expression provides a more similar weight to high and low conductivities. the petrophysical properties of the three samples are given in the following table.

Property	S9	S16	S22
Connected porosity ϕ (-)	0.48	0.49	0.43
Formation factor F (-)	4.1±0.3	5.9±0.1	4.4±0.5
Surface conductivity σ_S (S m^{-1})	(39±6) ×10^{-4}	(95±2) ×10^{-4}	(376±34) ×10^{-4}
Cementation exponent m (-)	1.9	2.5	1.8
Excess of charge Q_V (C m^{-3})	3.9×10^{6}	1.4×10^{7}	4.2×10^{7}
CEC (cmol kg^{-1})	1.4	5.3	12

Exercise 3.3. Interpretation of a geobattery experiment. From the self-potential data, we can see that there is no self-potential anomaly associated with the buried ISO1 while there is a negative self-potential anomaly associated with ISO2. To interpret these data you have to picture what is a vertical redox potential profile: positive in the vadose zone, negative below the water table and having a strong gradient through the capillary fringe. ISO1 experiences a relatively small redox potential gradient bit ISO2, which crossed the capillary fringe experience a much stronger redox potential gradient. Therefore the forcing term for ISO2 is much greater than for ISO1. This explains a much greater negative self-potential anomaly associated with the geobattery associated with ISO2.

Exercice 4.1. We consider a tube with a length of 50 cm where the porous medium has a height of 40 cm. We take a hydraulic conductivity $K = 10^{-4}$m s^{-1}, an electrical conductivity of 0.01 S m^{-1}, and a streaming current coupling coefficient $L = 10^{-4}$ A m^2. The numerical simulation shows that the self potential is linear with the hydraulic head. The head gradient is 1 (1m m^{-1}) and the electrical field is 0.0125 V m^{-1}.

Exercice 4.2. Correction,

1) $\Phi_T = \left\| \tilde{\mathbf{G}}\mathbf{m} - \tilde{\psi}^{obs} \right\|_2^2 = \left\| \begin{bmatrix} \mathbf{G} \\ \lambda I_m \end{bmatrix} \mathbf{m} - \begin{bmatrix} \psi^{obs} \\ 0 \end{bmatrix} \right\|_2^2 .$

2) The minimum of the objective function Φ_T is given by following expression:

$$\hat{\mathbf{m}}(\lambda) = \left(\mathbf{G}^T\mathbf{G} + \lambda^2 I\right)^{-1} \mathbf{G}^T \psi^{obs}$$

From these relations,

$\mathbf{G}\mathbf{G}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T$, $\mathbf{G}^T\mathbf{G} = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T$,

we see that the SVD of **G** depends to the eigenvalue decompositions of the symmetric positive semi-defnite matrices $\mathbf{G}^T\mathbf{G}$ and $\mathbf{G}\mathbf{G}^T$, then the minimum solution can be written as:

$$\hat{\mathbf{m}}(\lambda) = \left(\frac{\sigma_i^2}{\sigma_i^2 + \lambda^2} \right) \frac{u_i^T \psi^{obs}}{\sigma_i} v_i \ .$$

Exercise 4.3. The results of the inversion is shown in Figure 9.2.

Exercise 6.1. Figure 9.3 represents the comparison between of the observed and computed self potential data.

Exercice 7.1. The solution of the groundwater flow equation allows to obtain the hydraulic pressure and Darcy velocity which was used to estimate the electrical streaming density. The electric problem is solved after the compute of the source term using numerical approximations of the divergence operator of the Darcy velocity distribution times the effective excess volumetric charges. The areas of the ground water infiltration are associated with the negative self-potential anomalies. The results are shown in Figure 9.4.

Exercise 8.1. Drained and Undrained self-potential Response to Loading. 1. The first part of this problem is to show that the mass of the pore fluid per unit volume of porous material obeys a diffusion equation and to determine the diffusivity. 1. a. Solution:

$$(K-\frac{2}{3}\mu)\left(\varepsilon_{kk}\delta_{ij}\right)_{,j}+2\mu\varepsilon_{ij,j}-\alpha\left(p\delta_{ij}\right)_{,j}=0\,.$$

$$(K-\frac{2}{3}\mu)\varepsilon_{kk,i}+\mu\left(u_{i,jj}+u_{j,ij}\right)-\alpha p_{,i}=0\,.\quad \text{using } \varepsilon_{ij}=(1/2)\left(u_{i,j}+u_{j,i}\right)$$

$$(K-\frac{2}{3}\mu)\varepsilon_{kk,i}+\mu u_{i,jj}+\mu\varepsilon_{kk,i}-\alpha p_{,i}=0\,.\quad \text{using } \varepsilon_{ij}=(1/2)\left(u_{i,j}+u_{j,i}\right)$$

This yields

$$(K+\frac{1}{3}\mu)\varepsilon_{kk,i}+\mu u_{i,jj}-\alpha p_{,i}=0\,.$$

b. Solution:

$$\left(\rho_f\frac{k}{\eta_f}p_{,i}\right)_{,i}=\dot{m}$$

$$\rho_f\frac{k}{\eta_f}\nabla^2 p-\dot{m}=0$$

c. Solution: We start with

$$\Delta m\equiv m-m_0=\alpha\rho_f\left(\varepsilon_{kk}+\frac{\alpha p}{K_u-K}\right).$$

We take the Laplacian of this equation,

$$\nabla^2(m-m_0)=\alpha\rho_f\left(\nabla^2\varepsilon_{kk}+\frac{\alpha}{K_u-K}\nabla^2 p\right).$$

$$\left(\frac{K_u-K}{\alpha\rho_f}\right)\nabla^2 m=(K_u-K)\nabla^2\varepsilon_{kk}+\alpha\nabla^2 p\,.$$

d. Solution $(K+\frac{1}{3}\mu)\varepsilon_{kk,ii}+\mu u_{i,jji}-\alpha p_{,ii}=0$

$$(K+\frac{1}{3}\mu)\nabla^2\varepsilon_{kk}+\mu\nabla^2\varepsilon_{kk}=\alpha\nabla^2 p$$

$$(K+\frac{4}{3}\mu)\nabla^2\varepsilon_{kk}=\alpha\nabla^2 p$$

e. Solution. We compare the following two equations

$$\left(\frac{K_u-K}{\alpha\rho_f}\right)\nabla^2 m=(K_u-K)\nabla^2\varepsilon_{kk}+\alpha\nabla^2 p$$

$$(K+\frac{4}{3}\mu)\nabla^2\varepsilon_{kk}=\alpha\nabla^2 p\,.$$

This yields,

$$\left(\frac{K_u-K}{\alpha\rho_f}\right)\nabla^2 m-(K_u-K)\nabla^2\varepsilon_{kk}-(K+\frac{4}{3}\mu)\nabla^2\varepsilon_{kk}=0$$

$$\left(\frac{K_u - K}{\alpha\rho_f}\right)\nabla^2 m - (K_u + \frac{4}{3}\mu)\nabla^2\varepsilon_{kk} = 0$$

$$\frac{1}{(K_u + \frac{4}{3}\mu)}\left(\frac{K_u - K}{\alpha\rho_f}\right)\nabla^2 m - \nabla^2\varepsilon_{kk} = 0$$

We need to replace the deformation as a function of the fluid pressure.

We found $\nabla^2\varepsilon_{kk} = \dfrac{\alpha}{(K + \frac{4}{3}\mu)}\nabla^2 p$

Therefore

$$\frac{1}{(K_u + \frac{4}{3}\mu)}\left(\frac{K_u - K}{\alpha\rho_f}\right)\nabla^2 m - \frac{\alpha}{(K + \frac{4}{3}\mu)}\nabla^2 p = 0$$

Finally we need to replace the Laplacian of the fluid pressure by

$$\rho_f \frac{k}{\eta_f}\nabla^2 p - \dot{m} = 0 \Rightarrow \nabla^2 p = \frac{\eta_f}{\rho_f k}\dot{m} \ .$$

This yields $\dfrac{(K + \frac{4}{3}\mu)}{(K_u + \frac{4}{3}\mu)} k \left(\dfrac{K_u - K}{\eta_f \alpha^2}\right)\nabla^2 m - \dot{m} = 0$

$$c\nabla^2 m - \frac{\partial m}{\partial t} = 0 .$$

$$c = \left(\frac{K_u - K}{\alpha^2}\right)\left(\frac{k}{\eta_f}\right)\left(\frac{K + \frac{4}{3}\mu}{K_u + \frac{4}{3}\mu}\right).$$

2. The deformation can be decomposed into two parts: an undrained response (primary consolidation) where a fluid pressure p_u is created (which is equal to the Skempton coefficient B time the load P) and then start the secondary consolidation corresponding to the drained regime in which fluid flow out of the pores below the load in response to the increment in the confining stress. The self-potential response follows a similar response: an underained response associated iwth the fluid pressure builup p_u. This increase is followed by a relaxation of the potential associated with the flow of the pore water in response to the load (secondary consolidation). Having a network of self-potential stations, we can easily use a stochastic approaches (like the AMA algorithm) to invert the poroelastic parameters.