CHAPTER 21

ENSEMBLE SQUARE ROOT FILTERS

Exercise 21.1. Recall from sec. 20.5 that observations can be assimilated sequentially if their errors are independent. In this exercise, you will derive how this is done for the square root filter. In the case of one observation, (21.18) can be written as

$$\mathbf{D} = \mathbf{I} - \beta \mathbf{w}\mathbf{w}^T, \tag{21.1}$$

where

$$\mathbf{w} = \mathbf{X}^{BT}\mathbf{H}_t^T \quad \text{and} \quad \beta = \left(\mathbf{w}^T\mathbf{w} + \mathbf{R}_t\right)^{-1}. \tag{21.2}$$

Since only one observation is under consideration, β is a scalar. The matrix $\mathbf{D}$ differs from the identity matrix by a rank-1 matrix. It is natural to assume that the square root also differs from the identity by a term proportional to the same rank-1 matrix. Therefore, we seek a symmetric square root of $\mathbf{D}$ of the form

$$\mathbf{I} - \beta \mathbf{w}\mathbf{w}^T = \left(\mathbf{I} - \delta \mathbf{w}\mathbf{w}^T\right)\left(\mathbf{I} - \delta \mathbf{w}\mathbf{w}^T\right), \tag{21.3}$$

where δ is a parameter to be determined. Expanding the right side and simplifying yields

$$\mathbf{I} - \beta \mathbf{w}\mathbf{w}^T = \mathbf{I} - 2\delta \mathbf{w}\mathbf{w}^T + \delta^2 \mathbf{w}\mathbf{w}^T\mathbf{w}\mathbf{w}^T \tag{21.4}$$

$$0 = \left(\mathbf{w}^T\mathbf{w}\delta^2 - 2\delta + \beta\right)\mathbf{w}\mathbf{w}^T. \tag{21.5}$$

This equation is satisfied for arbitrary $\mathbf{w}$ only if the quadratic equation for δ vanishes, which requires

$$\delta = \frac{1 \pm \sqrt{1 - \mathbf{w}^T\mathbf{w}\beta}}{\mathbf{w}^T\mathbf{w}} = \frac{1 \pm \sqrt{\beta \mathbf{R}_t}}{\mathbf{w}^T\mathbf{w}} = \beta\left(1 \mp \sqrt{\frac{\mathbf{R}}{\mathbf{R} + \mathbf{w}^T\mathbf{w}}}\right)^{-1}. \tag{21.6}$$

This gives two solutions, but only the solution

$$\delta = \frac{1 - \sqrt{1 - \mathbf{w}^T\mathbf{w}\beta}}{\mathbf{w}^T\mathbf{w}} = \frac{1 - \sqrt{\beta \mathbf{R}_t}}{\mathbf{w}^T\mathbf{w}} = \beta\left(1 + \sqrt{\frac{\mathbf{R}}{\mathbf{R} + \mathbf{w}^T\mathbf{w}}}\right)^{-1}. \tag{21.7}$$

produces a positive semi-definite square root matrix. Show that this is true. □

Exercise 21.2. Prove the Schur Product theorem (theorem 21.1). Hint: show that the Schur product between a positive semi-definite matrix and a rank-1 matrix is positive semi-definite. Then, use the spectral factorization theorem to argue that any symmetric, positive semi-definite matrix can be represented by a sum of rank-1 matrices. □