Ocean Dynamics and the Carbon Cycle

Expanded questions and answers

The questions and answers are designed to consolidate the material addressed in the book and, in a few cases, provide some extension. To that end, we repeat the questions here and include a detailed, full set of answers. Our aim is to ensure that all students are able to work through the answers assuming a basic understanding of mathematics (including calculus) together with some understanding of physics and chemistry. This document will be updated to take on board any slips that we have missed.

Ric Williams and Mick Follows, June 2011

Chapter 2.

Q2.1. Heat storage of the atmosphere and ocean.
Estimate the thickness of the ocean that holds as much heat as the overlying atmosphere, where the amount of heat Q required to raise the temperature of the atmosphere or ocean by ΔT is given by

$$Q = \rho C_p A D \Delta T, \tag{2.2}$$

where ρ is density (kg m^{-3}), C_p is heat capacity (J kg^{-1}K^{-1}), A is horizontal area (m^2), and D is the vertical scale (m).

Assume $\rho \sim 1$ kg m^{-3} for the atmosphere and 10^3 kg m^{-3} for the ocean, $C_p \sim 1000$ J kg^{-1}K^{-1} for the atmosphere and 4000 J kg^{-1}K for the ocean, a vertical scale, D, of 10 km for the atmosphere (where the bulk of the atmosphere resides), $\Delta T = 1$ K and a horizontal area $A = 1$ m^2.

Answer:
For the parameters given with $\Delta T = 1$ K, in the atmosphere,

$$Q = \rho C_p A D \Delta T \sim (1 \text{ kg m}^{-3})(1000 \text{ J kg}^{-1}\text{K}^{-1})(1 \text{ m}^2)(10^4 \text{m})(1 \text{ K}) = 10^7 \text{J}.$$

For the ocean, the same heat storage is obtained with

$$Q \sim (1000 \text{ kg m}^{-3})(4000 \text{ J kg}^{-1}\text{K}^{-1})(1 \text{ m}^2)(2.5 \text{ m})(1 \text{ K}) = 10^7 \text{J}.$$

Thus, 2.5 m of ocean holds as much heat as the overlying atmosphere.

Q2.2. Radiative heating and equilibrium temperature.
(a) For a planet with no atmosphere, derive how the equilibrium temperature, T in Kelvin, depends on the incident solar radiation, S_c and the albedo, α, the fraction of reflected sunlight,

$$T = \left(\frac{(1-\alpha)S_c}{4\sigma_{sb}} \right)^{1/4}. \tag{2.3}$$

where σ_{sb} is the Stefan-Boltzman constant. Assume a radiative balance, as depicted in Fig. 2.25, where (i) the net solar radiation is absorbed over a circular disc with a cross-sectional area of the

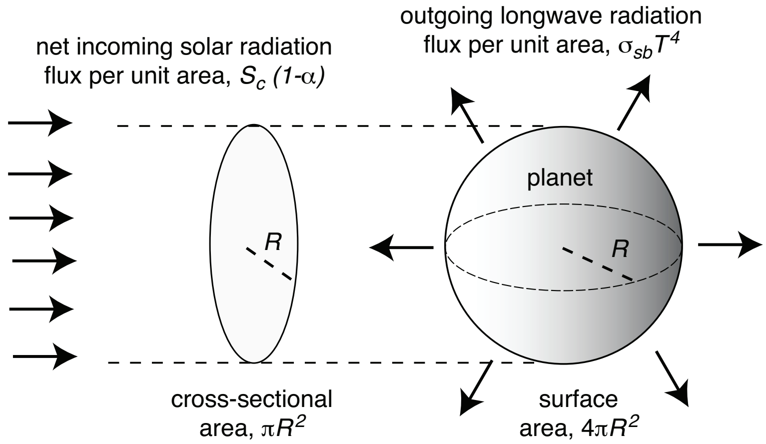

Figure 2.25: A schematic figure of the net incident solar radiation per unit horizontal area, the incident minus the reflected, $S_c(1-\alpha)$, which is absorbed over the cross-sectional area of a planet with radius R. The outgoing longwave radiation, $\sigma_{sb}T^4$, is radiated over the entire surface area of the planet.

planet and (ii) the outgoing longwave radiation per unit horizontal area in W m^{-2} is given by the Stefan-Boltzman law, $\sigma_{sb}T^4$, integrated over the surface area of the planet.
(b) Estimate this equilibrium temperature in Kelvin for Venus, Earth and Mars assuming that S_c is 2600, 1400 and 590 W m^{-2}, and their albedos, α, are 0.8, 0.3 and 0.15 respectively, and $\sigma_{sb} = 5.7 \times 10^{-8}$W m^{-2}K^{-4}. How do these temperatures compare with their respective observed surface values of typically 750K, 280K and 220K? Why might there be a mismatch in some cases?

(c) If the planet is now assumed to have an atmosphere that is transparent to solar radiation, but absorbs and re-radiates long wave radiation, then a local radiative balance suggests that the absorbed solar and long wave radiation at the ground balances the outgoing long wave radiation. The surface temperature is then given by

$$T = \left(\frac{(1-\alpha)S_c}{2\sigma_{sb}}\right)^{1/4}. \qquad (2.4)$$

Use this relationship to estimate the implied temperature contrast between the tropics and the high latitudes. For simplicity, in the tropics, assume that the incident radiation is given by S_c, while at the high latitudes, the incident radiation is given by $S_c/3$. How does this estimate compare with the actual meridional temperature contrast of typically 30K for the Earth?

Answer:
(a) (i) the solar radiation absorbed by a planet is given by

$$S_c \pi R^2 (1-\alpha),$$

where S_c is the solar constant, R is the radius of the planet and πR^2 is the circular area intersecting the Sun's rays, and α is the albedo.
(ii) the outgoing longwave radiation is given by

$$4\pi R^2 \sigma_{sb} T^4,$$

where $\sigma_{st} = 5.67 \times 10^{-8}$W m^{-2}K^{-4}. The factor 4 is due to the longwave radiation being emitted over an entire sphere with area $4\pi R^2$, whereas the solar radiation is only absorbed over a circular disc of area, πR^2;
(iii) equate relations in (i) and (ii) for a steady state, so that

$$S_c \pi R^2 (1-\alpha) = 4\pi R^2 \sigma_{sb} T^4,$$

divide by πR^2 on each side,

$$S_c(1-\alpha) = 4\sigma_{sb} T^4,$$

and re-arrange to obtain a prediction for the equilibrium temperature only depending on the solar constant and albedo,

$$T = \left(\frac{S_c(1-\alpha)}{4\sigma_{sb}} \right)^{1/4}.$$

This prediction for the temperature ignores the effect of the atmosphere, so can be viewed as a prediction for the temperature at the top of the atmosphere.

(b) The predicted equilibrium temperatures are:
219 K for Venus using $S_c = 2600$ W m^{-2} and $\alpha = 0.8$;
256 K for the Earth using $S_c = 1400$ W m^{-2} and $\alpha = 0.3$;
217 K for Mars using $S_c = 590$ W m^{-2} and $\alpha = 0.15$.
The actual surface temperature is warmer than these predicted equilibrium temperatures due to the extra surface heating from the absorption and re-radiation of longwave radiation in the atmosphere. There is only a slight warming for Mars due to its thin atmosphere, a warming of typically 30K for the Earth and a very large warming for Venus due to the high CO_2 content of its atmosphere (despite its extensive cloud cover and high albedo).

(c) Use

$$T = \left(\frac{S_c(1-\alpha)}{2\sigma_{sb}} \right)^{1/4},$$

with S_c for the tropics and $S_c/3$ for the high latitudes (and values in Q2.2b) to obtain a predicted surface temperature of 305 K in the tropics and 231 K in the high latitudes, giving a temperature difference of 73 K. Hence, a local radiative heat balance implies a pole-equator temperature contrast more than twice as large as the observed contrast.

Q2.3. Anthropogenic heating of the ocean by the increase in atmospheric CO_2.
Increasing atmospheric CO_2 leads to increasing radiative heating, $\Delta\mathcal{H}$ (in W m^{-2}), which varies logarithmically with the increase in mixing ratio for atmospheric CO_2 (as the effect of increasing CO$_2$ on the absorption and emission of longwave radiation gradually saturates),

$$\Delta\mathcal{H} = \alpha_r \ln\left(X_{CO_2}(t)/X_{CO_2}(t_0) \right), \qquad (2.5)$$

where $\alpha_r = 5.4$ W m^{-2} depends on the chemical composition of the atmosphere and $X_{CO_2}(t_0)$ and $X_{CO_2}(t)$ are the mixing ratios for CO_2 at times t_0 to t.

(a) Estimate the increase in implied radiative heating, $\Delta\mathcal{H}$, over the 50 years between 1958 and 2008 assuming an increase in X_{CO_2} from 315 ppmv to 386 ppmv; compare your answer with Fig. 1.11b.

(b) Given these estimates of radiative heating, then estimate how much the upper ocean might warm over 50 years. Assume that the temperature rise of the ocean is given from a simple heat balance by

$$\Delta T \sim \frac{\overline{\Delta\mathcal{H}}\,\mathcal{T}}{\rho C_p h},$$

where $\overline{\Delta\mathcal{H}}$ is the average extra heating over the time period, $\mathcal{T}$, of 50 years (convert to seconds) and h is the thickness of the upper ocean, taken as 500 m; ρ and C_p are as in Q2.1. Compare this estimate with the reported change for the global warming of the Earth over the last 50 years (IPCC, 2007).

Answer:
(a)

$$\Delta\mathcal{H} = \alpha_r \ln\left(\frac{X_{CO_2}(t)}{X_{CO_2}(t_0)}\right) = (5.4\ \text{W m}^{-2}) \ln\left(\frac{385\ \text{ppmv}}{315\ \text{ppmv}}\right) \sim 1.1\ \text{W m}^{-2}.$$

(b) Use $\overline{\Delta\mathcal{H}}$ as half the answer in (a), as an average, so that

$$\Delta T \sim \frac{\overline{\Delta\mathcal{H}}\,\mathcal{T}}{\rho C_p h} \sim \frac{(1.1\ \text{W m}^{-2}/2)(50 \times 365 \times 24 \times 60^2\text{s})}{(10^3\text{kg m}^{-3})(4 \times 10^3\text{J kg}^{-1}\text{K}^{-1})(500\ \text{m})} \sim 0.4\ \text{K}.$$

Thus, $\Delta T \sim 0.4^o$C after 50 years for a thickness of 500 m.

Q2.4. Atmospheric zonal jets and angular momentum.
Consider a tube of air circling the Earth at its equator that is uniformly displaced poleward, as depicted in Fig. 2.26. The angular momentum of the tube is given by

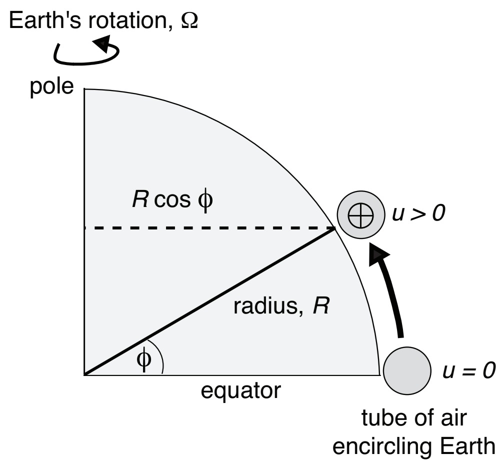

Figure 2.26: A schematic figure depicting a tube of air (dark shading) encircling the Earth along a latitude circle with the Earth rotating at an angular velocity Ω. The tube is at a distance $R\cos\phi$ from the rotational axis where R is the radius and ϕ is the latitude. As the tube moves from the equator towards the pole, the tube increases its zonal velocity u, so as to conserve angular momentum.

$$L_{ang} = (u + \Omega R \cos\phi)\, R\cos\phi, \tag{2.6}$$

where u is the zonal velocity, Ω is the angular velocity, R is the radius of the Earth, and ϕ is the latitude. $R\cos\phi$ represents the effective radius of the tube to its rotational axis, $\Omega R\cos\phi$ represents the velocity of the spinning Earth relative to a fixed point in space and u represents the velocity of the air relative to the Earth.

(a) derive an expression giving the zonal velocity, u, as a function of latitude, ϕ, by assuming that angular momentum L_{ang} is conserved and the initial zonal velocity at the equator is zero.

(b) calculate the implied zonal velocity for every 10° from the equator to 30°N for the Earth assuming $\Omega = 2\pi/$day and $R = 6340$ km. What are the implications of your result?

Answer:
(a) The angular momentum is given by

$$L_{ang} = (u + \Omega R \cos\phi)\, R\cos\phi.$$

At the equator, latitude $\phi = 0$ and assume a zonal flow of $u = 0$, then leads to

$$L_{ang} = \Omega R^2.$$

Assume angular momentum is conserved, so at all latitudes is the same as the value at the equator, then

$$L_{ang} = (u + \Omega R \cos\phi)\, R\cos\phi = \Omega R^2.$$

Re-arrange to obtain for the zonal velocity,

$$u = \frac{\Omega R}{\cos\phi}\left(1 - \cos^2\phi\right),$$

then use the trigonometric relation, $1 = \sin^2\phi + \cos^2\phi$, to re-express the zonal velocity as

$$u = \frac{\Omega R \sin^2\phi}{\cos\phi}.$$

(b) The predicted zonal wind speed is

$$u = \Omega R\left(\frac{\sin^2\phi}{\cos\phi}\right) = \left(\frac{2\pi}{24\times 60^2\text{s}}\right)(6340\times 10^3\text{m})\left(\frac{\sin^2\phi}{\cos\phi}\right).$$

Thus, obtain $u = 0$ m s^{-1} at the equator for $\phi = 0^o$, 14 m s^{-1} at 10°N, 57 m s^{-1} at 20°N and 133 m s^{-1} at 30°N. Hence, obtain a prediction of zonal velocities increasingly strongly with latitude with very fast, westerly jets of air formed. In practice, these zonal jets go unstable, leading to a poleward eddy transfer of heat.

Chapter 3.

Q3.1. Stirring of tracers.

(a) Conduct a simple dye experiment. Release a few drops of food dye in a glass container filled with water, which is at rest. Watch how the dye spreads over the next minute or so, look at the surface pattern and the vertical spreading.

Repeat the exercise, but this time make the water smoothly rotate in the container before adding the dye. Again watch how the dye spreads in the horizontal and vertical. How do the dye patterns differ in each case?

(b) In a similar manner on a grander scale, an iron-fertilisation experiment led to a phytoplankton bloom with subsequent stirring over the open ocean as illustrated in Fig. 3.19: iron was artificially

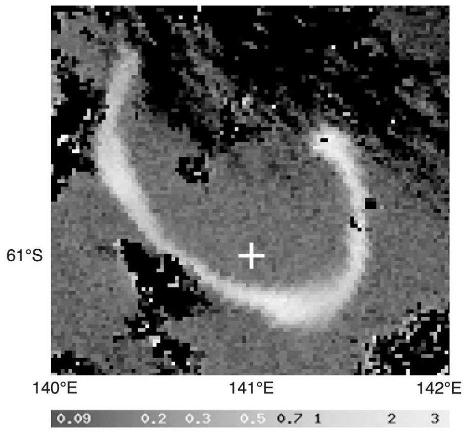

Figure 3.19: Snapshot of sea surface chlorophyll bloom formed from the Southern Ocean iron release experiment taken from the NASA SeaWiFS ocean colour satellite on 23 March 1999. There is a patch of elevated chlorophyll concentrations (light shading) extending in a filament 150 km long after 6 weeks; see Abraham et al. (2000) and Martin (2003) for further details. Image courtesy of Steve Groom, NEODAAS, Plymouth.

supplied to a patch of ocean 7 km in diameter, which led to a local phytoplankton bloom reaching unusually high chlorophyll *a* concentrations of 3 mg m^{-3}. The bloom rapidly expanded through a combination of stirring and diffusion of the patch, as well as the growth of phytoplankton: the fertilised patch expanded to length of 30 km long after 9 days and further to 150 km long after 42 days. Based on these length scales, estimate the effective strain rate, γ_e, for the two periods from (i) initial release to 9 days; and (ii) from 9 days to 42 days, where

$$\Delta x(t) = \Delta x_o \exp(\gamma_e t),$$

where $\Delta x(t)$ and Δx_o are the length scales of the patch at the elapsed time t and the initial time. Do you expect the strain rate to increase or decrease as the length scale inflates?

Answer
(a) In the non-rotating case, the dye spreads out from the point source in a relatively uniform manner. In the rotating case, the dye is stretched out horizontally in narrow filaments.

(b) (i) $\gamma_e = (9 \text{ days})^{-1} \ln (30 \text{ km}/7 \text{ km}) = 0.16 \text{ day}^{-1}$;
(ii) $\gamma_e = (33 \text{ days})^{-1} \ln (150 \text{ km}/30 \text{ km}) = 0.05 \text{ day}^{-1}$.
How the strain rate varies depends on the scales of the underlying eddies: as the patch expands above the scale of the eddies, the strain rate is likely to decrease.

Q3.2. Patchy tracers distributions and the flow pattern.

Tracers often have a patchy distribution, either confined in blobs or stretched out in narrow filaments, which reflects the effect of the flow pattern. Consider the spreading of a blob of tracer (Fig 3.20a) either by a rotational or convergent flow (Fig 3.20b,c). In (b), the rotational flow merely leads to the

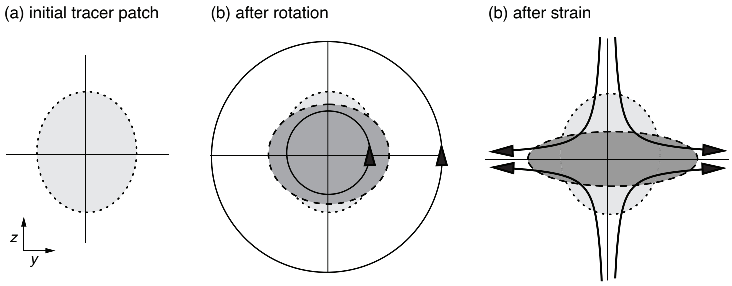

Figure 3.20: A schematic figure depicting (a) an elliptical patch of tracer (light shading) and the displacement of the patch (dark shading) after advection involving either (b) rotation or (c) pure strain (streamlines are full lines).

patch being rotated and there is no change in symmetry, while in (c), the tracer patch is compressed by the flow in the y-axis and stretched in the x-axis. Whether rapid stirring occurs or not can be diagnosed from the gradients in velocity, where the flow follows streamlines as marked in Fig 3.20, full lines.

(a) Assuming the streamfunction, ψ, for the flows in Fig 3.20b,c are given by (i) $\psi = a(x^2 + y^2)/2$ and (ii) $\psi = -axy$, then derive expressions for the velocities, $u \equiv -\partial\psi/\partial y$ and $v \equiv \partial\psi/\partial x$ for both cases.

(b) Based on these velocities, evaluate the vorticity, ζ,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{3.32}$$

and the strain rate, γ, by

$$\gamma^2 = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2, \tag{3.33}$$

for both cases.

(c) Whether there is rapid stirring depends on the relative size of the strain rate, γ, and the vorticity, ζ, a measure of the rotation, as given by the parameter,

$$\gamma^2 - \zeta^2. \tag{3.34}$$

Assuming that the spacing between tracer contours increases at a rate given by $\exp(\pm(\gamma^2 - \zeta^2)^{1/2}t)$, discuss the implications for how the tracer spreads for cases (i) and (ii).

Answer
(a) for (i) $\psi = a(x^2 + y^2)/2$,

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(a(x^2+y^2)/2) = -\frac{a}{2}\frac{\partial}{\partial y}(x^2+y^2) = -(a/2)(0+2y) = -ay,$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(a(x^2+y^2)/2) = \frac{a}{2}\frac{\partial}{\partial x}(x^2+y^2) = (a/2)(2x+0) = ax;$$

for (ii) $\psi = -axy$,

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial}{\partial y}(-axy) = ax\frac{\partial}{\partial y}(y) = ax,$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(-axy) = -ay\frac{\partial}{\partial x}(x) = -ay;$$

(b) for (i) $u = -ay$ and $v = ax$,

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(ax) - \frac{\partial}{\partial y}(-ay) = a\frac{\partial}{\partial x}(x) + a\frac{\partial}{\partial y}(y) = a + a = 2a,$$

$$\gamma^2 = \left(\frac{\partial}{\partial x}(-ay) - \frac{\partial}{\partial y}(ax)\right)^2 + \left(\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(-ay)\right)^2 = (0+0)^2 + (a-a)^2 = 0;$$

for (ii) $u = ax$ and $v = -ay$,

$$\zeta = \frac{\partial}{\partial x}(-ay) - \frac{\partial}{\partial y}(ax) = 0 + 0 = 0,$$

$$\gamma^2 = \left(\frac{\partial}{\partial x}(ax) - \frac{\partial}{\partial y}(-ay)\right)^2 + \left(\frac{\partial}{\partial x}(-ay) + \frac{\partial}{\partial y}(ax)\right)^2 = (a+a)^2 + (0+0)^2 = (2a)^2.$$

(c) Any flow can be considered as being made up of a contributions from rotation and pure strain. When vorticity dominates over the strain rate, tracer contours remain relatively confined and there is hardly any stirring. In contrast, when the strain rate dominates over the vorticity, chaotic stirring occurs and the horizontal spacing between tracer contours increase at a rate proportional to $\exp(\pm(\gamma^2 - \zeta^2)^{1/2}t)$.

In (i), the rotational case, the strain rate is given by $\gamma = 0$ and the vorticity by $\zeta = 2a$, so that $\gamma^2 - \zeta^2 = -2a < 0$. The horizontal spacing between tracer contours increase at a rate proportional to
$\exp(\pm(\gamma^2 - \zeta^2)^{1/2}t) = \exp(\pm i2at)$ where the imaginary number, $i \equiv (-1)^{1/2}$. The exponential is related to the trigonometric functions by $\exp(it) \equiv \cos t + i \sin t$. Hence, for $\gamma^2 - \zeta^2 < 0$, then there

are oscillatory solutions in time, implying that the tracer patch remains confined, as illustrated by the rotational flow in Fig. 3.20b.

Conversely, in (ii), the pure strain case, the strain rate is given by $\gamma = 2a$ and vorticity by $\zeta = 0$, so that $\gamma^2 - \zeta^2 = (2a)^2 > 0$. The horizontal spacing between tracer contours increase at a rate proportional $\exp(\pm(\gamma^2 - \zeta^2)^{1/2}t) = \exp(\pm 2at)$, which gives solutions amplifying or decaying in time. Hence, for $\gamma^2 - \zeta^2 > 0$, filaments of tracers are expected to be drawn out by the flow, as illustrated by the convergent flow in Fig. 3.20c.

Therefore, tracers are expected either to be confined in blobs or drawn out in filaments according to the relative size of the strain rate and vorticity.

Q3.3. Time-varying tracer fluxes.

(a) Construct two time series for velocity, $u(t)$, and tracer concentration, $c(t)$, based on random choices for each variable. For example, take a dice throw it 50 times and record the values for u and then repeat and record the values for c.

(b) Evaluate the time-mean value for velocity $\overline{u}$ and the time-mean value of tracer $\overline{c}$.

(c) Construct a time-series for $u'(t) = u(t) - \overline{u}$ and $c'(t) = c(t) - \overline{c}$.

(d) Evaluate the separate contributions to the products of velocity and tracer concentrations for each time, $u(t)\ c(t)$, $\overline{u}\ \overline{c}$, $u'(t)\ \overline{c}$, $\overline{u}\ c'(t)$, $u'(t)\ c'(t)$.

Check that each time that $u(t)c(t) = \overline{u}\overline{c} + u'(t)\overline{c} + \overline{u}c'(t) + u'(t)c'(t)$.

(e) Evaluate the time-averaged contributions of each of the terms, $\overline{u(t)\ c(t)}$, $\overline{\overline{u}\ \overline{c}}$, $\overline{u'(t)\ \overline{c}}$, $\overline{\overline{u}\ c'(t)}$, and $\overline{u'(t)\ c'(t)}$.

Identify which terms contribute to the time-averaged flux, $\overline{u(t)\ c(t)}$, and identify what each term represents?

Answer

$\overline{u(t)\ c(t)} = \overline{u}\ \overline{c} + \overline{u'(t)\ c'(t)}$ with $\overline{u'(t)\ \overline{c}} = 0$ and $\overline{\overline{u}\ c'(t)} = 0$. Thus, the time-averaged flux of tracer depends on the time-mean advection of the time-mean tracer, $\overline{u}\ \overline{c}$, plus the time-averaged contribution from the time-varying correlations between the velocity and tracer, $\overline{u'(t)\ c'(t)}$.

Q3.4. Bolus velocity and Stokes' drift in shallow water waves.

(a) For shallow water waves, there is local balance between the temporal acceleration and the horizontal pressure gradient (dependent on the thickness of the water column, h), such that

$$\frac{\partial v}{\partial t} = -g\frac{\partial h}{\partial y}, \tag{3.35}$$

where g is gravity. Assuming a sinusoidal wave form for the velocity in the water column associated with the wave,

$$v(y,t) = v_o \sin(ky - \omega t), \tag{3.36}$$

then show that the thickness of the water column, h, varies in a similar sinusoidal manner:

$$h(y,t) = \overline{h} + h_o \sin(ky - \omega t), \tag{3.37}$$

where $\overline{h}$ is the time-mean thickness of the column, h_o is the amplitude of the oscillating wave, k is the wavenumber, ω is the angular frequency for the wave. Identify how v_o and h_o are related.

Given the sinusoidal variations for v and h, how does the volume flux vh vary in magnitude and direction in the crest and trough of the wave?

(b) Integrate over a wave period τ at a fixed position $y = 0$, and show that the time-integrated volume flux using (3.36) and (3.37) is given by

$$\int_o^\tau v(0,t)h(0,t)dt = \int_o^\tau v_o \sin(-\omega t)(\overline{h} + h_o \sin(-\omega t))dt = \frac{v_o h_o}{2}\tau. \tag{3.38}$$

Hence, infer the direction of the volume flux associated with the wave motion.

(c) For the special case of shallow water waves approaching the shore, show how the implied bolus velocity, $v^* = \overline{h'v'}/\overline{h}$, from (3.38) is equivalent to the Stokes' drift velocity, $v_o^2/2c$, from (3.28) assuming a wave speed $c = (g\overline{h})^{1/2}$ for shallow water waves, and the relationship between v_o and h_o (from part (a)).

Answer

(a) Assume the sinusoidal change in velocity, $v(y,t) = v_o \sin(ky - \omega t)$, then differentiate in time,

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial t} v_o \sin(ky - \omega t) = -\omega v_o \cos(ky - \omega t),$$

substitute into (3.35), to obtain

$$\frac{\partial h}{\partial y} = \frac{\omega v_o}{g} \cos(ky - \omega t).$$

Then integrate with y on each side, so that

$$\int_0^y \frac{\partial h}{\partial y} dy = h(y,t) - h(0,t),$$

and

$$\frac{\omega v_o}{g} \int_0^y \cos(ky - \omega t)dy = \frac{\omega v_o}{gk} \left[\sin(ky - \omega t)\right]_0^y = \frac{\omega v_o}{gk} \left(\sin(ky - \omega t) - \sin(0 - \omega t)\right),$$

such that

$$h(y,t) = \overline{h} + \frac{\omega v_o}{kg} \sin(ky - \omega t). \tag{3.39}$$

where $\overline{h} = h(0,t) - h_o \sin(-\omega t)$ is the background thickness of the water column. The expressions for h in (3.37) and (3.39) are equivalent as long as the amplitude of the velocity and thickness oscillations in the wave are given by $v_o = g h_o / c$, and the phase speed related to the angular frequency and wave number of the wave by $c \equiv \omega / k$.

(b) Expand

$$\int_o^\tau v(0,t)h(0,t)dt = \int_o^\tau v_o \sin(-\omega t)(\overline{h} + h_o \sin(-\omega t))dt = v_o \int_o^\tau \left(\overline{h}\sin(-\omega t) + h_o \sin^2(-\omega t)\right) dt.$$

Use the trigonometric relationship $\sin^2(x) = (1/2)(1 - \cos(2x))$ to rewrite the integral as

$$v_o \int_o^\tau \left(\overline{h}\sin(-\omega t) + (h_o/2)(1 - \cos(-2\omega t))\right) dt,$$

which gives

$$v_o \left[-\frac{\overline{h}}{-\omega}\cos(-\omega t) + \frac{h_o t}{2} - \frac{h_o}{-4\omega}\sin(-2\omega t) \right]_o^\tau,$$

and then remembering that the angular frequency of the wave is related to the period by $\omega = 2\pi/\tau$, then the time-integrated volume flux in (3.38) is given by

$$v_o \left(\frac{\overline{h}}{\omega}(1-1) + \frac{h_o \tau}{2} + \frac{h_o}{4\omega}(0-0) \right) = \frac{v_o h_o}{2}\tau.$$

This eddy volume flux, $v_o h_o/2$, is in the same direction as the wave speed, there is a greater volume flux, vh, carried beneath the crest of the wave.

(c) The bolus velocity is given by $v^* \equiv v_o h_o/(2\overline{h})$. For this special case of shallow-water waves, $v_o = g h_o/c$ (from Q3.4a), the bolus velocity can be re-written as

$$v^* \equiv \frac{v_o h_o}{2\overline{h}} = \frac{c v_o^2}{2g\overline{h}},$$

and then substituting the phase speed for shallow-water waves, $c = (g\overline{h})^{1/2}$,

$$v^* = \frac{c v_o^2}{2c^2} = \frac{v_o^2}{2c},$$

which is identical to the definition of the Stokes' drift velocity, $v_{Stokes} = v_o^2/(2c)$. While this formal equivalence does not hold for more complicated flows, the example illustrates how the bolus velocity makes an important contribution to the Stokes' drift velocity.

Chapter 4.

Q4.1. Apparent accelerations

When particles are viewed in a rotating frame, they appear to be deflected by apparent accelerations. Consider the case of a particle initially moving only in the x-direction on the rotating Earth, which is deflected by the Coriolis acceleration. The particle travels a distance of 100 m in the x-direction in a time of either (i) 10 s or (ii) 10^4 s.

For each case, (a) calculate the Coriolis acceleration to the right of the motion in the y-direction given by $-fu$ where u is the zonal velocity in the x-direction and f is the Coriolis parameter; $f = 2\Omega \sin\phi$ where $\Omega = 2\pi/\text{day}$ and ϕ is the latitude, which assume here is at 45°N.

(b) calculate the displacement in the y-direction using $s = \frac{1}{2}at^2$ from the Coriolis acceleration, where s is the displacement, a is the Coriolis acceleration in the y-direction, and t is time.

(c) Comparing the x and y displacements, calculate the angle of flight (relative to the initial path).

(d) Compare your answers for (i) and (ii), and discuss in which cases the Coriolis effect appears to be more important.

(e) How might the situation change in the Southern hemisphere?

Answer
(a) (i)

$$-fu = -\left(\frac{2 \times 2\pi \sin 45^o}{24 \times 60^2 \text{s}}\right)\left(\frac{100 \text{ m}}{10 \text{ s}}\right) = -1 \times 10^{-3} \text{m s}^{-2},$$

(ii)

$$-fu = -\left(\frac{2 \times 2\pi \sin 45^o}{24 \times 60^2 \text{s}}\right)\left(\frac{100 \text{ m}}{10^4 \text{ s}}\right) = -1 \times 10^{-6} \text{m s}^{-2},$$

Larger Coriolis acceleration for faster velocity in case (i).

(b) The displacement, $s = \frac{1}{2}at^2 = \frac{1}{2}(-fu)t^2$.
In (i), $s = -0.05$ m and in (ii), $s = -51.4$ m, with a negative sign denoting a southward displacement.

(c) The angle of displacement from the initial motion, $\theta = \tan^{-1}(s/(ut))$.
In (i), $\theta = -0.03^o$ and in (ii), $\theta = -27.2^o$. Again a negative sign denoting a southward displacement from the initial easward motion.

Hence, larger displacement to the right of the motion and greater angle of deflection occurs for case (ii) with the longer timescale, t. Thus, Coriolis acceleration is important when the timescale of motion is comparable or longer than a rotational period. Deflection to the left of the motion in the Southern hemisphere.

Q4.2. Thermal wind balance across the Antarctic Circumpolar Current
Thermal wind balance relates the vertical shear in geostrophic velocity to the horizontal density

gradient, which for the eastward flow is given by

$$\frac{\partial u}{\partial z} = \frac{g}{\rho f}\frac{\partial \rho}{\partial y}.$$

(a) Derive thermal-wind balance by differentiating geostrophic balance (4.4a) with depth and substituting hydrostatic balance (4.2).

(b) Consider the density variations across the Drake Passage in the Southern Ocean (Fig. 4.21). Estimate the change in eastward velocity Δu_g associated with the northward change in density, $\Delta\rho$,

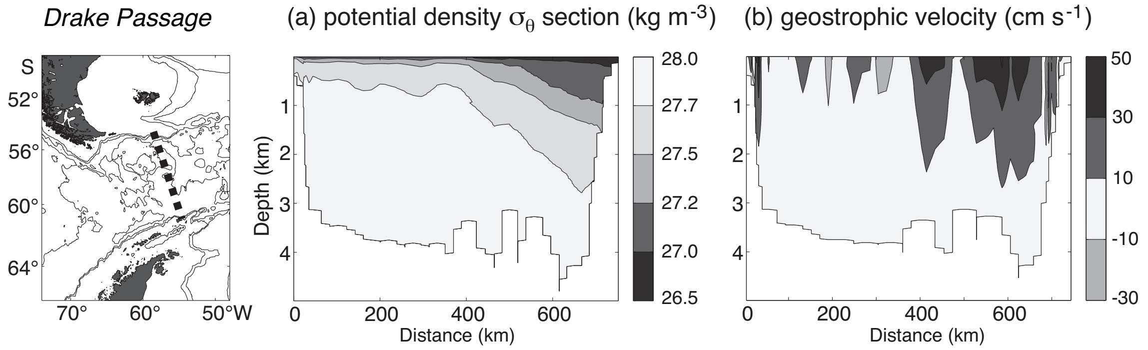

Figure 4.21: Section through Drake Passage between South America and Antarctica in the Southern Ocean for (a) potential density referenced to the sea surface minus 1000 kg m^{-3}, σ_θ, and (b) eastward geostrophic velocity (cm s^{-1}), relative to an assumed zero flow on the sea floor versus depth together with a map of bathymetry and position of section (dashed line) in the left panel. The distance along the section increases northward. Data collected between 30 December 1997 and 7 January 1998 with a maximum station spacing of 17 km. Data supplied by Brian King; further details, see Cunningham et al. (2003).

over a depth scale Δz using a differenced-version of thermal wind,

$$\Delta u_g \sim \frac{g}{\rho f}\frac{\Delta\rho}{\Delta y}\Delta z.$$

Take $\Delta\rho$ at $y = 600$ km in Fig. 4.21a over a north-south distance $\Delta y \sim 200$ km and a depth change $\Delta z \sim 2$ km with $f \sim -10^{-4}$s^{-1}. Check the units and sign of your answer. Compare your answer to the observed geostrophic velocity in Fig. 4.21b.

Answer

(a) Starting with geostrophic balance (4.4a), apply a vertical differential to each side,

$$\frac{\partial u}{\partial z} = -\frac{1}{f}\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial P}{\partial y}\right),$$

assume that the vertical variations of ρ are relatively small compared with the vertical variation of P, so take ρ outside the differential on the left-hand side,

$$\frac{\partial u}{\partial z} = -\frac{1}{\rho f}\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial y}\right).$$

Re-order the differentials on the right-hand side using the general rule for second partial differentials, $\frac{\partial^2}{\partial z \partial y} \equiv \frac{\partial^2}{\partial y \partial z}$, and substitute for $\partial P/\partial z$ using hydrostatic balance (4.2), such that

$$\frac{\partial u}{\partial z} = -\frac{1}{\rho f}\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial y}\right) = -\frac{1}{\rho f}\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}\right) = \frac{g}{\rho f}\frac{\partial \rho}{\partial y}.$$

(b) Assuming a horizontal density contrast, $\Delta\rho \sim -0.3$ kg m^{-3} over a north-south distance $\Delta y \sim 200$ km and a depth change $\Delta z \sim 2$ km implies a velocity change over this depth scale of typically

$$\Delta u_g \sim -\frac{(10 \text{ m s}^{-2})(-0.3 \text{ kg m}^{-3})(2 \times 10^3 \text{m})}{(10^3 \text{kg m}^{-3})(-10^{-4}\text{s}^{-1})(2 \times 10^5 \text{m})} \sim 0.3 \text{ m s}^{-1},$$

where $f \sim -10^{-4}$s^{-1}. This estimate of the eastward velocity is comparable to the diagnosed geostrophic flow assuming no flow along the seafloor (Fig. 4.21b); in practice, there is a flow along the sea floor which increases the eastward velocities and transport through Drake Passage.

Q4.3. Scaling of terms in the momentum equation for a Gulf Stream ring

(a) Consider the flow associated with an ocean eddy formed by the meandering of the Gulf Stream. Assume that the typical magnitude for the current speed is given by $U \sim 0.5$ m s^{-1} and a horizontal length scale $L \sim 100$ km and vertical height scale, $H \sim 500$ m, then estimate the (i) advective timescale given by L/U; and (ii) an upper bound for the vertical velocity from $W < UH/L$.

(b) The x-component of the unforced, momentum equation (4.1a) is given by

$$\frac{Du}{Dt} - 2\Omega v \sin\phi + 2\Omega w \cos\phi + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0.$$

Crudely estimate the magnitude of the first three terms, assuming that the rate of change following the motion is typically given by $Du/Dt \sim U^2/L$, the horizontal velocities are $u \sim v \sim U$ and the vertical velocity $w \sim W$; assume the angular velocity $\Omega = 2\pi/86400$s and a latitude $\phi \sim 35^o$N.

Hence, identify which term balances the horizontal pressure gradient.

(c) Show how the relative importance of the temporal acceleration and the Coriolis acceleration is given by the non-dimensional Rossby number,

$$Ro = \frac{U}{fL}.$$

Calculate how large the Rossby number is for the ocean ring.

Answer
(a) For the ocean ring, a plausible horizontal length scale $L \sim 100$ km implies
(i) an advective timescale given by

$$\frac{L}{U} \sim \frac{10^5 \text{ m}}{0.5 \text{ m s}^{-1}} \sim 2 \times 10^5 \text{ s} \sim 2 \text{ days},$$

and (ii) an upper bound for the vertical velocity is given by

$$W < \frac{UH}{L} \sim \frac{(0.5\text{ m s}^{-1})(0.5\text{ km})}{(100\text{ km})} \sim 2.5 \times 10^{-3}\text{m s}^{-1}.$$

(b) The typical magnitude of the different terms in the momentum equation are given by

$$\frac{Du}{Dt} \sim \frac{U^2}{L} \sim \frac{(0.5\text{ m s}^{-1})^2}{10^5\text{s}} = 2.5 \times 10^{-6}\text{m}^2\text{s}^{-3};$$

$$2\Omega w \cos\phi \sim 2\Omega W \cos\phi \sim (2.5 \times 10^{-3}\text{m s}^{-1})(1.1 \times 10^{-4}\text{s}^{-1}) \sim 3 \times 10^{-7}\text{ms}^{-2};$$

and

$$-fv \sim 2\Omega U \sin\phi \sim (0.8 \times 10^{-4}\text{s}^{-1})(0.5\text{ m s}^{-1}) \sim 4 \times 10^{-5}\text{ms}^{-2}.$$

Hence, the largest of these terms in the x-component of the momentum equation is $-fv$, which is typically one order of magnitude larger than the local acceleration, Du/Dt. Thus, the dominant balance in the unforced momentum equation is then

$$-fv + \frac{1}{\rho}\frac{\partial P}{\partial x} \simeq 0.$$

(c)

$$\left(\frac{Du}{Dt}\right)\left(\frac{1}{fv}\right) \sim \left(\frac{U^2}{L}\right)\left(\frac{1}{fU}\right) = \frac{U}{fL},$$

which is the same as the Rossby number, Ro.
For the Gulf Stream ring,

$$Ro = \frac{U}{fL} \sim \frac{(0.5\text{ m s}^{-1})}{(8 \times 10^{-5}\text{s}^{-1})(10^5\text{m})} \sim 0.06 << 1.$$

Thus, the horizontal components of the momentum equation reduce to geostrophic balance with an accuracy of typically 10% or better.

Q4.4. Divergence and curl.

(a) For the velocity fields for a circulating flow and a reversing jet, depicted in Fig. 4.23, speculate on whether there is (i) any horizontal divergence, i.e. whether more fluid leaves a unit area than enters the unit area (where more fluid leaving a region is defined as positive divergence) and (ii) any rotation of the velocity field, i.e. defined by how a paddle wheel placed in the flow will rotate (where an anti-clockwise rotation is defined as a positive rotation). In each case, identify the sign of your answer.

(b) More formally, now evaluate the horizontal divergence, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, and the relative vorticity measuring the rotation of the fluid, $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ corresponding to the velocity fields depicted in Fig. 4.23. Assume that the eastward velocity, u, and northward velocity, v, vary in x and y in the following manner: (i) $u = y - x$ and $v = -x - y$; and (ii) $u = 2 - 0.5y^2$ and $v = 0$.

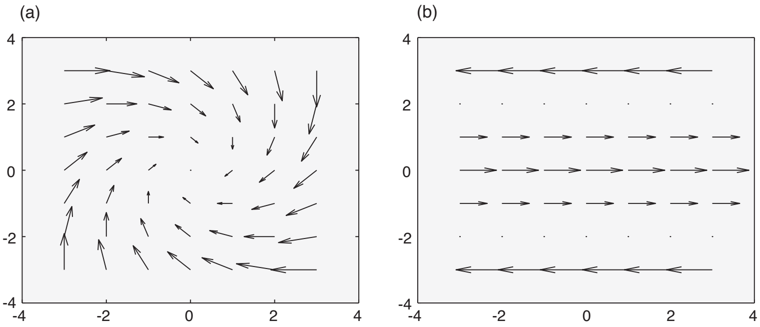

Figure 4.23: Velocity fields for (a) a circulating flow and (b) reversing jet. The horizontal divergence is denoted by whether more fluid leaves a horizontal area than enters it. The relative vorticity, ζ, is denoted by whether a paddle wheel placed in the flow rotates in a anti-clockwise manner.

Answer
(a) (i) For the circulating flow, the horizontal velocity converges towards the centre and so there is negative divergence. A paddle wheel placed in the flow will rotate in a clockwise sense.
(ii) For the reversing jet, the horizontal flow does not converge and so is non-divergent. A paddle wheel placed in the flow rotates in a positive sense (anti-clockwise) for positive y and a negative sense for negative y on either side of the jet.

(b) for case (i) $u = y - x$ and $v = -x - y$, so that the horizontal divergence is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(y - x) + \frac{\partial}{\partial y}(-x - y) = (0 - 1) + (0 - 1) = -2,$$

and relative vorticity by

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-x - y) - \frac{\partial}{\partial y}(y - x) = (-1 + 0) - (1 + 0) = -2,$$

and for case (ii) $u = 2 - 0.5y^2$ and $v = 0$, so that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(2 - 0.5y^2) = 0,$$

and

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial}{\partial y}(2 - 0.5y^2) = 0.5\frac{\partial}{\partial y}(y^2) = y.$$

Hence, in (i) the horizontal divergence is negative and the relative vorticity is negative, and in (ii), there is no divergence and the vorticity changes sign across the jet. The signs of these answers are consistent with (a).

Chapter 5.

Q5.1 How much carbon is in the microbes of the ocean?

Marine bacteria are typically on the order of 1 μm^3 in volume and have a carbon content of about 50×10^{-15}g C cell^{-1}. Bacteria are found throughout the whole water column with a population density of about 10^5cells ml^{-1}. The smallest phytoplankton, *Prochlorococcus*, are of similar size and carbon content and there are also as many as 10^5 cells ml^{-1} in the the surface waters of the subtropical gyres, about half the ocean surface area, but restricted to the upper 250 m depth. Larger organisms typically occur at lower number densities, so most of the living organic carbon in the ocean is in the form of these smallest cells.
(a) Approximately how many bacterial cells are in the global ocean? How many *Prochlorococcus* cells? Assume the volume of the global ocean is about 1.4×10^{18}m^3.
(b) Make an order of magnitude estimate of the amount of carbon in living microbes in the global ocean.
(c) If each *Prochlorococcus* cell divides once a day or less, what is an upper bound on the global rate of primary production (Pg C yr^{-1}) by this organism?

Answer
(a) Number of bacteria cells in the ocean $\sim$ $(1.4 \times 10^{18}$ m$^3)(10^5$cells ml$^{-1})(10^6$ml m$^{-3})$ $= 1.4 \times 10^{29}$ cells $\sim 10^{29}$ cells.
If *Prochlorococcus* occupy half the surface area of ocean to a depth of about 250 m, with the full depth being 5000 m, then the ocean volume occupied $(1.4 \times 10^{18}$ m$^3)(0.5)(250$ m$/5000$ m$)$ $= 3.5 \times 10^{16}$m^3.
Number of *Prochlorococcus* cells in the ocean $\sim$ $(3.5 \times 10^{16}$ m$^3)(10^5$ cells ml$^{-1})(10^6$ml m$^{-3})$ $= 3.5 \times 10^{27}$ cells $\sim 10^{27}$ cells.
(b) Global biomass (standing stock) of bacteria $\sim (1.4 \times 10^{29}$ cells$)(50 \times 10^{-15}$ gC cell$^{-1}) = 7$ Pg C.
Global biomass (standing stock) of *Procholorococcus* $\sim$ $(3.5 \times 10^{27}$cells$)(50 \times 10^{-15}$ gC cell$^{-1})$ $= 0.18$ Pg C.
(c) If each *Procholorococcus* cell reproduced once a day, annual primary production by *Prochlorococcus* $\sim (365$ day$)(0.18$ Pg C day$^{-1}) = 66$ Pg C.

Q5.2 Analogy of Michaelis-Menten type, two-stage process.

Consider a simple every-day analogy involving children collecting marbles (inspired by Runge et al., 2006). A teacher releases a large number of marbles which spill across the floor in a gym, the teacher asks the children to collect the marbles one by one and drop them into a single bucket. There are two stages, the first involves a child finding a single marble on the floor and the second involves the child carrying the marble to the bucket and dropping it in, which is then repeated.

We can write a pseudo-reaction to describe this game. To begin with, there are M uncollected marbles and S school children searching for an uncollected marble. Marbles being carried by the children are represented by $M \cdot S$ and marbles released into the bucket are represented by M_B. This process of collection and transfer of marbles is represented by

$$M + S \rightarrow M \cdot S \rightarrow M_B + S, \tag{5.36}$$

where the total number of school children, S_T, is given by the sum of the children searching for a marble, S, and those carrying a marble, $M \cdot S$,

$$S_T = S + M \cdot S \tag{5.37}$$

The rate of change of the number of uncollected marbles, M, is described by a loss, proportional to the number of uncollected marbles and the number of school children searching for a marble,

$$\frac{\partial M}{\partial t} = -k_{find} M \; S, \tag{5.38}$$

where the product $(k_{find} S)^{-1}$ represents the time for a single marble to be found and the product $(k_{find} M)^{-1}$ as the time for an individual child to find a marble.

(a) Assume that the rate of change of the number of marbles in the bucket, M_B, is given by the source, depending on the number of marbles being carried by a child divided by the time scale, $\mathcal{T}_{drop}$, to return the marble to the bucket and drop the marble in,

$$\frac{\partial M_B}{\partial t} = \frac{M \cdot S}{\mathcal{T}_{drop}}. \tag{5.39}$$

Then show that the rate of change in the number of marbles being dropped into the bucket, M_B, is related to the number of uncollected marbles, M, by

$$\frac{\partial M_B}{\partial t} = \frac{S_T}{\mathcal{T}_{drop}} \frac{M}{(\mathcal{T}_{drop} k_{find})^{-1} + M}. \tag{5.40}$$

(b) Consider the limit when there are very few marbles are on the floor. What is the process limiting the rate of increase in the marble being dropped in the bucket? How is the rate of change of M_B written in this limit?
(c) Now consider the opposing limit when there are a lot of marbles on the floor. What is the process limiting the rate of increase in the marble being dropped in the bucket? How is the rate of change of M_B written in this limit?

Answer
(a) There are several steps:
1. If the number of children carrying a marble is unchanging, then the magnitude of the sink and source have to be equal in (5.38) and (5.39), or equivalently the rate of change of the intermediate state is given by,

$$\frac{\partial}{\partial t} M \cdot S = k_{find} M \; S - \frac{M \cdot S}{\mathcal{T}_{drop}} = 0,$$

such that

$$k_{find} M \; S = \frac{M \cdot S}{\mathcal{T}_{drop}}.$$

2. Combine with the total number of school children, S_T, as this quantity is conserved,

$$S_T = S + M \cdot S,$$

to re-express the number of children searching for a marble, S, so that

$$k_{find}M\ S = k_{find}M\ (S_T - M \cdot S) = \frac{M \cdot S}{\mathcal{T}_{drop}}.$$

Re-arranging provides

$$k_{find}MS_T = M \cdot S \left(\frac{1}{\mathcal{T}_{drop}} + k_{find}M \right),$$

so that

$$M \cdot S = \frac{MS_T}{(k_{find}\mathcal{T}_{drop})^{-1} + M}.$$

3. Combining with (5.39) allows the rate of change of the marbles in the bucket,

$$\frac{\partial M_B}{\partial t} = \frac{M \cdot S}{\mathcal{T}_{drop}},$$

to be expressed as

$$\frac{\partial M_B}{\partial t} = \frac{S_T}{\mathcal{T}_{drop}} \frac{M}{(k_{find}\mathcal{T}_{drop})^{-1} + M}.$$

(b) When there are very few marbles are on the floor, $M << (k_{find}\mathcal{T}_{drop})^{-1}$, so that

$$\frac{\partial M_B}{\partial t} \approx \frac{S_T}{\mathcal{T}_{drop}} \frac{M}{(k_{find}\mathcal{T}_{drop})^{-1}} = S_T\ k_{find}M.$$

Hence, the rate limiting process is the time for a child to find a marble.

(c) When there are many marbles are on the floor, $M >> (k_{find}\mathcal{T}_{drop})^{-1}$, so that

$$\frac{\partial M_B}{\partial t} \approx \frac{S_T}{\mathcal{T}_{drop}} \frac{M}{M} = \frac{S_T}{\mathcal{T}_{drop}}.$$

Hence, the rate limiting process is the time for a child to carry the marble to the bucket and drop it in.

These relationships and limit cases are exactly analogous to the Michaelis-Menten description of an enzymatic reaction (5.11).

Q5.3. Nutrient diffusion towards the cell.

Consider the down-gradient diffusion of nutrient molecules, $\mathcal{N}$, towards a spherical cell of radius R. The transport or area-integrated flux towards the cell (mol s^{-1} cell^{-1}) through any sphere of radius $r > R$, can be described as

$$\int F(r)dA = 4\pi r^2 \kappa \frac{\partial \mathcal{N}}{\partial r}, \tag{5.41}$$

where κ is a molecular diffusivity (m^2 s^{-1}), $F(r)$ is the diffusive flux per unit area (mol s^{-1} m^{-2}) and $\int dA$ is the surface area of the cell (m^{2}cell^{-1}) taken to be a sphere of radius R.

Assume that a quasi-equilibrium state is reached in which the cell is acquiring nutrients at a constant rate and the transport of nutrient towards the cell through any sphere around the cell is also constant.

By continuity $\int F(R)dA$ is the rate of transport of nutrient into the cell which is facilitated by the cell-wall transporters.

(a) Derive an expression for the cell's rate of nutrient uptake in terms of $R, \mathcal{N}(R) = \mathcal{N}_o, \mathcal{N}_\infty$ and κ by re-arranging (5.41) and integrating from the radius R to far away from the cell.

(b) Under what circumstances might the near-cell concentration of the nutrient become almost depleted such that $\mathcal{N}_o \ll \mathcal{N}_\infty$?

(c) For the regime discussed in (b), if the average cellular content of $\mathcal{N}$ is constant, $Q_\mathcal{N}$ (mol cell^{-1}), write a simple expression for the population growth rate, μ (s^{-1}) in terms of cell radius.

(d) If you initialised a batch culture in the laboratory with two individual cells, one 5 microns in diameter and the other 50 microns, would the number density of the smaller or larger cells increase more rapidly?

Answer

(a) If the cellular uptake of nutrients is in an equilibrium with the surrounding medium and transport towards the cell is governed by molecular diffusion, the normal flux, $F(r)$ in mol m^{-2} s^{-1}, through any sphere of radius r towards a spherical cell of radius R can be described by

$$F(r) = \kappa \frac{d\mathcal{N}}{dr} \ . \tag{5.42}$$

Integrate this flux over a surface area per cell (m^{2}cell^{-1}) given by the sphere, $4\pi r^2$, to find the transport of nutrient (mol s^{-1} cell^{-1}) towards the cell surface at radius r ,

$$\mathrm{Transport(r)} = \int F(r)\ dA = 4\pi r^2 \kappa \frac{d\mathcal{N}}{dr} \ . \tag{5.43}$$

Re-arrange and integrate this expression to find the relationship between the transport towards each the cell, $\mathrm{Transport(r)}$, and the cell radius,

$$\frac{\mathrm{Transport(r)}}{4\pi\kappa} \int r^{-2} dr = \int d\mathcal{N} \ . \tag{5.44}$$

Assuming that there is an equilibrium (i.e. that the transport through all spheres, radius r are equal) and that the transport that arrives at the cell is absorbed, the transport towards the cell is equivalent to the uptake across the cell surface, $\mathrm{Transport(R)} = V_{uptake}$ (mol cell^{-1} s^{-1}). Using (5.44) with boundary conditions that $\mathcal{N} = \mathcal{N}_o$ when $r = R$ (just outside the cell wall) and $\mathcal{N} = \mathcal{N}_\infty$ when $r = \infty$ (in the medium, distant from the cell) we find an expression for the uptake,

$$V_{uptake} = 4\pi\kappa R \left(\mathcal{N}_\infty - \mathcal{N}_o\right) \ . \tag{5.45}$$

(b) If uptake across the cell wall is efficient, relative to the diffusive transfer, the near cell concentration will be low relative to the distant concentration $\mathcal{N}_\infty \gg \mathcal{N}_o$, and

$$V_{uptake} \sim 4\pi\kappa R \mathcal{N}_\infty \ . \tag{5.46}$$

(c) If the quota of $\mathcal{N}$ per cell is constant, $Q_{\mathcal{N}}$ (mol cell^{-1}) then dividing the uptake, V_{uptake} (mol cell^{-1} s^{-1}) by the cell quota gives the division rate, μ in s^{-1}:

$$\mu = \frac{V_{uptake}}{Q_{\mathcal{N}}}. \tag{5.47}$$

(d) For spherical cells of constant nutrient density, $\rho_{\mathcal{N}}$ (mol m^{-3}), the amount of nutrient per cell is given by $Q_{\mathcal{N}} = (4/3)\pi R^3 \rho_{\mathcal{N}}$. Using (a) and (c), obtain

$$\mu = \frac{4\pi\kappa R\,(\mathcal{N}_\infty - \mathcal{N}_o)}{(4/3)\pi R^3 \rho_{\mathcal{N}}} = \frac{3\kappa(\mathcal{N}_\infty - \mathcal{N}_o)}{\rho_{\mathcal{N}} R^2}.$$

Hence, μ varies as R^{-2} and population growth decreases with increasing cell radius. In this hypothetical experiment, the population of smaller cells will grow faster than larger cells because diffusive transport is more limiting for larger cells.

Q5.4. Nutrient content and growth of phytoplankton.

The chemostat is an experimental apparatus used to study the physiology of phytoplankton and bacteria. The vessel is filled with a nutrient replete medium (e.g. filtered seawater) and a seed population of the organism of interest. The vessel is stirred and aerated, and temperature and light are regulated. A nutrient replete medium is introduced at a continuous flow rate ψ with concentration of the limiting nutrient element $\mathcal{N}_{in}$. The volume of medium, V, is held constant by an equal rate of outflow (see Figure 4.23). The number density of cells, X, and the nutrient concentration, $\mathcal{N}$ in the outflow are monitored and the system is run to equilibrium. The biomass of phytoplankton, B, is the

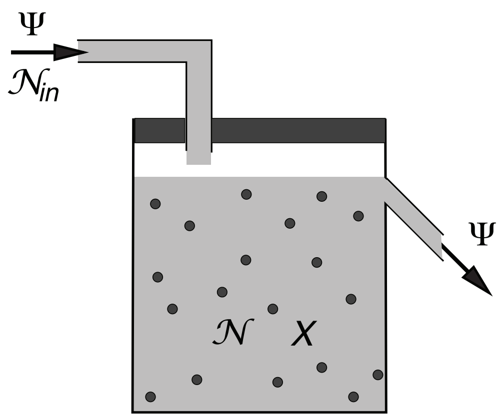

Figure 4.23: Schematic view of chemostat apparatus

product of the number density of cells in the vessel, X, and the average "cell quota", $Q_{\mathcal{N}}$ (quantity of nutrient element $\mathcal{N}$ per cell), of the cells in the vessel:

$$B = Q_{\mathcal{N}} X\ . \tag{5.48}$$

Conservation equations can be written for the nutrient,

$$\frac{d\mathcal{N}}{dt} = -\rho_{\mathcal{N}} X - D\left(\mathcal{N} - \mathcal{N}_{in}\right), \tag{5.49}$$

the biomass,

$$\frac{dB}{dt} = \rho_{\mathcal{N}} X - DB, \tag{5.50}$$

and the number density in the vessel,

$$\frac{dX}{dt} = \mu X - DX \tag{5.51}$$

where the "dilution rate", D, is determined by the flow rate and the volume of medium, $D = \psi/V$. The inflowing medium contains no phytoplankton, so that $X_{in} = 0$ and $B_{in} = 0$, whereas there is an input of the limiting nutrient, $\mathcal{N}_{in}$. The variables are: D (s^{-1}), the dilution rate; $\mathcal{N}$ (mol m^{-3}), the concentration of the limiting nutrient in the vessel; $Q_{\mathcal{N}}$ (mol cell^{-1}), the "cell quota" of element $\mathcal{N}$; X (cell m^{-3}), the number density of cells in medium; V (m^3), the volume of medium in vessel; $\rho_{\mathcal{N}}$ (mol cell^{-1}s^{-1}), the cellular uptake rate of of dissolved nutrient $\mathcal{N}$; μ (s^{-1}), the exponential growth rate of population; and ψ (m^{3}s^{-1}), the rate of inflow/outflow.

(a) Without measuring the composition of the cells directly, how would you estimate the cell quota, $Q_{\mathcal{N}}$, of element $\mathcal{N}$ in the cells at equilibrium?
(b) How would you control the experimental system to examine the relationship between growth rate, μ, and cell quota, $Q_{\mathcal{N}}$?
(c) Using a chemostat, Burmaster (1979) evaluated the relationship between exponential population growth rate and cell quota of phosphorus, Q_P, under equilibrium conditions, as illustrated in Fig. 5.8. Why is the intercept with the x-axis, not at $Q_P = 0$?
(d) The cell quota of phosphorus varies by an order of magnitude across the set of experiments. What underlying processes might this reflect? Why would a high cell quota be associated with higher growth rate?
(e) Briefly discuss in what physical regimes (if any) a chemostat system might be a useful analogy for understanding the regulation an oceanic phytoplankton population?

Answer

(a) We can evaluate the cell quota by running the experiment to equilibrium (i.e. no temporal changes in cell density, etc) and solving the system of equations at steady state (i.e. $d/dt = 0$ in all). From equation (5.44) we find $\rho_{\mathcal{N}} = DB/X$, then substituting for B from (5.42) $\rho_{\mathcal{N}} = DQ_{\mathcal{N}}$. Use this expression to eliminate $\rho_{\mathcal{N}}$ from (5.43) and find that $Q_{\mathcal{N}} = (\mathcal{N}_{in} - \mathcal{N})/X$. Hence, the cell quota $Q_{\mathcal{N}}$ (mol cell^{-1}) can be evaluated from known or measurable quantities from the laboratory experiment.

(b) The steady state ($d/dt = 0$) of equation (5.45) tells us that $\mu = D$; the population growth rate, μ (s^{-1}) and dilution rate D (s^{-1}) are equivalent. Thus the growth rate of the organisms can be controlled by regulation of the dilution rate, i.e. adjusting the flow of medium through the system.

(c) The experiment reveals that there is a minimum cell quota of phosphorus; even when population growth is zero the cells have a finite phosphorus quota. Living cells, even if not reproducing, need some phosphorus for their DNA, phospholipids and other key macro-molecules. Without some phosphorus they would not be viable and die.

(d) The cell quota of phosphorus may increase with growth rate for three reasons: (i) A fast growing population will have a larger fraction of cells about to divide than a slow growing population. Cells which are about to divide are typically about twice the size of a non-dividing cell and thus have a higher amount of phosphorus per cell. However, following this logic we expect a fast growing population of cells to possibly double the cell quota of phosphorus whereas the figure indicates an order of magnitude difference. (ii) Rapid growth and division demand a higher intra-cellular concentration of RNA, the phosphorus-rich molecules which build new proteins. (iii) The cells may have an "internal store", perhaps a vacuole filled with nutrient rich water or a reserve of a storage compound. This reservoir may be quite large in terms of the cells minimum phosphorus content and, if easily accessible, allows the cell to temporarily bypass the slower acquisition of phosphorus (or other element) from outside the cell. This internal store also provides a mechanism by which some cells can maintain division for several generations after the concentration of nutrient in the medium is completely depleted.

(e) The chemostat is somewhat of an analogy to a tropical upwelling regime, where there is relatively little seasonality so an equilibrium viewpoint might be relevant. Upwelling provides nutrients from below and continuity of volume demands that lateral spreading of waters act as an analogy of the outflow of the chemostat.

Chapter 6.

Q6.1. Sources, sinks and residence time of ocean tracers.
Sodium, Na^+, is one of the major conservative ions in the ocean. Sodium is delivered to the sea in rivers and lost through burial, evaporation or in sea-salt aerosol associated with spray. The concentration of sodium in the ocean, $[Na^+]$, is measured to be 470 $\times 10^{-3}$ mol kg^{-1}. The average concentration of sodium in rivers flowing into the sea, $[Na^+]_R$, is significantly lower, about 280$\times 10^{-6}$ mol kg^{-1}. The global delivery of river water to the oceans, ψ_R is about 1 Sv (10^6 m^3 s^{-1}). Assume the density of seawater $\rho_0 = 1024.5$ kg m^{-3}, and an ocean volume $V_0 = 1.4 \times 10^{18}$m^3.
(a) What is the source, S_{Na+} in moles yr^{-1}, of sodium to the global ocean?
(b) What is the global ocean inventory, I_{Na+}, of sodium ions (moles)?
(c) What is the residence time spent by a sodium ion in the global ocean, τ_{Na+} in years?
(d) If the the inventory of sodium in the ocean in not changing, what can we say about the rate at which losses are occurring from the ocean?

Answer
(a) $S_{Na+} = [Na^+]_R \, \psi_R \, \rho_0 = 9.0 \times 10^{12}$ mol yr^{-1}.

(b) $I_{Na+} = [Na^+] \, V_0 \, \rho_0 = 6.7 \times 10^{20}$moles.

(c) The inventory has dimensions of moles, the source has dimensions moles time^{-1}. Thus dividing the inventory by the source reveals a timescale (the residence time), $\tau_{Na+} = I_{Na+}/S_{Na+} \sim 75$ million years.

(d) If the inventory is unchanging in time, then the rate of loss must balance the source rate.

Q6.2. How much carbon is in the ocean?
Dissolved inorganic carbon is defined as

$$DIC = [CO_2^*] + [HCO_3^-] + [CO_3^{2-}], \tag{6.54}$$

and the carbonate species are related by thermodynamic equilibria,

$$K_1' = \frac{[HCO_3^-][H^+]}{[CO_2^*]} \quad , \quad K_2' = \frac{[CO_3^{2-}][H^+]}{[HCO_3^-]} \; . \tag{6.55}$$

(a) Find the equilibrium relationship between DIC and $[CO_2^*]$ in terms of K_1', K_2' and $[H^+]$.
(b) Assume that $[CO_2^*] = K_0 pCO_2^{at}$ together with $pH = 8.18$, pCO_2^{at}= 278μatm, and a mean ocean temperature $= 3.9^o$C, then estimate how much carbon (Pg C) is in the global ocean as DIC.
(c) What other contributions does this estimate miss?
Here use the following values: $T = 3.9^o$C, $S = 34.5$ g kg^{-1}, $K_0 = 5.4 \times 10^{-2}$ mol kg^{-1} atm^{-1}, $K_1' = 8.8 \times 10^{-7}$ mol kg^{-1}, $K_2' = 4.8 \times 10^{-10}$ mol kg^{-1}, ocean volume $V_0 = 1.4 \times 10^{18}$m^3, reference density of seawater $\rho_0 = 1024.5$ kg m^{-3}.

Answer.
(a)

$$DIC = [CO_2^*] \left(1 + \frac{K_1'}{[H^+]} + \frac{K_1' K_2'}{[H^+]^2} \right) .$$

(b) Assume surface ocean DIC is close to equilibrium with pCO_2^{at} in regions where deep waters are formed, $[CO_2^*] = K_0 pCO_2^{at} = 15.0 \times 10^{-6}$ mol kg^{-1}. $[H^+]$ = 10^{-pH} = 6.61$\times 10^{-9}$mol kg^{-1}, and using the solution from (i), evaluate DIC = 2158$\times 10^{-6}$mol kg^{-1}. Global inventory of dissolved inorganic carbon =$DIC\ \rho_0\ V_0\ M_c$ = 37100 Pg C, where M_C=12 g mole^{-1} is the atomic mass of carbon.

(c) Does not account for biological pumps or the fossil fuel input of CO_2.

Q6.3. Solving the carbonate system.
Using the definitions for DIC in (6.54) and the thermodynamic equilibria, K_1' and K_2' in (6.55), then outline how the carbonate system can be solved to evaluate $[H^+], [CO_2^*], [CO_3^{2-}]$, and $[HCO_3^-]$. Assume that you know the values of dissolved inorganic carbon, DIC, and the total alkalinity, A_T, and that $A_T \sim A_C = [HCO_3^-] + 2[CO_3^2-]$.

Answer.
There are four equations and four unknowns. Use the thermodynamic equilibrium expressions to eliminate $[HCO_3^-]$ and $[CO_3^{2-}]$ from the definitions of DIC and A_C, then divide the two expressions to find an equation for DIC/A_C in terms of K_1', K_2' and $[H^+]$. Rearrange to find a quadratic in $[H^+]$ and solve to find

$$[H^+] = \frac{1}{2}\left((\gamma-1)K_1' + \left\{(1-\gamma)^2 K_1'^2 - 4K_1'K_2'(1-2\gamma)\right\}^{\frac{1}{2}}\right),$$

where $\gamma = DIC/A_C$. Other variables can be expressed in terms of $[H^+]$.

Q6.4. Air-sea flux of heat and carbon dioxide

Surface heating and cooling lead to air-sea differences in the effective partial pressure of carbon dioxide, or the effective concentrations of dissolved inorganic carbon, DIC. In turn, these differences drive, and are eroded by, air-sea gas exchange. Consider following a water column with a surface mixed layer of thickness, h (m), and assume there is no biological activity.
(a) Write an expression for the rate of temperature change in the mixed layer, $\partial T/\partial t$ in K s^{-1}, due to a heat flux across the sea surface, $\mathcal{H}$ in W m^{-2}; see Chapter 4.
(b) Write an expression for the rate of change of DIC in the mixed layer in terms of the air-sea flux of carbon dioxide, $\mathcal{F}_C$ in mol m^{-2} s^{-1}; split DIC into its saturated and disequilibrium components, $DIC = DIC_{sat} + \Delta DIC$.
(c) Taking advantage of the almost linear relationship between saturated dissolved inorganic carbon, DIC_{sat}, and temperature (see Fig. 6.6c), with slope $\partial DIC_{sat}/\partial T \simeq \gamma_T$, relate the rate of change in DIC_{sat} in the mixed layer to a surface heat flux, $\mathcal{H}$ (W m^{-2}).
(d) Hence, write down a relationship between air-sea heat and carbon fluxes. What is the relationship when the disequilibrium carbon concentration ΔDIC, is not changing in time? This relationship defines the "potential carbon flux" due to changes in temperature and solubility, $\mathcal{F}_C^{pot}$.
(e) Estimate the "potential carbon flux" across the sea surface (i.e. the flux that would occur in response to a heat gain or loss in order to maintain air-sea equilibrium in partial pressures) in response to a surface warming of 100 W m^{-2} in the tropical Atlantic and surface cooling of -200 W m^{-2} in the vicinity of the Gulf Stream (see Fig. 4.17); assume that $C_p = 4000$ J kg^{-1} K^{-1} and $\gamma_T = -9.0$ μmol kg^{-1} K^{-1}.
(f) Why do measured air-sea carbon fluxes (Fig. 6.16) differ from the "potential carbon flux" (Fig.

6.18)?

Answer

(a) Following equation (4.13), the surface temperature change following a moving water column is described by the vertical flux divergence of heat. Assuming an isolated, column of water where there is no flux through the base of the well mixed layer, the flux divergence is related to the surface heat flux, $\mathcal{H}$, and scaled into temperature units using the density, ρ, and heat capacity, C_p, of seawater:

$$\frac{DT}{Dt} = \frac{1}{\rho C_p}\frac{\mathcal{H}}{h}. \tag{6.56}$$

(b) The rate of change in DIC following a moving water column is described by the vertical flux divergence of carbon. In an isolated, column of water where there is no flux through the base or sides of the well mixed layer, and no biological or freshwater influence, the carbon flux divergence depends only on the air-sea flux of carbon, $\mathcal{F}_C$:

$$\frac{D}{Dt}DIC = \frac{D}{Dt}DIC_{sat} + \frac{D}{Dt}\Delta DIC = \frac{\mathcal{F}_C}{\rho h}. \tag{6.57}$$

(c) Assuming a linear relationship between DIC_{sat} and T,

$$\frac{D}{Dt}DIC_{sat} = \frac{\partial DIC_{sat}}{\partial T}\frac{DT}{Dt} = \gamma_T\frac{DT}{Dt}. \tag{6.58}$$

Combining (6.57) and (6.58)

$$\gamma_T\frac{DT}{Dt} + \frac{D}{Dt}\Delta DIC = \frac{\mathcal{F}_C}{\rho h}. \tag{6.59}$$

(d) Combining (6.59) and (6.56) we find

$$\gamma_T\frac{\mathcal{H}}{\rho C_p h} + \frac{D}{Dt}\Delta DIC = \frac{\mathcal{F}_C}{\rho h}, \tag{6.60}$$

and when $D(\Delta DIC)/Dt = 0$, the potential carbon flux is

$$\mathcal{F}_C^{pot} = \gamma_T\frac{\mathcal{H}}{C_p}. \tag{6.61}$$

(e) (i) Tropical potential carbon flux: the surface heat flux in the tropics, $\mathcal{H} \sim$ 100 W m^{-2} (i.e. into the ocean), so that $\mathcal{F}_C^{pot} = (-9 \times 10^{-6}$mol kg^{-1}K$^{-1})(100$ W m$^{-2})(4000$ J kg^{-1} K$^{-1})^{-1} = -0.23 \times 10^{-6}$ mol m^{-2} s^{-1} or -7 mol m^{-2} y^{-1} (out of the ocean).
Gulf Stream potential carbon flux: the surface heat flux over the Gulf Stream, $\mathcal{H} \sim -200$ W m^{-2} (i.e. out of the ocean), so that $\mathcal{F}_C^{pot} = 0.45 \times 10^{-6}$ mol m^{-2} s^{-1} or 14 mol m^{-2} y^{-1} (into the ocean).

(f) Local carbon fluxes are not exactly described by the potential carbon flux because lateral and vertical mixing, biological processes and freshwater fluxes may also be significant. In addition, the long timescale for equilibration of carbon across the sea surface (on the order of one year) means that water parcels may be swept along way down stream before an air-sea carbon flux fully equilibrates in response to a surface heat flux.

Chapter 7.

Q7.1. Seasonality in temperature.

Consider the seasonal change in temperature for a body of water in a well-mixed sea of constant thickness, D, where the temperature evolution is given by

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C_p}\frac{\mathcal{H}(t)}{D}. \tag{7.5}$$

and the surface heat flux is assumed to vary as $\mathcal{H}(t) = -\mathcal{H}_o \cos(2\pi t/\mathcal{T})$ where the time t is 0 at the start of the year and $\mathcal{T}$ at the end of the year.
(a) When is $\partial T/\partial t$ most positive and negative?
(b) When is T likely to be largest and smallest over the year?
(c) Show that the seasonal temperature range is given by $\frac{\mathcal{H}_o}{\rho C_p D}\frac{\mathcal{T}}{2\pi}$.
(d) How large is the implied seasonal cycle in temperature for a water thickness of $D = 100$ m, typical for a shelf sea, a surface heat flux of $\mathcal{H}_o = 200$ W m^{-2}, $\rho \sim 1000$ kg m^{-3}, $C_p \sim 4000$ J kg^{-1} K^{-1} and $\mathcal{T}$ is the number of seconds in a year. Check the units of your answer.

Answer
(a) Substitute $\mathcal{H}(t) = -\mathcal{H}_o \cos(2\pi t/\mathcal{T})$ into (7.5),

$$\frac{\partial T}{\partial t} = \frac{1}{\rho C_p}\frac{\mathcal{H}(t)}{D} = -\frac{1}{\rho C_p}\frac{\mathcal{H}_o \cos(2\pi t/\mathcal{T})}{D},$$

then $\partial T/\partial t$ is most positive at $t = \mathcal{T}/2$, the summer solstice, and most negative at $t = \mathcal{T}$, the winter solstice.

(b) T is largest at $t = 3\mathcal{T}/4$, the autumn equinox after the surface heat input ceases and is smallest at $t = \mathcal{T}/4$, the spring equinox after the surface cooling ceases.

(c) Integrate (7.5) using $\int_0^t \cos(at)dt = \frac{1}{a}\sin(at)$, so that

$$T(t) - T(0) = \int_0^t -\frac{1}{\rho C_p}\frac{\mathcal{H}_o \cos(2\pi t/\mathcal{T})}{D}dt = -\frac{\mathcal{H}_o}{\rho C_p D}\int_0^t \cos(2\pi t/\mathcal{T})dt.$$

Then use $\int_0^t \cos(at)dt = \frac{1}{a}\sin(at)$, so that

$$T(t) - T(0) = -\frac{\mathcal{H}_o}{\rho C_p D}\left[\frac{\sin(2\pi t/\mathcal{T})}{2\pi/\mathcal{T}}\right]_0^t = -\frac{\mathcal{H}_o}{\rho C_p D}\frac{\mathcal{T}}{2\pi}\sin(2\pi t/\mathcal{T}).$$

Hence, the seasonal range in temperature is typically the order of magnitude of

$$\frac{\mathcal{H}_o}{\rho C_p D}\frac{\mathcal{T}}{2\pi}.$$

(d) For a shallow shelf sea, the typical seasonal temperature range is

$$\frac{\mathcal{H}_o}{\rho C_p D}\frac{\mathcal{T}}{2\pi} \sim \frac{(200\ \text{W m}^{-2})(365 \times 24 \times 60^2\text{s})}{(10^3\text{kg m}^{-3})(4 \times 10^3\text{J kg}^{-1}\text{K}^{-1})(100\ \text{m})(2\pi)} = 2.5\ \text{K}.$$

Units of $(\mathcal{H}_o\mathcal{T})/(\rho C_p D)$ are

$$\frac{(\text{J s}^{-1}\text{m}^{-2})(\text{s})}{(\text{kg m}^{-3})(\text{J kg}^{-1}\text{K}^{-1})(\text{m})} \equiv \text{K}.$$

Q7.2. Summer mixed-layer thickness over the open ocean.

During the summer, the thickness of the mixed layer depends on the competition between wind mixing and surface heat input, and can often be predicted by

$$h = \frac{2mu_*^3}{g\alpha_T\mathcal{H}/(\rho C_p)}. \tag{7.6}$$

(a) Show that dimensionally the two sides of the equation are balanced where the friction velocity u_* is in m s^{-1}, g in m s^{-2}, α_T in K^{-1}, $\mathcal{H}$ in J s^{-1}m^{-2}, ρ in kg m^{-3}, C_p in J kg^{-1}K^{-1} and m is non dimensional.

(b) For a wind speed, u_a, of 10 m s^{-1}, estimate the magnitude of the surface stress, $\tau = \rho_a c_d u_a^2$, and diagnose the related friction velocity, u_* defined by $\tau = \rho u_*^2$, assuming a drag coefficient, c_d of 1.14×10^{-3} and air density, ρ_a of 1.2 kg m^{-3}.

(c) Predict the summer thickness of the mixed layer, h, from (7.6) assuming a wind speed of 10 m s^{-1} and a surface heat flux of $\mathcal{H} \sim 200$ W m^{-2} together with $\alpha_T \sim 2 \times 10^{-4}$ K^{-1}, $\rho \sim 10^3$kg m^{-3}, $g = 9.81$ m s^{-2} and $m = m_e\,(\rho/(\rho_a c_d))^{1/2}$ with an efficiency of wind mixing, m_e, of 1.5×10^{-3}.

(d) How much does h alter if either (i) the wind speed doubles or (ii) the heat flux into the ocean doubles?

Answer
(a) Units of h in m, and

$$\frac{2mu_*^3}{g\alpha_T\mathcal{H}/(\rho C_p)} \sim \frac{(\text{m s}^{-1})^3(\text{kg m}^{-3})(\text{J kg}^{-1}\text{K}^{-1})}{(\text{m s}^{-2})(\text{K}^{-1})(\text{J s}^{-1}\text{m}^{-2})} = \text{m}.$$

(b)

$$\tau = \rho_a c_d u_a^2 = (1.2 \text{ kg m}^{-3})(1.14 \times 10^{-3})(10 \text{ m s}^{-1})^2 = 0.14 \text{ N m}^{-2}$$

(with N$\equiv$kg m s^{-2});

$$u_* = \left(\frac{\tau}{\rho}\right)^{1/2} = \left(\frac{0.14 \text{ N m}^{-2}}{10^3\text{kg m}^{-3}}\right)^{1/2} = 0.012 \text{ m s}^{-1}.$$

(c)

$$m = m_e\left(\frac{\rho}{\rho_a c_d}\right)^{1/2} \sim 1.5 \times 10^{-3}\left(\frac{10^3\text{kg m}^{-3}}{(1.2 \text{ kg m}^{-3})(1.14 \times 10^{-3})}\right)^{1/2} = 1.28,$$

$$\begin{aligned} h &= \frac{2mu_*^3}{g\alpha_T\mathcal{H}/(\rho C_p)} \\ &\sim \frac{(2)(1.5 \times 10^{-3})(1.2 \times 10^{-2}\text{m s}^{-1})^3(10^3\text{kg m}^{-3})(4 \times 10^3\text{J kg}^{-1}\text{K}^{-1})}{(9.81 \text{ m s}^{-2})(2 \times 10^{-4}\text{K}^{-1})(200 \text{ m})} = 42 \text{ m}. \end{aligned}$$

(d) (i) as $h \propto u_*^3$ and $u_* \propto u_a$, then doubling of wind speed leads to h increasing by a factor $2^3 = 8$, (ii) as $h \propto \mathcal{H}^{-1}$, then doubling of heat flux leads to h increasing by a factor 0.5.

Q7.3. Mechanical forcing from winds and tides in a shelf sea.

In the shelf seas, there is often a front separating regions of stable and unstable stratification, which occurs at a depth given by

$$D = \frac{2}{g\alpha_T \mathcal{H}/C_p}\left(m_e c_d \rho_a u_a^3 + m_b c_b \rho u_b^3\right), \tag{7.7}$$

which is based upon the competition between the rate of input of mechanical energy available for mixing from the wind, $m_e c_d \rho_a u_a^3$, and the tides, $m_b c_b \rho u_b^3$, versus the stratifying effect of a surface heat flux, $\mathcal{H}$; m_b is the efficiency in using tidal inputs of energy for mixing and c_b is the drag coefficient for the bottom, and u_b is a bottom velocity.

(a) Show that the previous mixed-layer balance (7.6) in Q7.2 can be re-expressed as (7.7) assuming that (i) the mixed layer thickness, h, becomes the same as the depth of the water column, D, and (ii) the mechanical energy input available for mixing from the wind, mu_*^3, is augmented to include the mechanical input from the tides, $m_b c_b u_b^3$.

(b) Compare the relative importance of the mechanical energy inputs available for mixing for a wind speed of 10 m s^{-1} and a bottom current of (i) for a strong tide, 1 m s^{-1} or (ii) a weak tide, 0.1 m s^{-1}; assume $m_b = 4 \times 10^{-3}$ and $c_b = 2.5 \times 10^{-3}$ (and values from Q7.2).

(c) For both the strong and weak tides, predict the thickness D of the well-mixed water column from (7.7) if there is a surface heat flux of 200 W m^{-2}.

Answer

(a) Start from (7.6) in Q7.2, use $D = h$ and include $m_b c_b u_b^3$ to obtain

$$D = \frac{2}{g\alpha_T \mathcal{H}/(\rho C_p)}\left(mu_*^3 + m_b c_b u_b^3\right).$$

Use $m = m_e\left(\rho/(\rho_a c_d)\right)^{1/2}$ and $u_* = (\tau/\rho)^{1/2} = (\rho_a c_d/\rho)^{1/2} u_a$, so that D is expressed as

$$D = \frac{2}{g\alpha_T \mathcal{H}/(\rho C_p)}\left(m_e\left(\frac{\rho_a c_d}{\rho}\right)^{-1/2}\left(\frac{\rho_a c_d}{\rho}\right)^{3/2} u_a^3 + m_b c_b u_b^3\right),$$

which can be written as

$$D = \frac{2}{g\alpha_T \mathcal{H}/C_p}\left(m_e c_d \rho_a u_a^3 + m_b c_b \rho u_b^3\right).$$

(b) Ratio of tidal input/wind input of mechanical energy available for mixing is given by

$$\frac{m_b c_b \rho u_b^3}{m_e c_d \rho_a u_a^3}.$$

For case (i), this ratio is

$$\frac{(4\times10^{-3})(2.5\times10^{-3})(10^3\text{kg m}^{-3})(1\text{ m s}^{-1})^3}{(1.5\times10^{-3})(1.14\times10^{-3})(1.2\text{ kg m}^{-3})(10\text{ m s}^{-1})^3} = 4.87 \sim 5.$$

For case (i), this ratio is

$$\frac{(4\times10^{-3})(2.5\times10^{-3})(10^3\text{kg m}^{-3})(0.1\text{ m s}^{-1})^3}{(1.5\times10^{-3})(1.14\times10^{-3})(1.2\text{ kg m}^{-3})(10\text{ m s}^{-1})^3} = 4.87\times10^{-3} \sim 5\times10^{-3}.$$

Hence, tidal mixing dominates for strong currents and wind mixing for weak currents.

(c) Use

$$D = \frac{2C_p}{g\alpha_T\mathcal{H}}\left(m_e c_d \rho_a u_a^3 + m_b c_b \rho u_b^3\right).$$

For case (i),

$$D = \frac{(2)(4000)}{(9.81)(2\times10^{-4})(200)}\left\{(1.5\times10^{-3})(1.14\times10^{-3})(1.2)(10)^3 + (2.5\times10^{-3})(10^3)(1)^3\right\} = 246,$$

and for case (ii), $D = 42$, with units given by

$$\frac{(\text{J kg}^{-1}\text{K}^{-1})}{(\text{m s}^{-2})(\text{K}^{-1})(\text{J s}^{-1}\text{ m}^{-2})}\left((\text{kg m}^{-3})(\text{m s}^{-1})^3\right) = \text{m}.$$

Hence, combination of tidal and wind mixing versus surface heating leads to the water column being well mixed over a depth of (i) 246 m and (ii) 42 m, leading to the tidally-mixed front being located at this depth.

Q7.4 Inter-annual variability and longterm warming in the shelf seas.

Winter temperature anomalies in the shelf seas are illustrated in Fig. 7.21, full line over the European Shelf. These thermal anomalies are primarily due to the effect of the surface forcing, rather than

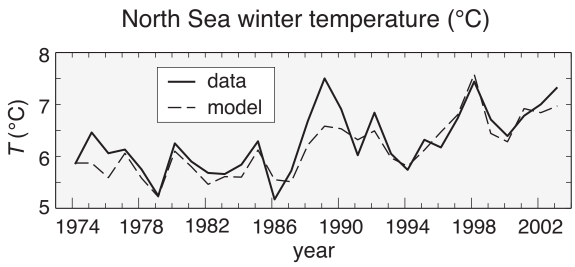

Figure 7.21: Observed and modelled time series of February temperature (^{o}C) in the North Sea (56.3°N, 1.7°W) of the European shelf from 1974 to 2003. The simulation (dashed line) is from a one-dimensional mixed layer model driven by meteorological and tidal forcing (Sharples et al., 2006). The time series reveals both interannual variability and a longer-term warming trend. Data supplied by Jonathan Sharples.

horizontal exchange with the open ocean, since they are predicted reasonably well using a one-dimensional mixed-layer model (like Box 7.1.2) driven by meteorological and tidal forcing (Fig. 7.21,

dashed line).
(a) Estimate the warming trend over nearly the three decades of data in Fig. 7.21.

(b) Assuming a one-dimensional heat balance (7.5), then estimate the surface heat flux, $\mathcal{H}$, needed to explain this warming trend; assume $\rho \sim 1027$ kg m^{-3}, $C_p \sim 4000$ J kg^{-1} K^{-1}, and D is the depth of 60 m.

(c) How would your answer in (a) have altered if applied to a shorter record? What then are the implications for inferring long-term climate change?

Answer
(a) a warming trend of

$$\frac{7.4^o\text{C} - 5.9^o\text{C}}{(2003 - 1974)\text{yr}} = 0.0517^o\text{C yr}^{-1} \sim 0.5^o\text{C per decade.}$$

(b) a surface heat flux of

$$\begin{aligned} \mathcal{H} &= \rho C_p D \frac{\partial T}{\partial t} \\ &\sim (1027 \text{ kg m}^{-3})(4000 \text{ J kg}^{-1}\text{K}^{-1})(60 \text{ m}) \frac{(0.0517 \text{ K})}{(365 \times 24 \times 60^2 \text{s})} \sim 0.4 \text{ W m}^{-2}. \end{aligned}$$

(c) there is much larger inter-annual variability, a warming of more than 2^oC in 4 years between 1986 and 1990 or a cooling of -1^oC in 2 years between 1998 and 2000. Need a record of at least several decades to detect a robust signal of climate warming.

Chapter 8.

Q8.1. What does the Stommel model suggest for the width of the western boundary current?
Based on the Stommel closure for the western boundary layer, the meridional velocity is determined by the balance between how fluid changes planetary vorticity, $Df/Dt = \beta v$, and dissipation of vorticity by bottom friction,

$$\beta v \simeq -r\zeta, \tag{8.24}$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Assuming that west-east gradients are greater than north-south gradients, (8.24) can be approximated to

$$\beta v \simeq -r\frac{\partial v}{\partial x}. \tag{8.25}$$

(a) Using scale analysis, estimate the characteristic width of the western boundary current. Assume $\beta = 10^{-11}$m^{-1}s^{-1} and a spin down timescale for the currents of $r^{-1} \sim 10$ days.
(b) What is the strength of the western boundary current assuming that the current returns the interior Sverdrup transport, $v_{int}HL_{int} \sim -50 \times 10^6$m^{3}s^{-1} (Fig. 8.13c) and a depth of $H \sim 10^3$m ,

$$v_{bdry}HL_{bdry} + v_{int}HL_{int} = 0, \tag{8.26}$$

Answer
(a) the meridional velocity in the boundary current has a typical magnitude v_{bdry} and a width L_{bdry}, then implies $\partial v/\partial x \sim |v_{bdry}/L_{bdry}|$. From a scaling of the terms in (8.25),

$$\beta v_{bdry} \sim -r\frac{\partial v}{\partial x} \sim r\frac{v_{bdry}}{L_{bdry}},$$

and suggests a width of the boundary current of

$$L_{bdry} \sim \frac{r}{\beta},$$

which is typically 100 km for $\beta \sim 10^{-11}$m^{-1}s^{-1} and a spin down timescale for the currents of $r^{-1} \sim 10$ days. However, this choice for the spin down timescale is poorly known and should be viewed as device to obtain a western boundary solution.

(b) From conservation of volume transport,

$$v_{bdry} = -\frac{v_{int}HL_{int}}{HL_{bdry}},$$

then assuming the volume transport in the gyre interior, $v_{int}HL_{int} = -50 \times 10^6$m^{3}s^{-1}, the thickness of moving water, $H \sim 10^3$m, and the width of the western boundary (from part (a)), $L_{bdry} \sim 10^5$ m, then

$$v_{bdry} \sim -\frac{(-50 \times 10^6\text{m}^3\text{s}^{-1})}{(10^3\text{m})(10^5\text{m})} = 0.5 \text{ m s}^{-1}.$$

These relations for the boundary current strength and width should be viewed with some caution, although they do suggest that the boundary current narrows as β increases or as the frictional dissipation decreases.

Q8.2. What does the Munk model suggest for the width of the western boundary current?
Munk (1950) provided an alternative to the Stommel (1948) model for ocean gyres, where the meridional flow in the western boundary is controlled by a horizontal diffusion of relative vorticity,

$$\beta v = K_h \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \zeta, \tag{8.27}$$

where K_h is the horizontal diffusivity. Apply the scale analysis diagnose how the width of the western boundary varies with β and K_h. What is the typical value for the boundary width for $\beta = 10^{-11}\text{m}^{-1}\text{s}^{-1}$ and $K_h \sim 10^3\text{m}^2\text{s}^{-1}$?

Answer
Scaling as in Q8.1 applied to (8.27) gives

$$\beta v_{bdry} \sim K_h \frac{\partial^2}{\partial x^2} \zeta \sim \frac{K_h}{L_{bdry}^2} \frac{v_{bdry}}{L_{bdry}},$$

which implies

$$L_{bdry} \sim \left(\frac{K_h}{\beta} \right)^{1/3}.$$

For $\beta = 10^{-11}\text{m}^{-1}\text{s}^{-1}$ and a plausible value for the horizontal diffusivity, $K_h = 10^3\text{m}^2\text{s}^{-1}$,

$$L_{bdry} \sim \left(\frac{10^3\text{m}^2\text{s}^{-1}}{10^{-11}\text{m}^{-1}\text{s}^{-1}} \right)^{1/3} = (10^{14}\text{m}^3)^{1/3} = 46.4 \text{ km} \sim 50 \text{ km}.$$

Again, this scaling should be viewed with some caution, although K_h is a better known parameter than r^{-1} as in the Stommel model. The important point is that the width of the western boundary decreases as the rotational control from β increases.

Q8.3. Derive the vorticity equation
The momentum equations for a layer of constant density and variable thickness, h, are given by

$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x} + \frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}, \tag{8.28}$$

$$\frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y} + \frac{1}{\rho_0}\frac{\partial \tau_y}{\partial z}, \tag{8.29}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\partial/\partial x + v\partial/\partial y$ and represents the rate of change following a fluid parcel, and τ is the frictional stress.

Form a vorticity equation, taking $\partial/\partial x$(8.29)-$\partial/\partial y$(8.28) to obtain

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right), \tag{8.30}$$

where $Df/Dt = \beta v$ and represents the advection of planetary vorticity and $\zeta = \partial v/\partial x - \partial u/\partial y$ is the relative vorticity.

Answer
In forming the vorticity equation:
(i) When taking $\partial/\partial x$(8.29)-$\partial/\partial y$(8.28), terms involving h gradients cancel;
(ii) $\partial/\partial x(fu) = f\partial u/\partial x$ as f does not vary with x, while $\partial/\partial y(fv) = f\partial v/\partial y + v\partial f/\partial y$ as f varies with y;
(iii) the order of two differentials can be re-ordered, such that $\partial/\partial x(\partial/\partial z) \equiv \partial/\partial z(\partial/\partial x)$.

In more detail, to obtain the vorticity equation, follow these steps:
1. Start by taking $\partial/\partial x$(8.29),

$$\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) + \frac{\partial}{\partial x}(fu) = \frac{\partial}{\partial x}\left(-g\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial \tau_y}{\partial z}\right). \tag{8.31}$$

Remembering the definition of the material derivative,

$$\frac{Dv}{Dt} \equiv \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y}, \tag{8.32}$$

and the general rule for differentiating the product of two variables, $\partial/\partial x(AB) \equiv A\partial B/\partial x + B\partial A/\partial x$, then the first term in (8.31) becomes

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{Dv}{Dt}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} \\ &= \frac{D}{Dt}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y}.\end{aligned} \tag{8.33}$$

For the second term in (8.31), f does not vary with x, so that

$$\frac{\partial}{\partial x}(fu) = f\frac{\partial u}{\partial x}. \tag{8.34}$$

For the third term in (8.31),

$$\frac{\partial}{\partial x}\left(-g\frac{\partial h}{\partial y}\right) = -g\frac{\partial^2 h}{\partial x \partial y}. \tag{8.35}$$

For the fourth term in (8.31), ρ_0 is not assumed to vary and the order of the second differentials can be swapped, so that

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho_0}\frac{\partial \tau_y}{\partial z}\right) = \frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial \tau_y}{\partial z}\right) = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x}\right). \tag{8.36}$$

Hence, substituting (8.33) to (8.36) into (8.31) gives

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + f\frac{\partial u}{\partial x} = -g\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x}\right). \tag{8.37}$$

2. Now apply $\partial/\partial y$(8.28) to obtain

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) - \frac{\partial}{\partial y}(fv) = -g\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}\right). \tag{8.38}$$

The first term in (8.38) can be written as

$$\frac{\partial}{\partial y}\left(\frac{Du}{Dt}\right) = \frac{D}{Dt}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y}. \tag{8.39}$$

For the second term in (8.38), f *does* vary with y, so that

$$\frac{\partial}{\partial y}(fv) = f\frac{\partial v}{\partial y} + v\frac{df}{dy}. \tag{8.40}$$

For the third term in (8.38),

$$\frac{\partial}{\partial y}\left(-g\frac{\partial h}{\partial x}\right) = -g\frac{\partial^2 h}{\partial x \partial y}, \tag{8.41}$$

and the fourth term in (8.38) can be re-written as

$$\frac{\partial}{\partial y}\left(\frac{1}{\rho_0}\frac{\partial \tau_x}{\partial z}\right) = \frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial \tau_x}{\partial z}\right) = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_x}{\partial y}\right). \tag{8.42}$$

Hence, substituting (8.39) to (8.42) into (8.38) gives

$$\frac{D}{Dt}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} - f\frac{\partial v}{\partial y} - v\frac{df}{dy} = -g\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_x}{\partial y}\right). \tag{8.43}$$

3. Now take (8.37) - (8.43), so that the first term on the right-hand side cancel, as

$$-g\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial y}\right) + g\frac{\partial}{\partial y}\left(\frac{\partial h}{\partial x}\right) = 0,$$

to give

$$\frac{D}{Dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right),$$

then defining the relative vorticity, $\zeta \equiv \partial v/\partial x - \partial u/\partial y$, gives

$$\frac{D}{Dt}\zeta + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + v\frac{df}{dy} = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right),$$

and, finally, using $Df/Dt \equiv v df/dy$ leads to the vorticity equation,

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right). \tag{8.44}$$

Hence, the rate of change of the absolute vorticity, $\zeta + f$, (first term) is controlled by the absolute vorticity multiplied by the horizontal divergence (second term) and the vertical divergence in the frictional twisting acceleration (third term).

Q8.4. Rossby waves

Over timescales longer than a day, the ocean supports a particular class of waves called Rossby waves or planetary waves, where the restoring force is provided by the meridional change in planetary vorticity over the globe; the motion in these planetary waves can be understood in terms of the fluid conserving potential vorticity, $Q = (\zeta + f)/h$.

(a) Firstly, on a small scale, ignore any variations in the thickness of the water column, h, and consider conservation of absolute vorticity, $q = \zeta + f$. Consider a string of hypothetical particles, labeled A to D positioned from east to west, which are able to move and assume that q is conserved for each of them (Fig. 8.24a). If particle B is perturbed northward to a larger f, explain why a

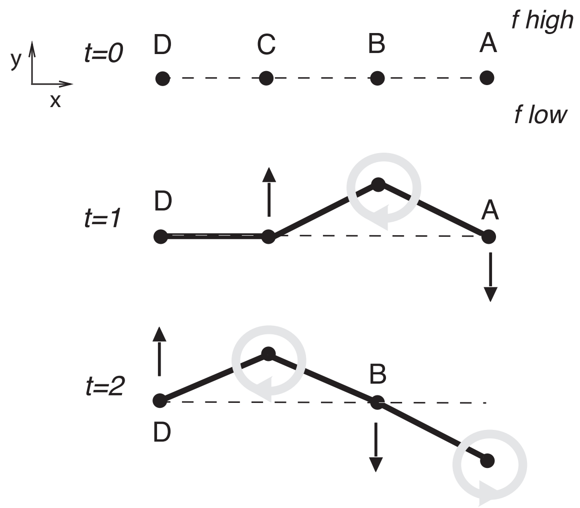

Figure 8.24: A schematic figure depicting how Rossby waves propagate westward due to the northward increase in planetary vorticity, f A plan view of a hypothetical string of particles labeled A to D. A time sequence of the string of particles is shown from $t = 0$ where they are orientated along a latitude circle. If particle B is displaced northward at a later time ($t = 1$), then a circulation with a negative relative vorticity is acquired around B. This anti-cyclonic circulation around particle B (marked by grey arrow) displaces particle C northward and particle A southward ($t = 2$).

circulation with a negative relative vorticity is created. Why is particle C then displaced northward and particle A southward? Explain what subsequently occurs.

(b) Secondly, consider the large-scale limit where relative vorticity, ζ is small and f/h is conserved. Again as particle B is displaced northward, then f increases, how does the thickness h vary? What is the circulation around particle B based upon geostrophy, $v = \frac{g}{f}\partial h/\partial x$? Explain how this flow affects the movement of the particle C and A? What is the similarity with case (a).

Answer
(a) Absolute vorticity, $q = \zeta + f$, is assumed to be conserved. If particle B initially has no relative vorticity and is at a planetary vorticity $f_{initial}$, then

$$q_{final} = \zeta_{final} + f_{final} = f_{initial},$$

which implies $\zeta_{final} = f_{initial} - f_{final}$. If particle B moves northward, then $f_{final} > f_{initial}$, so that particle B acquires $\zeta_{final} < 0$. This secondary circulation around particle B then displaces particle

C northward and particle A southward. In turn, particle C has to acquire $\zeta < 0$, which then deflects particle D northward and returns particle B southward. This process continues to repeat itself with the northward displacement propagating westward as a wave motion.

(b) Large-scale potential vorticity, f/h, is assumed to be conserved, such that

$$\frac{f_{final}}{h_{final}} = \frac{f_{initial}}{h_{initial}},$$

which implies that

$$h_{final} = h_{initial}\frac{f_{final}}{f_{initial}}.$$

If B moves northward, then $f_{final} > f_{initial}$, so that the thickness of the layer increases, $h_{final} > h_{initial}$. If there is increased layer thickness, then from geostrophy,

$$v = \frac{g}{f}\frac{\partial h}{\partial x},$$

there is a northward flow to the west of B (as $\partial h/\partial x > 0$) and a southward flow to the east (as $\partial h/\partial x < 0$). This induced circulation displaces C northward and A southward. Consequently, a thickness anomaly propagates from B to C and further to the west. In both cases, the Rossby waves have a phase (linking lines of crests or troughs) which propagate westward. To understand the energy transferred by Rossby waves and how they might account for the spin up of western boundary currents, see Gill (1982).

Chapter 9.

Q9.1. Slantwise convection involves the exchange of two fluid parcels where a light parcel rises and a dense parcel sinks. (a) Identify whether potential energy is released and converted to kinetic energy for the following possible exchange paths marked as (i) to (iv) in Fig. 9.20a across part of Drake Passage in the Southern Ocean.

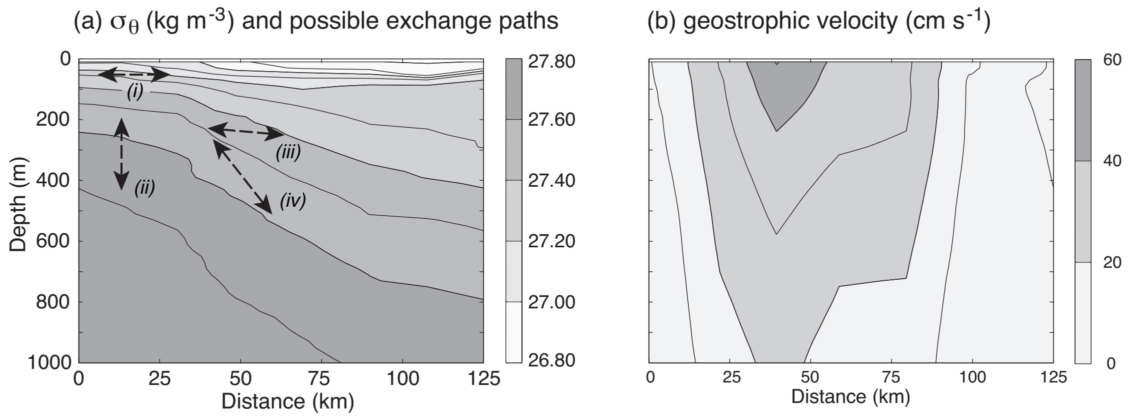

Figure 9.20: A meridional section crossing part of Drake Passage between South America and Antarctica in the Southern Ocean for (a) potential density (contours every 0.1 kg m^{-3}) referenced to the sea surface minus 1000 kg m^{-3}, σ_θ, and (b) eastward geostrophic velocity (contours every 10 cm s^{-1}) versus depth. Possible exchange paths are marked as (i) to (iv) in (a). Data supplied by Brian King.

(b) For the case where there might be a release of potential energy, then estimate how large the velocity might be from the exchange of two fluid parcels, labelled A and B,

$$\overline{\rho}\Delta v^2 = g\Delta\sigma\Delta z = g(\sigma_A - \sigma_B)(z_A - z_B), \tag{9.30}$$

where σ_A and σ_B are the initial and final potential densities, and z_A and z_B are the initial and final vertical positions respectively (taken as negative depth); $g = 9.81$m s^{-2} and $\overline{\rho}$ is the mean density of parcels A and B. Take your estimates from Fig. 9.20a and assume exchange on a horizontal scale of 25 km.
(c) How does your estimate for Δv compare with a scaling estimate of the eddy velocity U_{eddy} from the background velocity shear,

$$U_{eddy} \sim \frac{H}{2}\left|\frac{\partial u}{\partial z}\right|,$$

where the vertical scale $H \sim 1$ km and the velocity shear is estimated from the data in Fig. 9.20b.

Answer
(a) in (i) $\Delta z = 0$, so $\Delta v^2 = 0$; in (ii) and (iv), lighter fluid overlies denser fluid along the exchange paths, so $\Delta v^2 < 0$; in (iii) denser fluid overlies lighter fluid along the exchange path, so $\Delta v^2 > 0$.

(b) For path (iii), fluid parcels chosen to have values $\sigma_A = 27.45$ at $z = -200$ m and $\sigma_B = 27.4$ at $z = -250$ m, which implies
$\Delta\sigma = \sigma_A - \sigma_B = 0.05$ kg m^{-3};
$\Delta z = z_A - z_B = (-200 \text{ m}) - (-250 \text{ m}) = 50$ m.
The estimate of the change in velocity squared from the energy conversion in (9.30) is given by

$$\Delta v^2 = \frac{g}{\rho}\Delta\sigma\Delta z = \frac{(9.81 \text{ m s}^{-2})}{(1027.45 \text{ kg m}^{-3})}(0.05 \text{ kg m}^{-3})(50 \text{ m}) = 0.024 \text{ m}^2\text{s}^{-2},$$

which implies $\Delta v = 0.15$ m s^{-1}.

(c) The velocity shear is typically,

$$\frac{\partial u}{\partial z} \sim \frac{0.3 \text{ m s}^{-1}}{1000} \text{ m},$$

implying that

$$U_{eddy} \sim \frac{H}{2}\left|\frac{\partial u}{\partial z}\right| \sim \left(\frac{1000 \text{ m}}{2}\right)\left(\frac{0.3 \text{ m s}^{-1}}{1000 \text{ m}}\right) = 0.15 \text{ m s}^{-1},$$

in accord with the rough estimate in (b), rather fortuitously the same given the approximations made.

Q9.2. Consider the direction of the poleward heat flux associated with a velocity and temperature perturbation given by $v'(x,t) = v_o \sin(kx - \omega t)$ and
either (i) $T_1'(x,t) = T_o \cos(kx - \omega t)$ or (ii) $T_2'(x,t) = T_o \sin(kx - \omega t)$,
where the wave number $k = 2\pi/\lambda$ and λ is the wavelength in the x direction.

(a) Sketch how the products $v'T_1'$ and $v'T_2'$ vary with x from 0 to λ for $t = 0$. Hence, speculate whether the eddy temperature flux, $v'T'$ averaged in x from 0 to λ, is positive, zero or negative.

(b) More formally, estimate the eddy temperature flux by integrating the expression,

$$\overline{v'T'} = \frac{1}{\lambda}\int_0^\lambda v'(x,t)T'(x,t)dx,$$

for cases (i) and (ii) at $t = 0$. Apply general trigonometric identities, $2\sin\alpha\cos\alpha = \sin(2\alpha)$ and $2\sin^2\alpha = 1 - \cos(2\alpha)$, and integral relations, $\int_a^b \sin(\alpha x)dx = -\frac{1}{\alpha}[\cos(\alpha x)]_a^b$ and $\int_a^b \cos(\alpha x)dx = \frac{1}{\alpha}[\sin(\alpha x)]_a^b$.

Answer
(a) in (i) v' and T' are out of phase, so expect either their product is small or vanishes, while in (ii) v' and T' are in phase, so that their product is positive.

(b) at $t = 0$,

$$\begin{aligned}
\overline{v'T_1'} &= \frac{v_oT_o}{\lambda}\int_0^\lambda \cos(kx)\sin(kx)dx = \frac{v_oT_o}{2\lambda}\int_0^\lambda \sin(2kx)dx, \\
&= \frac{v_oT_o}{2\lambda}\left[\frac{\cos(2kx)}{-2k}\right]_0^\lambda = \frac{v_oT_o}{8\pi}(1 - \cos(4\pi)) = 0,
\end{aligned}$$

as $k = 2\pi/\lambda$.

$$\begin{aligned}\overline{v'T_2'} &= \frac{v_o T_o}{\lambda}\int_0^{\lambda}\sin(kx)\sin(kx)dx = \frac{v_o T_o}{2\lambda}\int_0^{\lambda}(1-\cos(2kx))dx,\\ &= \frac{v_o T_o}{2\lambda}\left[x - \frac{\sin(2kx)}{2k}\right]_0^{\lambda} = \frac{v_o T_o}{2\lambda}\left((\lambda - 0) + \left(\frac{\sin(4\pi)-\sin(0)}{-2k}\right)\right) = \frac{v_o T_o}{2}.\end{aligned}$$

Q9.3. Consider the movement of a particle in an eastward jet $u(y)$ when there is a propagating meander giving a northward flow, $v(x,t)$ (like in Fig. 9.19).

Assume that the position of the particle is given by x_p, y_p, such that its northward position is given by

$$y_p(\mathcal{T}) = \int_0^{\mathcal{T}} v_o\cos(k(x_p - ct))dt, \tag{9.31}$$

where $v(x,t) = v_o\cos(k(x_p - ct))$ and v_o is the amplitude of the northward velocity in the meander, $k = 2\pi/\lambda$ is the wave number, λ is the wavelength and c is the wave speed of the meander.

(a) Speculate on where the meridional displacement is likely to be greatest across the jet.

(b) If the eastward position of the particle is approximated by $x_p = ut$ (where for simplicity u is assumed constant for the particle), show that northward position of the particle from (9.31) is given by

$$y_p(\mathcal{T}) = \frac{v_o\sin(2\pi(u-c)/c)}{k(u-c)}, \tag{9.32}$$

note that the period $\mathcal{T} = 2\pi/(kc)$, since $c = \lambda/\mathcal{T}$.
(c) Roughly estimate the maximum value of $y_p(\mathcal{T})$ from (9.32) for three cases:
(i) when the wave speed exceeds the eastward jet, $u << c$;
(ii) when the wave speed is much less than the eastward jet, $u >> c$;
and (iii) when the wave speed is only slightly less than the eastward jet, $u = 5c/4$.
Compare your answers with your speculation in (a).

Answer
(a) Expect little meridional displacement in the core of the jet, but enhanced meridional displacement on the flanks of the jet where the zonal velocity is comparable to the wave speed, such that the particle moves downstream at a comparable speed as the propagating meander.

(b) Integrate the definition of the meridional velocity (9.31) and substitute $x_p = ut$,

$$\begin{aligned}y_p(\mathcal{T}) &= \int_0^{\mathcal{T}} v_o\cos(k(x_p - ct))dt = \int_0^{\mathcal{T}} v_o\cos(k(u-c)t)dt,\\ &= \left[\frac{v_o\sin(k(u-c)t)}{k(u-c)}\right]_0^{\mathcal{T}} = \frac{v_o\sin(k(u-c)\mathcal{T})}{k(u-c)},\end{aligned}$$

which applies the general integral relation,
$\int_0^{\mathcal{T}}\cos(\alpha t)dt = (1/\alpha)[\sin(\alpha t)]_0^{\mathcal{T}} = (1/\alpha)\sin(\alpha\mathcal{T})$, where α in this case is $k(u-c)$.

(c) $y_p(\mathcal{T})$ from (9.32) becomes

$$\frac{v_o \sin(2\pi(u-c)/c)}{k(u-c)},$$

which has the following limits:
(i) for $u << c$, becomes

$$\frac{v_o \sin(2\pi(-c)/c)}{k(-c)} = 0;$$

(ii) for $u >> c$, becomes

$$\frac{v_o \sin(2\pi u/c)}{ku} < \frac{v_o}{ku},$$

and (iii) for $u = 5c/4$ becomes

$$\frac{v_o \sin(2\pi(5/4-1))}{k(5/4-1)c} = \frac{4v_o}{kc} = \frac{5v_o}{ku}.$$

Thus, the greatest meridional displacement on the flank of the jet where jet speed is comparable to the wave speed.

Q9.4. Consider how the direction of eddy tracer fluxes are controlled starting with the tracer equation for a generic tracer, c,

$$\frac{\partial c}{\partial t} + \mathbf{u}\cdot\nabla c = F - D, \tag{9.33}$$

where $\mathbf{u}$ is the advective velocity along an isopycnal and F and D represent a source and sink of the tracer respectively.
(a) Separate each of the variables into time mean and time-varying components, such as $c = \overline{c} + c'$, and then apply a time average of (9.33) (where by definition $\overline{c'} \equiv 0$) to obtain an equation for the time-mean tracer, $\overline{c}$,

$$\frac{\partial \overline{c}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\overline{c} + \overline{\mathbf{u}'\cdot\nabla c'} = \overline{F} - \overline{D}. \tag{9.34}$$

Then (i) obtain a similar equation for the temporal variation in the tracer from (9.33) - (9.34);
(ii) Multiply your equation in (i) by c' and apply a time average to obtain the equation for tracer variance, $\overline{c'^2}/2$,

$$\frac{\partial}{\partial t}\frac{\overline{c'^2}}{2} + \overline{\mathbf{u}}\cdot\nabla\frac{\overline{c'^2}}{2} + \overline{\mathbf{u}'\cdot\nabla\frac{c'^2}{2}} + \overline{\mathbf{u}'c'}\cdot\nabla\overline{c} = \overline{F'c'} - \overline{D'c'}, \tag{9.35}$$

which can be written more concisely by combining the first three terms as a time integral following the flow, $D/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla$.
(iii) Re-arrange (9.35) so that the scalar product of the eddy tracer flux, $\overline{\mathbf{u}'c'}$, and the background tracer gradient, $\nabla\overline{c}$, is the only term on the left hand side to obtain

$$\overline{\mathbf{u}'c'}\cdot\nabla\overline{c} = -\overline{\frac{D}{Dt}\frac{c'^2}{2}} + \overline{F'c'} - \overline{D'c'}. \tag{9.36}$$

(b) Hence, using (9.36), identify when the eddy tracer flux, $\overline{\mathbf{u}'c'}$, is directed down the background tracer gradient? Assume that the forcing of tracer perturbations, $\overline{F'c'}$, is relatively small compared to the other terms. Conversely, identify when the eddy tracer flux is directed up the background tracer gradient?

Answer
(a) Apply the following steps:
1. Start with the tracer equation (9.33), split each variable into a time mean and time deviation, such that $c = \overline{c} + c'$, $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, $F = \overline{F} + F'$, $D = \overline{D} + D'$, to obtain

$$\frac{\partial \overline{c}}{\partial t} + \frac{\partial c'}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{c} + \overline{\mathbf{u}} \cdot \nabla c' + \mathbf{u}' \cdot \nabla \overline{c} + \mathbf{u}' \cdot \nabla c' = \overline{F} + F' - \overline{D} - D',$$

then apply a time mean, so that

$$\overline{\frac{\partial \overline{c}}{\partial t}} + \overline{\frac{\partial c'}{\partial t}} + \overline{\overline{\mathbf{u}} \cdot \nabla \overline{c}} + \overline{\overline{\mathbf{u}} \cdot \nabla c'} + \overline{\mathbf{u}' \cdot \nabla \overline{c}} + \overline{\mathbf{u}' \cdot \nabla c'} = \overline{\overline{F}} + \overline{F'} - \overline{\overline{D}} - \overline{D'},$$

and recall the definition of a time average, $\overline{\overline{c}} \equiv \overline{c}$ and $\overline{c'} \equiv 0$, so that the time-mean tracer equation is obtained,

$$\frac{\partial \overline{c}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{c} + \overline{\mathbf{u}' \cdot \nabla c'} = \overline{F} - \overline{D}.$$

2. Subtract the time-mean tracer equation (9.34) from the instantaneous tracer equation (with variables separated into a time mean and time deviation) to obtain

$$\left(\frac{\partial \overline{c}}{\partial t} + \frac{\partial c'}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{c} + \overline{\mathbf{u}} \cdot \nabla c' + \mathbf{u}' \cdot \nabla \overline{c} + \mathbf{u}' \cdot \nabla c' \right) - \left(\frac{\partial \overline{c}}{\partial t} + \overline{\mathbf{u}} \cdot \nabla \overline{c} + \overline{\mathbf{u}' \cdot \nabla c'} \right)$$
$$= \left(\overline{F} + F' - \overline{D} - D' \right) - \left(\overline{F} - \overline{D} \right),$$

to obtain the equation for the temporal variation in tracer

$$\frac{\partial c'}{\partial t} + \overline{\mathbf{u}} \cdot \nabla c' + \mathbf{u}' \cdot \nabla \overline{c} + \mathbf{u}' \cdot \nabla c' - \overline{\mathbf{u}' \cdot \nabla c'} = F' - -D'.$$

3. Multiply the equation for the temporal variation in tracer by c' (using $c' \frac{\partial c'}{\partial t} \equiv \frac{\partial}{\partial t} \frac{c'^2}{2}$)

$$\frac{\partial}{\partial t} \frac{c'^2}{2} + \overline{\mathbf{u}} \cdot \nabla \frac{c'^2}{2} + c' \mathbf{u}' \cdot \nabla \overline{c} + \mathbf{u}' \cdot \nabla \frac{c'^2}{2} - c' \overline{\mathbf{u}' \cdot \nabla c'} = F'c' - -D'c',$$

and apply a time average to obtain the tracer variance equation,

$$\frac{\partial}{\partial t} \overline{\frac{c'^2}{2}} + \overline{\mathbf{u}} \cdot \nabla \overline{\frac{c'^2}{2}} + \overline{\mathbf{u}' \cdot \nabla \frac{c'^2}{2}} + \overline{\mathbf{u}'c'} \cdot \nabla \overline{c} = \overline{F'c'} - \overline{D'c'}.$$

4. This tracer variance equation can be written more concisely using the material derivative, $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$, either involving a time average following the time-mean flow,

$$\overline{\frac{D}{Dt}} \overline{\frac{c'^2}{2}} + \overline{\mathbf{u}' \cdot \nabla \frac{c'^2}{2}} + \overline{\mathbf{u}'c'} \cdot \nabla \overline{c} = \overline{F'c'} - \overline{D'c'},$$

where $\overline{\frac{D}{Dt}} = \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \nabla$, or as a time average following the flow,

$$\overline{\frac{D}{Dt} \frac{c'^2}{2}} + \overline{\mathbf{u}'c'} \cdot \nabla \overline{c} = \overline{F'c'} - \overline{D'c'},$$

which can be re-arranged to give

$$\overline{\mathbf{u}'c'} \cdot \nabla \overline{c} = -\overline{\frac{D}{Dt}\frac{c'^2}{2}} + \overline{F'c'} - \overline{D'c'}. \tag{9.36}$$

(b) The direction of the eddy tracer flux, $\overline{\mathbf{u}'c'}$, relative to the gradient in the time-mean tracer is given by the scalar product $\overline{\mathbf{u}'c'} \cdot \nabla \overline{c}$, the eddy flux is directed down the gradient of the time-mean tracer when the scalar product is negative and directed up gradient when it is positive.

Using the simplified version of (9.36) with forcing of tracer perturbations, $\overline{F'c'}$, neglected,

$$\overline{\mathbf{u}'c'} \cdot \nabla \overline{c} \simeq -\overline{\frac{D}{Dt}\frac{c'^2}{2}} - \overline{D'c'},$$

the eddy flux is directed down gradient, $\overline{\mathbf{u}'c'} \cdot \nabla \overline{c} < 0$, when either there is (i) strong eddy dissipation of tracer perturbations, $\overline{D'c'} >> 0$, such as for surface temperature dampened by air-sea heat fluxes or surface nitrate dampened by the biology, or (ii) a Lagrangian increase in tracer variance, $\overline{\frac{D}{Dt}\frac{c'^2}{2}} >> 0$, following the flow.

Conversely, the eddy flux is directed up gradient, $\overline{\mathbf{u}'c'} \cdot \nabla \overline{c} > 0$, when there is a Lagrangian decrease in tracer variance, $\overline{\frac{D}{Dt}\frac{c'^2}{2}} << 0$. Hence, there can be reversing directions in the eddy tracer flux, according to whether the tracer variance is growing or decaying; see model illustrations in Wilson and Williams (2004).

Chapter 10.

Q10.1. Ekman upwelling velocity

The Ekman upwelling velocity at the base of the Ekman layer is related to the wind stress by

$$w_{ek} = \frac{1}{\rho_0}\left(\frac{\partial}{\partial x}\frac{\tau_y^s}{f} - \frac{\partial}{\partial y}\frac{\tau_x^s}{f}\right). \tag{10.19}$$

(a) For the following cases, identify whether the wind forcing denotes a subtropical gyre with $w_{ek} < 0$ or a subpolar gyre with $w_{ek} > 0$:
(i) the eastward wind stress decreases northward in the northern hemisphere; (ii) the eastward wind stress increases northward in the northern hemisphere; (iii) the eastward wind stress decreases northward in the southern hemisphere; and (iii) the eastward wind stress increases northward in the southern hemisphere. In each case, ignore any changes in the northward wind stress.

(b) Derive the relationship for Ekman upwelling (10.19) based upon continuity of volume,

$$\frac{\partial u_{ek}}{\partial x} + \frac{\partial v_{ek}}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

and the definition of the horizontal Ekman velocities,

$$\int_{-h_{ek}}^{0} u_{ek} dz = \frac{\tau_y^s}{\rho_0 f} \quad \text{and} \quad \int_{-h_{ek}}^{0} v_{ek} dz = -\frac{\tau_x^s}{\rho_0 f}. \tag{10.20}$$

Answer
(a) Assume

$$w_{ek} = \frac{1}{\rho_0}\left(\frac{\partial}{\partial x}\frac{\tau_y^s}{f} - \frac{\partial}{\partial y}\frac{\tau_x^s}{f}\right) \sim -\frac{1}{\rho_0}\frac{\partial}{\partial y}\frac{\tau_x^s}{f}.$$

(i) $\partial\tau_x/\partial y < 0,\ f > 0$, subpolar gyre;
(ii) $\partial\tau_x/\partial y > 0,\ f > 0$, subtropical gyre;
(iii) $\partial\tau_x/\partial y < 0,\ f < 0$, subtropical gyre;
(iv) $\partial\tau_x/\partial y > 0,\ f < 0$, subpolar gyre.

(b) Integrate the continuity equation for the Ekman flow with depth over the surface Ekman layer,

$$\int_{-h_{ek}}^{0}\left(\frac{\partial u_{ek}}{\partial x} + \frac{\partial v_{ek}}{\partial y} + \frac{\partial w}{\partial z}\right) dz = 0.$$

For the third term, apply

$$\int_{-h_{ek}}^{0} \frac{\partial w}{\partial z} dz = w(0) - w(-h_{ek}).$$

For the first and second terms, assume h_{ek} is constant, so that the horizontal differentials can be taken outside the depth integral,

$$\int_{-h_{ek}}^{0} \frac{\partial u_{ek}}{\partial x} dz = \frac{\partial}{\partial x}\int_{-h_{ek}}^{0} u_{ek} dz.$$

Then the depth integral of continuity becomes

$$\frac{\partial}{\partial x}\int_{-h_{ek}}^{0} u_{ek}dz + \frac{\partial}{\partial y}\int_{-h_{ek}}^{0} v_{ek}dz + w(0) - w(-h_{ek}) = 0.$$

Assume that w at the sea surface is zero and substitute definitions of the horizontal Ekman velocities (10.20) to obtain

$$w(-h_{ek}) = \frac{\partial}{\partial x}\frac{\tau_y^s}{\rho_0 f} - \frac{\partial}{\partial y}\frac{\tau_x^s}{\rho_0 f},$$

then take ρ_0 outside the differentials (as is constant) and define the Ekman upwelling velocity, w_{ek} as the same as $w(-h_{ek})$, to obtain

$$w_{ek} = \frac{1}{\rho_0}\left(\frac{\partial}{\partial x}\frac{\tau_y^s}{f} - \frac{\partial}{\partial y}\frac{\tau_x^s}{f}\right).$$

Q10.2. Tritium-helium age

Tritium decays to helium with a half live of 12.3 years. The helium outgasses to the atmosphere within the surface mixed layer, but accumulates within the ocean interior. Consequently, a ventilation age, defined by the time since fluid was in the surface mixed layer, can be estimated from the ratio of the tritium and helium concentrations,

$$\mathcal{T} = 17.96 \text{ years} \ln\left(1 + \frac{^3He}{^3H}\right). \tag{10.21}$$

(a) Evaluate the ventilation age assuming (i) a tritium concentration, ^3H, of 2.5 TU and a helium concentration, ^3He, of 1.5 TU; and (ii) a tritium concentration of 0.5 TU and a helium concentration of 1 TU (1 tritium unit TU equals 1 tritium atom in 10^{18} hydrogen atoms).
(b) Assuming that tritium decays as

$$\frac{d}{dt}\,^3H = -\alpha\,^3H, \tag{10.22}$$

show that the tritium concentration decays as $^3H(t) = \,^3H(0)\exp(-\alpha t)$ where $\alpha = 1/(17.96$ years$)$.
(c) Given that tritium decays to helium, show that helium accumulates in the interior as $^3He(t) = \,^3H(0)(1 - \exp(-\alpha t))$. Assume that there is initially no helium present in the mixed layer.
(d) Hence, derive the ventilation age given in (10.21).

Answer
(a) (i)

$$\mathcal{T} = (17.96 \text{ years}) \ln\left(1 + \frac{1.5 \text{ TU}}{2.5 \text{ TU}}\right) = 8.4 \text{ years};$$

(ii)

$$\mathcal{T} = (17.96 \text{ years}) \ln\left(1 + \frac{1 \text{ TU}}{0.5 \text{ TU}}\right) = 19.7 \text{ years}.$$

(b) Integrate (10.22) with respect to time,

$$\int_{^3H(0)}^{^3H(t)} \frac{d\,^3H}{^3H} = -\int_0^t \alpha dt,$$

giving

$$\ln {}^3H(t) - \ln {}^3H(0) = \ln\left(\frac{{}^3H(t)}{{}^3H(0)}\right) = -\alpha t,$$

which can be re-arranged as

$${}^3H(t) = {}^3H(0)\exp(-\alpha t).$$

(c) Total amount of helium and tritium is conserved in the interior, so that

$${}^3H(t) + {}^3He(t) = {}^3H(0) + {}^3He(0),$$

with ${}^3He(0) = 0$ due to degassing, so that

$${}^3He(t) = {}^3H(0) - {}^3H(t) = {}^3H(0)(1 - \exp(-\alpha t)).$$

(d) find the ratio of ${}^3He(t)/{}^3H(t)$,

$$\frac{{}^3He(t)}{{}^3H(t)} = \frac{{}^3H(0)(1 - \exp(-\alpha t))}{{}^3H(0)\exp(-\alpha t)} = \exp(\alpha t) - 1,$$

then re-arrange to obtain the age, $\mathcal{T}$, given by the time since the fluid was in the mixed layer,

$$t = \alpha^{-1}\ln\left(1 + \frac{{}^3He(t)}{{}^3H(t)}\right),$$

where $\alpha^{-1} = 17.96$ years.

Q10.3. Potential vorticity and the mixed layer

The large-scale potential vorticity of fluid subducted beneath the mixed layer (as in Fig 10.10) is defined by

$$Q = -\frac{f}{\rho S(t)}\frac{D\sigma_m}{Dt} = \frac{f}{\rho(w_b + \frac{Dh}{Dt})}\frac{D\sigma_m}{Dt}, \tag{10.23}$$

where $D\sigma_m/Dt$ represents the rate of change of mixed-layer density following the flow, $S(t)$ is the instantaneous subduction rate (10.6), w_b is the vertical velocity at the base of the mixed layer and Dh/Dt is the rate of change of mixed-layer thickness following the flow (Williams, 1991); this relationship ignores any contribution from relative vorticity.

(a) For there to be subduction, $S(t) > 0$, how does the mixed-layer density have to vary in time?

(b) What are the criteria for low values of Q, such as occurring in mode waters?

(c) Derive the relationship for potential vorticity (10.23). Consider a particle initially within the mixed layer, which is then subducted into the thermocline and conserves its density. Derive (i) how the density difference between the particle and the overlying mixed layer, $\Delta\sigma$, increases with the time t since subduction; (ii) how the vertical spacing between the particle and the overlying mixed layer, Δz, increases with t; and then (iii) combine with the definition for potential vorticity, $Q = -(f/\rho)\Delta\sigma/\Delta z$, on the basin scale.

Answer
(a) Assume that the ratio of the potential vorticity and planetary vorticity, $Q/f > 0$, is positive throughout the stratified ocean. Then (10.23) implies that subduction only occurs, $S(t) > 0$, when the mixed layer becomes lighter, $\frac{D\sigma_m}{Dt} < 0$.

(b) Q is low whenever there is only a small downstream lightening of the mixed layer, $D\sigma_m/Dt < 0$, or a large subduction rate, $S(t)$, from either a large downward velocity at the base of the mixed layer, $w_b << 0$, or a pronounced shoaling of the mixed layer, $Dh/Dt << 0$.

(c) (i) The density difference between a subducted particle, σ_p, and the overlying mixed layer, $\sigma_m(t)$, is given by

$$\Delta\sigma = \sigma_p - \sigma_m(t) = -t\frac{D\sigma_m}{Dt},$$

where t measures the elapsed time since subduction, such that $\sigma_p = \sigma_m(0)$ at the point of subduction, $\sigma_m(t) = \sigma_m(0) + t\frac{D\sigma_m}{Dt}$ and σ_p is assumed to be conserved after subduction;
(ii) The vertical spacing between the subducted particle, $z_p(t)$, and the overlying mixed layer, $z_m(t)$, is given by the product of the subduction rate, $S(t)$, and the elapsed time, t, since subduction,

$$\Delta z = z_p(t) - z_m(t) = -tS(t),$$

where $S(t)$ is the instantaneous subduction rate;
(iii) then combine together with the definition of large-scale potential vorticity,

$$Q = -\frac{f}{\rho}\frac{\Delta\sigma}{\Delta z},$$

to obtain a relationship between the potential vorticity for subducted fluid and the Lagrangian change in mixed-layer density and the subduction rate:

$$Q = -\frac{f}{\rho}\left(-t\frac{D\sigma_m}{Dt}\right)\left(\frac{1}{-tS(t)}\right) = -\frac{f}{\rho S(t)}\frac{D\sigma_m}{Dt}.$$

Q10.4. Transport of warm water and the separation of the boundary current

Consider a single layer of moving fluid, representing the thermocline, overlying a dense abyss within in a wind-forced, subtropical gyre (as in Fig. 10.19 and in Box 10.2.6). The overall northward transport of warm water in this model, T_{north}, is given by the sum of the Ekman transport, $-\frac{x_e\overline{\tau_x}}{\rho f}$, and geostrophic transport, $\frac{g'}{2f}\left(h_e^2 - h_w^2\right)$,

$$\mathsf{T}_{north} = -\frac{x_e\overline{\tau_x}}{\rho f} + \frac{g'}{2f}\left(h_e^2 - h_w^2\right). \tag{10.24}$$

(a) Calculate the Ekman transport and the geostrophic transport in Sv assuming $x_e = 5000$ km, $\tau_x = 0.1$ N m^{-2}, $\rho = 10^3$kg m^{-3}, $f = 10^{-4}$s^{-1}, $g' = 0.01$ m s^{-2}, $h_w = 200$ m and $h_e = 600$ m.
(b) For the same northward transport, T_{north}, how does the layer thickness on the western boundary, h_w vary if the wind stress increases? What value has the wind stress to reach if the layer thickness

on the western boundary vanishes, $h_w = 0$?
(c) When $h_w = 0$, the transport of warm water is instead given by

$$\mathsf{T}_{north} = -(x_e - X_p(y))\frac{\overline{\tau_x}}{f} + \frac{g'}{2f}h_e^2, \tag{10.25}$$

where $X_p(y)$ defines the western outcrop of the warm water. If the northward transport of warm water remains conserved and the wind stress increases further north, how does the western outcrop of the warm water vary in position?

Answer
(a) Ekman transport given by

$$-\frac{x_e\overline{\tau_x}}{\rho f} \sim -\frac{(5\times10^6\text{m})(0.1\ \text{N m}^{-2})}{(10^3\ \text{kg m}^{-3})(10^{-4}\text{s}^{-1})} = -5\times10^6\text{m}^3\text{s}^{-1} = -5\ \text{Sv};$$

and geostrophic transport given by

$$\frac{g'}{2f}\left(h_e^2 - h_w^2\right) \sim \frac{(10^{-2}\text{m s}^{-2})}{(2)(10^{-4}\text{s}^{-1})}\left((600\ \text{m}^2)^2 - (200\ \text{m}^2)^2\right) = 16\times10^6\text{m}^3\text{s}^{-1} = 16\ \text{Sv}.$$

(b) The northward transport of warm water is given by $\mathsf{T}_{north} = -5$ Sv $+ 16$ Sv $= 11$ Sv.

If the T_{north} is unchanged, then an increase in wind stress with a larger southward Ekman transport requires a larger northward geostrophic transport, which requires h_w to decrease.

When the thickness on the western boundary first vanishes,

$$\mathsf{T}_{north} = -\frac{x_e\overline{\tau_x}}{\rho f} + \frac{g'}{2f}h_e^2,$$

which implies that the stress is given by

$$\begin{aligned}\overline{\tau_x} &= \frac{\rho f}{x_e}\left(\frac{g'}{2f}h_e^2 - \mathsf{T}_{north}\right)\\ &\sim \frac{(10^3\text{kg m}^{-3})(10^{-4}\text{s}^{-1})}{(5\times10^6\text{m})}\left(18\times10^6\ \text{m}^3\text{s}^{-1} - 11\times10^6\ \text{m}^3\text{s}^{-1}\right) = 0.14\ \text{N m}^{-2}.\end{aligned}$$

increasing wind stress; $\tau_x = 0.14$ N m^{-2} for $h_w = 0$.

(c) If the outcrop of the warm water has detached from the western boundary, the transport of warm water being conserved requires

$$\mathsf{T}_{north} = -(x_e - X_p(y))\frac{\overline{\tau_x}}{f} + \frac{g'}{2f}h_e^2.$$

If the wind stress increases, then the width of the warm water, $x_e - X_p(y)$ then decreases, leading to the western boundary current moving eastward.

Chapter 11.

Q11.1. Global export production over the oligotrophic, subtropical gyres.

Export production has been estimated to reach 2 mol C m^{-2}y^{-1} close to Hawaii in the North Pacific subtropical gyre (Emerson et al., 1997). Assuming that this export is representative of other subtropical oceans and that their collective surface area makes up 60% of the global ocean (with the ocean making 71% of the surface area of the Earth), then estimate the following:
(a) the surface area of the subtropical gyres over the globe (assume the Earth's radius of 6400 km);
(b) the area-integrated export production over the subtropical gyres (note that 1 mole of carbon is equivalent to 12 g of carbon);
and (c) what proportion of the global export estimated as 10 PgC y^{-1} is provided by export from the oligotrophic subtropical gyres?

Answer
(a) Surface area of the Earth given by $4\pi r^2$ with $r = 6400$ km, then surface area of subtropical gyres over the globe
$\sim (0.71)(0.6)(4\pi(6.4 \times 10^6)^2\text{m}^2) = 2.2 \times 10^{14}\text{m}^2$.

(b) Mean export over subtropical gyres
$\sim (2 \text{ mol C m}^{-2}\text{y}^{-1})(12 \text{ g mol}^{-1})(2.2 \times 10^{14}\text{m}^2)$
$= 5.3 \times 10^{15}\text{gC m}^{-2}\text{y}^{-1} \sim 5 \text{ Pg C m}^{-2}\text{y}^{-1}$.

(c) 5.3 PgC y$^{-1}/10$ PgC y$^{-1} \sim 1/2$, so that typically half the global export of carbon occurs over the subtropical gyres.

Q11.2. Nutrient transport in boundary currents

Consider the nutrient transport in a western boundary current and their likely down-stream fate.
(a) If a boundary current has an along stream velocity of 1 m s^{-1} and a nitrate concentration of 10 mmol N m^{-3} over a width of 100 km and a vertical scale of 500 m, then what is the along stream transport from the area integral of the product of velocity and concentration? Give your answer in units of mol N y^{-1}.
(b) If this along stream nutrient transport swept along a boundary current eventually is transferred along density surfaces into the down stream, winter mixed layer in the subpolar gyre, then what is the effective nutrient flux per unit horizontal area passing into the winter mixed layer? Assume the surface horizontal area of the winter mixed layer in the subpolar gyre is given by 4000 km by 3000 km. Give your answer in units of mol N m^{-2}y^{-1} and compare your answer to the nutrient transfer estimates in section 11.1.

Answer
(a) Along-stream transport is (1 m s^{-1})(10 mmol N m^{-3})(10^5m)(5×10^2m)$\sim 1.6 \times 10^{13}$mol N y^{-1}.

(b) Flux into the winter mixed layer is given by (transport)/(horizontal area) $\sim (1.6 \times 10^{13}$mol N y^{-1}) $(12 \times 10^{12}\text{m}^2)^{-1} \sim 1$ mol N m^{-2}y^{-1}.

Q11.3. Scale analysis of the Nitrate budget for the mixed layer

Consider a simplified nitrate budget over a mixed layer of thickness, h, given by

$$h\frac{\partial \mathcal{N}_m}{\partial t} = \mathcal{F}_{\mathcal{N}} + \Lambda(\mathcal{N}_{th} - \mathcal{N}_m)\frac{\partial h}{\partial t} - K_v \left.\frac{\partial \mathcal{N}}{\partial z}\right|_{z=-h} - \mathbf{U}_e \cdot \nabla \mathcal{N}_m - h\lambda \mathcal{N}_m, \tag{11.26}$$

where $\mathcal{N}$ represents nitrate here, $\mathcal{F}_{\mathcal{N}}$ is the air-sea flux into the mixed layer, $\Lambda(\mathcal{N}_{th} - \mathcal{N}_m)\frac{\partial h}{\partial t}$ is the entrainment flux, $-K_v\left(\frac{\partial \mathcal{N}}{\partial z}\right)_{z=-h}$ is the vertical diffusive input at the base of the mixed layer, and $-\mathbf{U}_e \cdot \nabla \mathcal{N}_m$ is the Ekman advective supply , and $-h\lambda\mathcal{N}_m$ represents biological consumption.

Identify (a) the dominant balances for the winter versus the summer, and (b) identify the relative importance of advection over a year? Make the following assumptions:
(i) The air-sea flux from deposition, $\mathcal{F}_{\mathcal{N}}$, typically reaches 0.01 mol N m^{-2}year^{-1};
(ii) The entrainment flux, $\Lambda(\mathcal{N}_{th} - \mathcal{N}_m)\frac{\partial h}{\partial t}$, only occurs when the mixed layer thickens (represented by $\Lambda = 1$ when $\partial h/\partial t > 0$ and otherwise 0) and $\mathcal{N}_{th} - \mathcal{N}_m$ represents the difference in nutrient concentration between the thermocline and mixed layer. Estimate this entrainment flux by

$$-\Lambda h \frac{\partial \mathcal{N}}{\partial z}\frac{\partial h}{\partial t},$$

where $h \sim 50$ m in summer and $h \sim 200$ m in winter with the deepening occurring over 6 months, and the vertical nutrient profile is assumed to be $\partial\mathcal{N}/\partial z \sim -10^{-5}$ mol N m^{-4};
(iii) Assume that the diffusive supply of nutrients has a vertical diffusivity of $K_v = 2 \times 10^{-5}$ m^{-2}s^{-1};
(iv) The Ekman advective transfer is simply taken from the meridional transfer,

$$-\mathbf{U}_e \cdot \nabla \mathcal{N}_m \sim -V_e \frac{\partial \mathcal{N}_m}{\partial y},$$

where the Ekman volume flux per unit length $V_e \sim \pm 1$ m^2 s^{-1} (equivalent to an Ekman velocity of ± 1 cm s^{-1} over a thickness of 100 m) and $\partial \mathcal{N}_m/\partial y \sim 10^{-9}$ mol N m^{-4};
(v) The biological consumption of nitrogen is simply represented here by an exponential decay of the mixed–layer nitrate with a decay timescale of $(1/\lambda)$. Assume that there is the decay timescale is typically the order of 15 days in summer, but there is no consumption in winter due to light limitation, and the mixed-layer nutrient concentration is typically, $\mathcal{N}_m \sim 0.5 \times 10^{-3}$ mol N m^{-3}.

Answer

For the individual terms:
(i) the air-sea flux of nitrogen,

$$\mathcal{F}_{\mathcal{N}} \sim 0.01 \text{ mol N m}^{-2}\text{y}^{-1};$$

(ii) the entrainment flux only occurs in winter when the mixed layer thickens,

$$-\Lambda h \frac{\partial \mathcal{N}}{\partial z}\frac{\partial h}{\partial t} \sim (250 \text{ m})(-10^{-5} \text{ mol N m}^{-4})\left(\frac{250 \text{ m} - 50 \text{ m}}{0.5 \text{ year}}\right) = 0.6 \text{ mol N m}^{-2}\text{y}^{-1};$$

(iii) the vertical diffusive flux,

$$-K_v \left.\frac{\partial \mathcal{N}}{\partial z}\right|_{z=-h} \sim -(2\times 10^{-5}\text{m}^2\text{s}^{-1})(-10^{-5} \text{ mol N m}^{-4}) \sim 2\times 10^{-10}\text{mol N m}^{-2}\text{s}^{-1} \sim 0.01 \text{ mol N m}^{-2}\text{y}^{-1};$$

(iv) the advective flux from the Ekman transfer,

$$-\mathbf{U}_e\cdot\nabla\mathcal{N}_m \sim -V_e\frac{\partial \mathcal{N}_m}{\partial y} \sim -(\pm 1\ \text{m}^2\text{s}^{-1})(10^{-9}\text{mol N m}^{-4}) = \pm 10^{-9}\text{mol N m}^{-2}\text{s}^{-1} \sim \pm 0.03\ \text{mol N m}^{-2}\text{y}^{-1};$$

and (v) the biological consumption is assumed only to occur in the summer,

$$-h\lambda\mathcal{N}_m \sim -(50\,\text{m})\left(\frac{1}{15\times 24\times 60^2\text{s}}\right)(0.5\times 10^{-3}\text{mol N m}^{-3}) \sim 2\times 10^{-8}\text{mol N m}^{-2}\text{s}^{-1} \sim -0.6\ \text{mol N m}^{-2}\text{y}^{-1}.$$

Hence, the seasonal nutrient balances are :

$$\begin{array}{lcccccc} & h\dfrac{\partial \mathcal{N}_m}{\partial t} = \mathcal{F}_{\mathcal{N}} & + \Lambda h\dfrac{\partial \mathcal{N}}{\partial z}\dfrac{\partial h}{\partial t} & - K_v\dfrac{\partial \mathcal{N}}{\partial z}\Big|_{z=-h} & - V_e\dfrac{\partial \mathcal{N}_m}{\partial y} & - h\lambda\mathcal{N}_m, \\ \text{Winter :} & 0.01 & 0.6 & 0.01 & \pm\, 0.03 & 0, \\ \text{Summer :} & 0.01 & 0 & 0.01 & \pm\, 0.03 & -\,0.6, \end{array}$$

values in units of mol N $\text{m}^{-2}\text{y}^{-1}$.

(a) On a seasonal basis, the mixed-layer nitrate budget is dominated by entrainment during winter, and biological export during summer.

(b) Advection in each season is an order of magnitude smaller than the leading processes. However, integrating over the entire year, the biological export and entrainment terms largely oppose each other leaving advection with a more significant role in setting the annual nitrate distribution.

Chapter 12.

Q12.1 Heat transport by the ocean.

Consider the relative importance of the heat transport by the ocean from the horizontal gyres and the vertical overturning using

$$\rho C_p \Delta\theta\, \psi, \tag{12.15}$$

where $\Delta\theta$ is a potential temperature difference, ψ is a volume transport (m^{3}s^{-1}), $\rho \sim 10^3$kg m^{-3} and $C_p \sim 4 \times 10^3$J kg^{-1}K^{-1}. Estimate the heat transport using (12.15) with typical values for the North Atlantic: (a) the gyre circulation with a west-east temperature contrast, $\Delta\theta \sim 2$ K across the basin and the volume transport, $\psi \sim 30 \times 10^6$m^{3}s^{-1}. Check the units of your answer.
(b) the overturning circulation with a vertical temperature contrast, $\Delta\theta \sim 15$ K and a volume transport, $\psi \sim 15 \times 10^6$m^{3}s^{-1}.
(c) What fraction of the total ocean heat transport is carried by the overturning?

Answer
(a) Heat transport by the horizontal gyres,

$$\rho C_p \Delta\theta\, \psi = (10^3\text{kg m}^{-3})(4 \times 10^3\text{J kg}^{-1}\text{K}^{-1})(2\text{ K})(30 \times 10^6\text{m}^3\text{s}^{-1}) = 2.4 \times 10^{14}\text{W} = 0.24\text{ PW}.$$

Units W=J s^{-1}.

(b) Heat transport by the overturning circulation,

$$\rho C_p \Delta\theta\, \psi = (10^3\text{kg m}^{-3})(4 \times 10^3\text{J kg}^{-1}\text{K}^{-1})(15\text{ K})(15 \times 10^6\text{m}^3\text{s}^{-1}) = 9 \times 10^{14}\text{W} = 0.9\text{ PW}.$$

(c) The fraction of the ocean heat transport carried by the overturning is given by

$$\frac{0.9\text{ PW}}{(0.9\text{ PW} + 0.24\text{ PW})} = 0.79.$$

Thus, nearly 80% of the ocean heat transport is carried by the overturning. This comparison is more appropriate for the Atlantic than the Pacific.

Q12.2 Overturning and pressure contrasts.

Consider the meridional transport by the geostrophic flow within a basin, which is defined by a west-east integral of the velocity across the basin and a depth integral from the surface to a level of no motion, $z = -d_{ref}$,

$$\int_{-d_{ref}}^{z=0} \int_{x_w}^{x_e} v_g dx dz, \tag{12.16}$$

where x_w and x_e defines the western and eastern sides of the basin.
(a) By assuming geostrophic flow, $v_g = (1/(\rho_o f))\partial P/\partial x$, show that (12.16) can be expressed as a pressure contrast across the basin,

$$\frac{1}{\rho_o f} \int_{-d_{ref}}^{z=0} (P(x_e) - P(x_w)) dz. \tag{12.17}$$

(b) For a poleward transport above the level of no motion, how does the pressure on each boundary compare with each other?
(c) How might the height of the sea surface, η, then vary across the basin? Assume that greater pressure at depth is achieved by a thicker water column with a nearly uniform density.
(d) If the pressure distribution remains unchanged along the eastern boundary, then if the overturning strengthens with a greater poleward upper transport, then does sea level rise or fall on the western boundary? For an application of this balance to relate overturning and sea level along the North American eastern coast, see Bingham and Hughes (2009).

Answer
(a) Start with the definition of the northward geostrophic flow,

$$v_g = \frac{1}{\rho_o f}\frac{\partial P}{\partial x},$$

substitute into the definition for the overturning transport,

$$\int_{-d_{ref}}^{z=0}\int_{x_w}^{x_e} v_g dxdz = \int_{-d_{ref}}^{z=0}\int_{x_w}^{x_e} \frac{1}{\rho_o f}\frac{\partial P}{\partial x}dxdz,$$

then apply

$$\int_{x_w}^{x_e}\frac{\partial P}{\partial x}dx = P(x_e) - P(x_w),$$

to obtain

$$\int_{-d_{ref}}^{z=0}\int_{x_w}^{x_e} v_g dxdz = \frac{1}{\rho_o f}\int_{-d_{ref}}^{z=0}(P(x_e) - P(x_w))dz. \qquad (12.17)$$

(b) Poleward upper flow is associated with $\partial P/\partial x > 0$, so greater pressure on the eastern boundary, x_e, relative to reduced pressure on the western boundary, x_w.

(c) Expect greater pressure to be associated with a thicker water column. Hence poleward upper flow (from (b)) is associated a reduced pressure on the western boundary and a thinner water column, compared with the eastern boundary.

(d) Increased overturning with a poleward upper transport is associated with a fall in sea level on the western boundary.

Q12.3. Stommel-Arons model of the deep ocean.

Assume that the cold, deep water is slowly upwelling and balancing a downward diffusion of heat:

$$w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2}. \qquad (12.18)$$

(a) Use scale analysis to relate w (m s^{-1}) and the diffusivity of heat, κ (m^{2}s^{-1}), and the depth scale, H (m).
(b) Relate v, w and κ by assuming linear vorticity balance, $\beta v = f\partial w/\partial z$, where β is in units of m^{-1}s^{-1} and f in s^{-1}.

(c) estimate the magnitude of w, v and the poleward volume transport, vHL_x, for $\kappa \sim 10^{-4}\text{m}^2\text{s}^{-1}$, $H \sim 3$ km and $L_x \sim 5000$ km. Quote the volume transport in sverdrups ($10^6\text{m}^3\text{s}^{-1}$).
(d) Sketch a plan view of the horizontal flow in the deep waters over the North Pacific, which is implied by the Stommel and Arons model.
(e) Observations suggest that $\kappa \sim 10^{-5}\text{m}^2\text{s}^{-1}$ is very small in the interior of the ocean from tracer-release experiments, but is enhanced near topography. Given your relations between v and κ, speculate on how the Stommel and Arons solution changes if there is a larger κ running along the eastern boundary of the Pacific and a zero value elsewhere.
(f) What are the strengths and weaknesses of the Stommel-Arons model?

Answer
(a) The vertical heat balance,

$$w\frac{\partial T}{\partial z} = \kappa\frac{\partial^2 T}{\partial z^2},$$

implies from scaling that

$$W\frac{\Delta T}{H} \sim \kappa\frac{\Delta T}{H^2},$$

so that the vertical velocity is given by

$$W \sim \frac{\kappa}{H}.$$

(b) The linear vorticity balance,

$$\beta v = f\frac{\partial w}{\partial z},$$

implies from scaling that the horizontal poleward velocity

$$V \sim \frac{fW}{\beta H} \sim \frac{f\kappa}{\beta H^2}.$$

(c) the vertical velocity,

$$W \sim \frac{\kappa}{H} \sim \frac{10^{-4}\text{m}^2\text{s}^{-1}}{3\times 10^3\text{m}} \sim 1/3 \times 10^{-7}\text{m s}^{-1} \sim 1\text{ m y}^{-1};$$

the horizontal velocity in the interior,

$$V \sim \frac{fW}{\beta H} \sim \frac{(10^{-4}\text{s}^{-1})(1/3\times 10^{-7}\text{m s}^{-1})}{(10^{-11}\text{m}^{-1}\text{s}^{-1})(3\times 10^3\text{m})} \sim 10^{-4}\text{m s}^{-1},$$

and the poleward volume transport,

$$vHL_x \sim (10^{-4}\text{m s}^{-1})(3\times 10^3\text{m})(5\times 10^6\text{m}) \sim 1.7\times 10^6\text{m}^3\text{s}^{-1} = 1.7\text{ Sv}.$$

(d) Stommel-Arons picture always has a poleward interior flow (increasing with larger f) and vanishing at the equator. In the North Pacific, dense bottom water is not supplied at the northern boundary, but instead from the Southern Ocean via a predicted deep western boundary current.

(e) If mixing is enhanced on the eastern boundary, then the poleward interior flow will be concentrated there, rather than over the middle parts of the basin. Then expect zonal flows over the mid-basin.

(f) Strengths of Stommel-Arons model: first prediction of deep western boundary currents. Weaknesses of the Stommel-Arons model: no account of topographical steering , assumption of uniform mixing no variations in deep flows with depth, and flows can be masked by eddies.

Q12.4. Consider the work done by the wind on the ocean.

(a) Over subtropical gyres, the winds provide an Ekman pumping, pushing down density surfaces and deforming the thermocline, which increases the potential energy. Gill et al. (1974) argued that this work done by the wind provides an increase in potential energy per unit horizontal area of 10^{-3}W m^{-2} over the subtropical gyres, which is assumed to sustain the formation of ocean eddies. Estimate the work done by the wind averaged over the area of the subtropical gyres; assume subtropical gyres make up 60% of the global ocean and the ocean makes up 71% of the surface area of the Earth, take the Earth's radius as 6400 km.
(b) Over the Southern Ocean, the winds are strongly aligned with the Antarctic Circumpolar Current. Estimate the work done by the wind on the ocean per unit horizontal area from the product of the wind stress and geostrophic current in the ocean, $\tau \cdot \mathbf{u}_g$, assuming the magnitudes of the wind stress $\tau \sim 0.15$ N m^{-2} and surface geostrophic flow are $\mathbf{u}_g \sim 0.1$ m s^{-1}; check your units (remember J≡N m≡kg m^{2}s^{-2}).
(c) Estimate the work done by the wind averaged over the Southern Ocean assuming that the latitudinal extent of the Antarctic Circumpolar Current is 20^o and the current encircles the globe at a latitude of typically 50^oS.
(d) Estimate the work done by the wind over the globe from the sum of your answers (a) and (c), and the fraction of the global work done by the wind occurring over the Southern Ocean.

Answer
(a) Work done by the winds over the subtropical gyres,

$$(10^{-3}\text{W m}^{-2})(0.6 \times 0.71 \times 4\pi(6.4 \times 10^6\text{m})^2) = 0.22 \times 10^{12}\text{W} = 0.22 \text{ TW}.$$

(b) Work done per unit horizontal area by the winds over the Southern Ocean,

$$(0.15 \text{ N m}^{-2})(0.1 \text{ m s}^{-1}) = 15 \times 10^{-3}\text{W m}^{-2}.$$

(c) Work done by the winds over the Southern Ocean,

$$(15 \times 10^{-3}\text{W m}^{-2})(20 \times 110 \times 10^3\text{m})(2\pi \cos 50^o \times 6.4 \times 10^6\text{m}) = 0.85 \times 10^{12}\text{W} = 0.85 \text{ TW}.$$

(d) Total work done 0.22 TW+0.85 TW=1.07 TW, so that the fraction of work done over the Southern Ocean compared with the total is 0.85 TW/1.07 TW ~ 0.79. So typically $\sim 80\%$ of the work done by winds occurs over the Southern Ocean; compare with altimetric diagnostics by Wunsch (1998).

Chapter 13.

Q13.1 Radiative heating and carbon emissions

The extra radiative heating from carbon dioxide, $\Delta\mathcal{H}$ (in W m^{-2}), increases logarithmically with the mixing ratio of atmospheric carbon dioxide, X_{CO_2}, where

$$\Delta\mathcal{H} = \alpha_r \ln\left(\frac{X_{CO_2}(t)}{X_{CO_2}(t_o)}\right), \tag{13.39}$$

with X_{CO_2} increasing from times t_o to t, and $\alpha_r = 5.4$ W m^{-2} depends on the chemical composition of the atmosphere.

(a) Why does the radiative heating vary logarithmically with increasing atmospheric carbon dioxide, rather than increase linearly? Provide a mechanistic explanation.

(b) On a millennial timescale, atmospheric carbon dioxide increases exponentially with carbon emissions, ΔI, (this long-term equilibrium state is depicted in Fig. 13.5),

$$X_{CO_2}(t) = X_{CO_2}(t_o)\exp\left(\frac{\Delta I}{I_B}\right) \tag{13.40}$$

with the buffered carbon inventory for the atmosphere and ocean, I_B, being typically 3500 PgC.

Estimate the atmospheric carbon dioxide $X_{CO_2}(t)$ at equilibrium for carbon emissions of $\Delta I = 1000, 2000, 3000$ and 4000 PgC. Plot $X_{CO_2}(t)$ versus ΔI. Assume X_{CO_2} is a pre-industrial value of 280 ppmv.

(c) Why does the atmospheric carbon dioxide vary exponentially with carbon emissions in (13.40)? Provide a mechanistic explanation.

(d) For a long-term equilibrium, show how extra radiative heating from carbon dioxide, $\Delta\mathcal{H}$, varies linearly with carbon emissions, ΔI, such that

$$\Delta\mathcal{H} = \frac{\alpha_r}{I_B}\Delta I. \tag{13.41}$$

Check the units of your expression; see Goodwin et al. (2009) for discussion of this relationship.

(e) Estimate how the extra radiative heating from carbon dioxide, $\Delta\mathcal{H}$, varies with carbon emissions of $\Delta I = 1000, 2000, 3000$ and 4000 PgC in (13.41). Plot $\Delta\mathcal{H}$ versus ΔI.

(f) The conventional carbon reserves are estimated to reach up to typically 5000 PgC (Rogner, 1997). What then are the implications for the radiative heating of the planet if all the conventional carbon reserves are utilised (without carbon capture from the atmosphere)? What further processes might eventually lead to a reduction of the atmospheric carbon dioxide?

Answer.
(a) The effect on CO_2 on the absorption and emission of longwave radiation gradually saturates.

(b) For an emission of $\Delta I = 1000$ PgC,

$$X_{CO_2} = (280 \text{ ppmv}) \exp\left(\frac{1000 \text{ PgC}}{3500 \text{ PgC}}\right) = 373 \text{ ppmv};$$

increasing for higher emissions to $496, 660$ and 878 ppmv.

(c) Increasing atmospheric carbon dioxide affects the partitioning of carbon in the ocean, increasing the amount of dissolved CO_2 at the expense of carbonate ions, which inhibits the further ocean uptake of carbon dioxide. Hence, an increasing fraction of the carbon emitted remains in the atmosphere.

(d) Substitute (13.40) into (13.39), so that

$$\Delta\mathcal{H} = \alpha_r \ln\left(\frac{X_{CO_2}(t)}{X_{CO_2}(t_o)}\right) = \alpha_r \ln\left(\frac{X_{CO_2}(t_o)\exp\left(\frac{\Delta I}{I_B}\right)}{X_{CO_2}(t_o)}\right) = \alpha_r \frac{\Delta I}{I_B}.$$

$\Delta\mathcal{H}$ in units of W m^{-2} and $\alpha_r \Delta I/I_B$ in units of

$$(\text{W m}^{-2})\frac{(\text{PgC})}{(\text{PgC})} = \text{W m}^{-2},$$

so are identical on either side of (13.41).

(e) For an emission of $\Delta I = 1000$ PgC, the extra heating

$$\Delta\mathcal{H} = (5.4 \text{ W m}^{-2})\left(\frac{1000 \text{ PgC}}{3500 \text{ PgC}}\right) = 1.5 \text{ W m}^{-2};$$

increasing for higher emissions to $3.1, 4.6$ and 6.2 W m^{-2}.

(f) Increase in heat flux of 7 to 8 W m^{-2} lasting for millennia, until sediment interactions and weathering leads to a reduction in atmospheric carbon dioxide.

Q13.2 Cycling of calcium ions

The average calcium ion concentration in rivers due to weathering is about 370 μmol kg^{-1} (Langmuir, 1997) and the riverine flux of freshwaters to the ocean about 1 Sv (1 Sv = 10^6 m^{3}s^{-1}).
(a) How many moles of calcium ions are delivered to the ocean each year?
(b) The average concentration of calcium ions in the global ocean is about 10 mmol kg^{-1}. What is the total inventory of calcium ions in the ocean? (Volume of ocean is 1.4$\times 10^{18}$m^3).
(c) What is the average life time of calcium ions in the ocean?
(d) If the burial rate is perturbed and decreases by 10%, but the delivery through weathering remains steady, how long would it take for ocean alkalinity to increase by 10μmol kg^{-1}.
(e) What would be the effect on atmospheric X_{CO_2}?

Answer.
(a) Annual river input of Ca^{2+} into the ocean,

$$(370 \times 10^{-6}\text{mol Ca}^{2+}\text{ kg}^{-1})(10^3\text{kg m}^{-3})(10^6\text{m}^3\text{s}^{-1}) = 3.7 \times 10^5\text{mol Ca}^{2+}\text{s}^{-1},$$

$$= 1.2 \times 10^{13}\text{mol Ca}^{2+}\ \text{y}^{-1} \sim 10^{13}\text{mol Ca}^{2+}\ \text{y}^{-1}.$$

(b) Inventory of Ca^{2+} in the ocean=concentration of $Ca^{2+}\times$mass of the ocean,

$$[\text{Ca}^{2+}]\rho_0 V_0 \sim (10 \times 10^{-3}\ \text{mol Ca}^{2+}\text{kg}^{-1})(10^3\text{kg m}^{-3})(1.4 \times 10^{18}\text{m}^3) \sim 1.4 \times 10^{19}\text{mol Ca}^{2+}.$$

(c) Lifetime of calcium ions given by,

$$\frac{\text{inventory of Ca}^{2+}}{\text{source of Ca}^{2+}} \sim \frac{(1.4 \times 10^{19}\text{mol Ca}^{2+})}{(1.2 \times 10^{13}\text{mol Ca}^{2+}\ \text{y}^{-1})} \sim 1.1 \text{ million years} \sim 1.1 \text{ million years.}$$

(d) 10% reduction in burial rate leads to a net rate of increase in ocean calcium ion concentration of

$$0.1 \times 10^{13}\text{mol Ca}^{2+}\ \text{y}^{-1}.$$

This net increase in the calcium ion supply leads to a corresponding rate of increase in alkalinity, which is a factor of 2 larger due to calcium having two positive charges,

$$0.2 \times 10^{13}\text{mol}A_T\ \text{y}^{-1}.$$

The timescale for the alkalinity to increase by 10μmol kg^{-1} is given by

$$\frac{\text{global change in}A_T}{\text{net rate of increase in}A_T} \sim \frac{(10 \times 10^{-6}\text{mol } A_T\ \text{kg}^{-1})(10^3\text{kg m}^{-3})(1.4 \times 10^{18}\text{m}^3)}{(0.2 \times 10^{13}\text{mol } A_T\ \text{y}^{-1})} \sim 7000 \text{ years.}$$

(e) An alkalinity increase leads to the ocean being able to hold more DIC, so reducing the amount of carbon held in the atmosphere. From Figure 13.8, we expect about 5 ppmv decrease in X_{CO_2} for a 10μmol kg^{-1} increase in alkalinity derived from calcium carbonate weathering.

Q13.3 Soft-tissue pump

The contribution to deep ocean DIC due to the soft tissue pump, C^{soft}, upwells into the mixed layer. Here, upon entering the mixed layer, this component is redefined as a contribution to the disequilibrium DIC, ΔC. This upwelling waters tend to drive the surface waters towards supersaturation and outgassing. Assume that the temperature, salinity and alkalinity of the recently upwelled waters are unchanged, and a fraction ϵ_C of the ΔC anomaly is subducted back into the thermocline or deep ocean. The remaining fraction, $1 - \epsilon_C$, is either lost to the atmosphere or consumed by phytoplankton; here $0 \leq \epsilon_C \leq 1$. Thus there is a contribution $\epsilon_C C^{soft}$ to the disequilibrium of the subducted waters. If C^{soft} changes, then assume that a linear change to the disequilibrium occurs.
(a) What is the meaning of $\epsilon_C = 0$ and $\epsilon_C = 1$? (b) Starting from (13.28) and substituting $\delta\Delta C = \epsilon_C \delta C^{soft}$, derive an expression for the relationship of atmospheric X_{CO_2} and $\mathcal{P}^*$ which accounts for this process. (c) How does the sensitivity to $\mathcal{P}^*$ change as ϵ_C increases from 0 to 1?

Answer.
(a) $\epsilon_C = 0$, all soft tissue pump contribution to upwelled DIC is either outgassed or biologically utilised before re-subduction. $\epsilon_C = 1$, none of soft tissue pump contribution is outgassed or biologically utilised.

(b) Follow these steps:
1. Start with the carbon inventory (13.5),

$$\delta I_{oa} = M\delta X_{CO_2} + V_o\rho_o\left(\delta\overline{C^{sat}} + \delta\overline{C^{reg}} + \delta\overline{\Delta C}\right).$$

Assume that $\delta I_{oa} = 0$ and ignore any changes in calcium carbonate cycling to obtain

$$0 = M\delta X_{CO_2} + V_o\rho_o\left(\delta\overline{C^{sat}} + \delta\overline{C^{soft}} + \delta\overline{\Delta C}\right).$$

2. Then assume that $\delta\Delta C = \epsilon_C\delta C^{soft}$ to obtain

$$0 = M\delta X_{CO_2} + V_o\rho_o\left(\delta\overline{C^{sat}} + (1+\epsilon_C)\delta\overline{C^{soft}}\right).$$

3. Assuming that there are no changes in temperature and alkalinity, then re-express $\delta\overline{C^{sat}}$ using (13.6) and (13.8) to obtain $\delta\overline{C^{sat}} = (\overline{C^{sat}}/(BX_{CO_2}))\delta X_{CO_2}$, which then gives

$$0 = M\delta X_{CO_2} + \frac{V_o\rho_o\overline{C^{sat}}}{BX_{CO_2}}\delta X_{CO_2} + V_o\rho_o(1+\epsilon_C)\delta\overline{C^{soft}},$$

or equivalently,

$$0 = \left(MX_{CO_2} + \frac{V_o\rho_o\overline{C^{sat}}}{B}\right)\frac{\delta X_{CO_2}}{X_{CO_2}} + V_o\rho_o(1+\epsilon_C)\delta\overline{C^{soft}},$$

which can be written using the definition of I_B in (13.12) as

$$0 = I_B\frac{\delta X_{CO_2}}{X_{CO_2}} + V_o\rho_o(1+\epsilon_C)\delta\overline{C^{soft}}.$$

4. Now express $\delta\overline{C^{soft}}$ in terms of the change in $\delta\overline{\mathcal{P}^*}$ using

$$\delta\overline{C^{reg}} = R_{CP}\ \overline{PO_4^{3-}}\ \delta\overline{\mathcal{P}^*},$$

to obtain

$$0 = I_B\frac{\delta X_{CO_2}}{X_{CO_2}} + V_o\rho_o(1+\epsilon_C)R_{CP}\ \overline{PO_4^{3-}}\ \delta\overline{\mathcal{P}^*}.$$

5. Now re-arranging as

$$\frac{\delta X_{CO_2}}{X_{CO_2}} = -\frac{V_o\rho_o(1+\epsilon_C)R_{CP}\ \overline{PO_4^{3-}}}{I_B}\delta\overline{\mathcal{P}^*},$$

then integrating gives

$$\ln(X_{CO_2}(\overline{\mathcal{P}^*})/X_{CO_2}(\overline{\mathcal{P}^*}_{ref})) = -\frac{V_o\rho_o(1+\epsilon_C)R_{CP}\ \overline{PO_4^{3-}}}{I_B}\Delta\overline{\mathcal{P}^*},$$

which gives

$$X_{CO_2}(\overline{\mathcal{P}^*}) = X_{CO_2}(\overline{\mathcal{P}^*}_{ref})\exp\left(-V_o\rho_o\ \ (1+\epsilon_C)R_{CP}\overline{PO_4^{3-}}\Delta\overline{\mathcal{P}^*}\ /\ I_B\right).$$

(c) From (b), when $\epsilon_C = 0$,

$$X_{CO_2}(\overline{\mathcal{P}^*}) = X_{CO_2}(\overline{\mathcal{P}^*}_{ref}) \exp\left(-V_o\rho_o \;\; R_{CP}\overline{PO_4^{3-}}\Delta\overline{\mathcal{P}^*} \;/\; I_B\right),$$

and when $\epsilon_C = 1$, then

$$X_{CO_2}(\overline{\mathcal{P}^*}) = X_{CO_2}(\overline{\mathcal{P}^*}_{ref}) \exp\left(-V_o\rho_o \;\; 2R_{CP}\overline{PO_4^{3-}}\Delta\overline{\mathcal{P}^*} \;/\; I_B\right).$$

Hence, exponential decay rate of X_{CO_2} with $\mathcal{P}^*$ increases by a factor of 2 as ϵ_C increases from 0 to 1.

Q13.4 What are the typical rates of density forcing and transformation? The rate at which water masses are transformed from one density class to another can be estimated given knowledge of the surface heat and freshwater fluxes.

(a) Estimate the effect of a surface cooling on the density gain over the North Atlantic. Assume the surface heat flux, $\mathcal{H} \sim -40$ W m^{-2}, then estimate the effective surface density flux, $\mathcal{D}_{in} = -\alpha_T\mathcal{H}/C_p$ for $\alpha_T = 2 \times 10^{-4}$ K^{-1} and $C_p = 4 \times 10^3$ J kg^{-1} K^{-1}. Check your units.

(b) Roughly estimate the implied transformation rate, G, using a differenced form of (13.35) (and ignoring the contribution of any diffusive fluxes, D_{diff}),

$$G \sim \frac{1}{\Delta\rho}\mathcal{D}_{in}A_{outcrop},$$

where the density interval, $\Delta\rho = 0.2$ kg m^{-3}, the effective surface density flux $\mathcal{D}_{in}$ is taken from (a), and is assumed to uniformly apply over a density outcrop with a surface area, $A_{outcrop}$, of 4000 km$\times 500$ km. Compare your estimate to Fig. 13.14a.

(c) For a steady state, ultimately this transformation of light to dense water needs to be offset by an opposing transformation of dense to light water, which can either be achieved by surface forcing for the density outcrop over another part of the globe or by a diffusive flux across the isopycnal.

Now assume that diapycnal mixing is important: the surface density flux into dense waters is assumed to be balanced by the diffusive flux across the isopycnal (based upon the two terms on the right-hand side of (13.35) balancing when integrated from the densest ρ at the surface to the ρ at the outcrop), such that

$$\mathcal{D}_{in}A_{surface} \sim -\kappa\frac{\partial\rho}{\partial z}A_{therm},$$

where $A_{surface}$ is the horizontal extent of waters denser than ρ at the sea surface, the diffusive flux into denser water is written as $\kappa\frac{\partial\rho}{\partial z}$, integrated over the horizontal area of the isopycnal in the thermocline and deep water, A_{therm}; see Walin (1982), Speer (1997) and Nurser et al. (1999) for a careful evaluation. Rearrange to solve for the diapycnal diffusivity, κ. Assume that the vertical gradient in potential density is $\partial\rho/\partial z \sim (1026.4 - 1028)$kg m$^{-3}/2\times 10^3$m, $A_{surface}$ is 4000 km$\times 1000$ km and the horizontal area of the isopycnal in the thermocline, A_{therm} is 4000 km$\times 20000$ km.

(d) In situ measurements of diapycnal mixing in the thermocline suggest that κ is often only 2×10^{-5}m^{2}s^{-1}. Compare with your answer in (c) and speculate on the implications?

Answer.
(a) Effective surface density flux,

$$\mathcal{D}_{in} \sim -(2 \times 10^{-4}\ \mathsf{K}^{-1})\frac{(-40\ \mathsf{W\ m}^{-2})}{(4 \times 10^3\ \mathsf{J\ kg}^{-1}\ \mathsf{K}^{-1})} \sim 2 \times 10^{-6}\ \mathsf{kg\ s}^{-1}\mathsf{m}^{-2}.$$

(b) Transformation,

$$G \sim \frac{1}{\Delta\rho}\mathcal{D}_{in}\Delta A \sim \frac{1}{(0.2\ \mathsf{kg\ m}^{-3})}(2 \times 10^{-6}\ \mathsf{kg\ s}^{-1}\mathsf{m}^{-2})(2 \times 10^{12}\mathsf{m}^2) \sim 20 \times 10^6\ \mathsf{m}^3\mathsf{s}^{-1}.$$

(c) Diapycnic diffusivity,

$$\kappa \sim -\frac{\mathcal{D}_{in}A_{surface}}{(\frac{\partial\rho}{\partial z}A_{therm})} \sim -\frac{(2 \times 10^{-6}\ \mathsf{kg\ s}^{-1}\mathsf{m}^{-2})(4 \times 10^{12}\mathsf{m}^2)}{(-8 \times 10^{-4}\mathsf{kg\ m}^{-4})(8 \times 10^{13}\mathsf{m}^2)},$$

which implies $\kappa \sim 1.25 \times 10^{-4}\mathsf{m}^2\mathsf{s}^{-1}$.

(d) In situ measurements of κ are an order of magnitude smaller. Either (i) the mixing might be occurring in localised regions and so in situ measurements of κ are not representative of the bulk estimate of κ from (c) or (ii) the surface density flux is not offset by diffusion in the thermocline, but instead by an opposing surface density flux in another region, such as the Southern Ocean.