Toward more sophisticated solution techniques

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Cambridge Texts in Applied Mathematics

1. Point-particle solutions
   - Point force
   - Point torque and stresslet
2. Integral and multipole representation
3. Resistance matrices
4. Particle motion
5. Slender body theory
6. Boundary integral method

Point force

Flow generated by a translating sphere in a quiescent fluid

$$\mathbf{u} = (\frac{\mathbf{I}}{r} + \frac{\mathbf{xx}}{r^3}) \cdot \frac{\mathbf{F^e}}{8\pi\mu} + (\frac{\mathbf{I}}{3r^3} - \frac{\mathbf{xx}}{r^5}) \cdot \frac{a^2\mathbf{F^e}}{8\pi\mu}$$

Stokeslet = flow created by a point force = $O(1/r)$

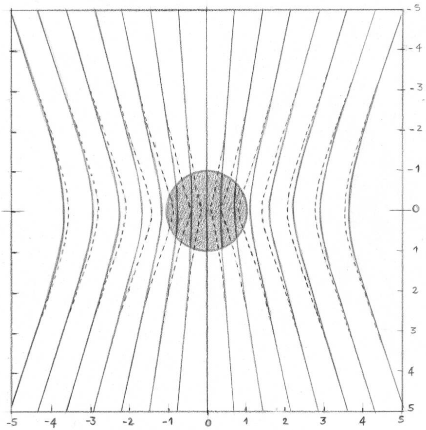
($a \to 0$ with $\mathbf{F^e} = -\mathbf{F^h} = 6\pi\mu a\mathbf{U}$ constant)

$$\mathbf{u}^{\mathrm{PF}} = (\frac{\mathbf{I}}{r} + \frac{\mathbf{xx}}{r^3}) \cdot \frac{\mathbf{F^e}}{8\pi\mu} \quad \text{with Oseen-Burgers tensor:} \quad \mathcal{G} = \frac{\mathbf{I}}{r} + \frac{\mathbf{xx}}{r^3}$$

Velocity field generated by the translating sphere = Stokeslet + degenerate quadrupole

$$\mathbf{u} = \frac{\mathbf{F^e}}{8\pi\mu} \cdot (1 + \frac{a^2}{6}\nabla^2)\,\mathcal{G}$$

Flow lines created by a translating sphere and a point force



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Point torque

Flow generated by a rotating sphere in a quiescent fluid

$$\mathbf{u} = \frac{\mathbf{T^e}}{8\pi\mu} \times \frac{\mathbf{x}}{r^3}$$

Rotlet = flow created by a point torque = $O(1/r^2)$

($a \to 0$ with $\mathbf{T^e} = -\mathbf{T^h} = 8\pi\mu a^3 \boldsymbol{\omega}$ constant)

$$\mathbf{u}^{\mathrm{Rotlet}} = \frac{\mathbf{T}^e}{8\pi\mu} \times \frac{\mathbf{x}}{r^3}$$

Velocity field generated by the rotating sphere ≡ Rotlet

$$\mathbf{u} = \mathbf{u}^{\mathrm{Rotlet}}$$

particle size not explicitly involved

Point stresslet

Disturbance flow around a sphere immersed in a straining motion

$$u_i = -\frac{S^h_{jk}}{8\pi\mu}\frac{3x_ix_jx_k}{r^5} - \frac{3a^2}{5}\frac{S^h_{jk}}{8\pi\mu}\left(\frac{\delta_{ij}x_k + \delta_{ik}x_j}{r^5} - 5\frac{x_ix_jx_k}{r^7}\right)$$

Flow created by a point stresslet $= O(1/r^2)$

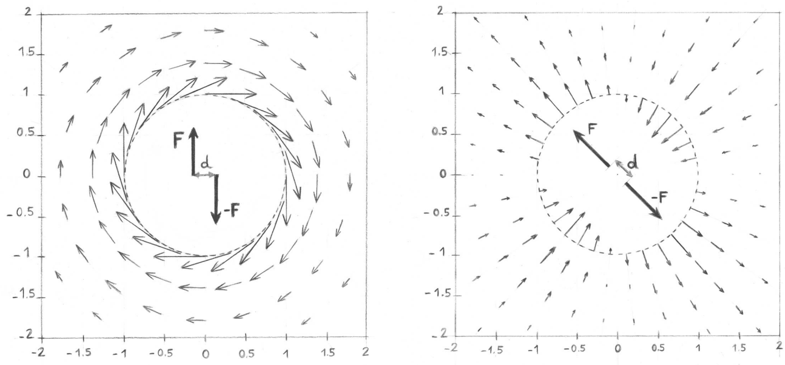
($a \to 0$ with $\mathbf{S^h} = \frac{20}{3}\pi\mu a^3\mathbf{E}^\infty$ constant)

$$u_i^{\mathrm{PS}} = -\frac{3x_ix_jx_k}{r^5}\frac{S^h_{jk}}{8\pi\mu}$$

Velocity field generated by the sphere immersed in a straining motion = flow created by a point stresslet + degenerate octopole

$$u_i = E^\infty_{ij}x_j + \frac{S^h_{jk}}{8\pi\mu}\left(1 + \frac{a^2}{10}\nabla^2\right)\frac{\partial \mathcal{G}_{ij}}{\partial x_k}$$

Flow fields produced by a point torque and a point stresslet



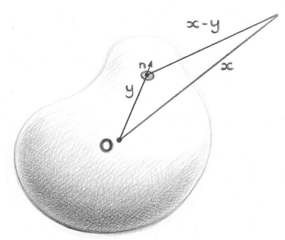
Integral representation

- Flow solution for a distribution of forces = superposition of the flow fields generated independently by each of the forces
- Disturbance flow created by a rigid particle = sum of the distribution of point forces imparted to the fluid on the surface of the particle $\boldsymbol{\sigma}\cdot\mathbf{n}dS$

$$u_i(\mathbf{x}) - u_i^{\infty}(\mathbf{x}) = \int_{S_p} \frac{\mathcal{G}_{ij}(\mathbf{x}-\mathbf{y})}{8\pi\mu}(-\sigma_{jk}n_k)(\mathbf{y})\ dS(\mathbf{y})$$

single layer of forces similar to single layer of charges in electrostatics

single layer potential



Far field

- $|\mathbf{x}| \gg |\mathbf{y}| \Rightarrow \mathcal{G}_{ij}(\mathbf{x}-\mathbf{y}) \sim \mathcal{G}_{ij}(\mathbf{x})$

$$
\begin{aligned}
u_i(\mathbf{x}) - u_i^{\infty}(\mathbf{x}) &= -\frac{\mathcal{G}_{ij}(\mathbf{x})}{8\pi\mu}\int_{S_p}(\sigma_{jk}n_k)(\mathbf{y})\ dS(\mathbf{y}) \\
&= -\frac{\mathcal{G}_{ij}(\mathbf{x})}{8\pi\mu}F_j^h
\end{aligned}
$$

- If the particle experiences a drag, the influence seen at great distances is that of a point force regardless of particle shape
- Analogous to the single layer potential of electrostatics reducing to the field of a point charge far away from the conductor

Multipole expansion

- $|\mathbf{x}| \gg |\mathbf{y}| \Rightarrow \mathcal{G}_{ij}(\mathbf{x}-\mathbf{y}) \sim G_{ij}(\mathbf{x}) - y_k \frac{\partial \mathcal{G}_{ij}}{\partial x_k}(\mathbf{x}) + \ldots$

$$u_i(\mathbf{x}) - u_i^{\infty}(\mathbf{x}) = -\frac{F_j^h}{8\pi\mu}\mathcal{G}_{ij}(\mathbf{x}) + \frac{M_{jk}}{8\pi\mu}\frac{\partial \mathcal{G}_{ij}}{\partial x_k}(\mathbf{x}) + \ldots$$

- Zeroth moment of the traction taken over the particle surface

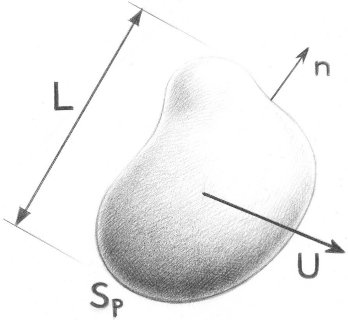
$$F_j^h = \int_{S_p} (\boldsymbol{\sigma}\cdot\mathbf{n})_j \, dS(\mathbf{y}) \quad \text{Drag force}$$

- First moment of the traction taken over the particle surface

$$M_{jk} = \int_{S_p} (\boldsymbol{\sigma}\cdot\mathbf{n})_j \, y_k dS(\mathbf{y}) = S_{jk} + A_{jk}$$

Stresslet and Torque with $T_i = -\epsilon_{ijk} A_{jk}$

Translating particle



Homogeneous Stokes equations

$$\nabla \cdot \mathbf{u} = 0$$

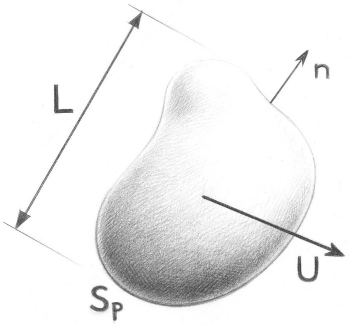
$$\mu \nabla^2 \mathbf{u} = \nabla p$$

with boundary conditions

$$\mathbf{u} = \mathbf{U} \qquad \mathbf{x} \in S_p$$

$$\mathbf{u} \text{ and } p \to 0 \qquad r = \|\mathbf{x}\| \to \infty$$

Linearity between force and velocity



$F^h \sim -(\mu \frac{U}{L})L^2 = -\mu L U$

Drag force on the particle

$$\mathbf{F^h} = \int_{S_p} \boldsymbol{\sigma} \cdot \mathbf{n}\, dS$$

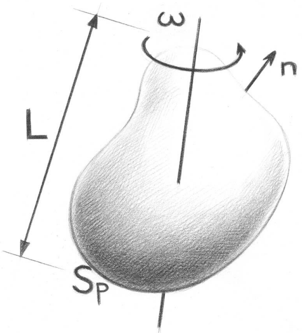
- $\boldsymbol{\sigma}$ = linear in $\mathbf{u}$
- $\mathbf{u}$ = linear in $\mathbf{U}$
- $\mathbf{F^h}$ linear in $\mathbf{U}$

$$\mathbf{F^h} = -\mathbf{R^{FU}} \cdot \mathbf{U}$$

$\mathbf{R^{FU}}$ resistance tensor $\sim \mu L$

depends on the shape of the body

Rotating particle



Homogeneous Stokes equations

$$\nabla \cdot \mathbf{u} = 0$$

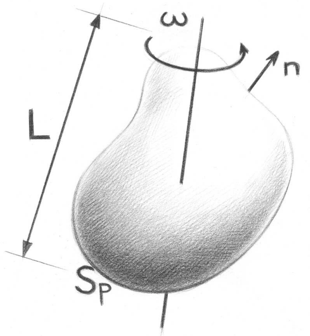
$$\mu \nabla^2 \mathbf{u} = \nabla p$$

with boundary conditions

$$\mathbf{u} = \boldsymbol{\omega} \times \mathbf{x} \qquad \mathbf{x} \in S_p$$

$$\mathbf{u} \text{ and } p \to 0 \qquad r = \|\mathbf{x}\| \to \infty$$

Linearity between torque and rotation velocity



$T^h \sim -L(\mu \frac{\omega L}{L}) L^2 = -\mu L^3 \omega$

Hydrodynamic torque on the particle

$$\mathbf{T^h} = \int_{S_p} \mathbf{x} \times (\boldsymbol{\sigma} \cdot \mathbf{n})\, dS$$

- $\boldsymbol{\sigma}$ = linear in $\mathbf{u}$
- $\mathbf{u}$ = linear in $\boldsymbol{\omega}$
- $\mathbf{T^h}$ = linear in $\boldsymbol{\omega}$

$$\mathbf{T^h} = -\mathbf{R^{T\omega}} \cdot \boldsymbol{\omega}$$

$\mathbf{R^{T\omega}}$ resistance tensor $\sim \mu L^3$

depends on the shape of the body

Resistance tensors

General linear relationship between the velocities and the hydrodynamic force and torque for a rigid particle translating at $\mathbf{U}$ and rotating with $\boldsymbol{\omega}$ through an otherwise motionless fluid:

$$\begin{pmatrix} \mathbf{F^h} \\ \mathbf{T^h} \end{pmatrix} = - \begin{pmatrix} \mathbf{R^{FU}} & \mathbf{R^{F\omega}} \\ \mathbf{R^{TU}} & \mathbf{R^{T\omega}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\omega} \end{pmatrix} = -\tilde{\mathbf{R}} \cdot \begin{pmatrix} \mathbf{U} \\ \boldsymbol{\omega} \end{pmatrix}$$

- $\mathbf{R^{FU}}$, $\mathbf{R^{F\omega}}$, $\mathbf{R^{TU}}$, $\mathbf{R^{T\omega}} \sim \mu$ and determined by the geometry of the particle
- $\mathbf{R^{FU}} \sim L$, $\mathbf{R^{T\omega}} \sim L^3$, both $\mathbf{R^{F\omega}}$ and $\mathbf{R^{TU}} \sim L^2$

Symmetry properties (1)

- $F_i^{h(1)}$ drag on particle translating at velocity $U_i^{(1)}$
- $F_i^{h(2)}$ drag on particle translating at velocity $U_i^{(2)}$
- Reciprocal theorem:

$$\int_{S_p} U_j^{(2)} \sigma_{ij}^{(1)} n_i dS = \int_{S_p} U_j^{(1)} \sigma_{ij}^{(2)} n_i dS$$

$$\therefore U_j^{(2)} F_j^{h(1)} = U_j^{(1)} F_j^{h(2)}$$

- Linearity: $F_j^{h(1)} = -R_{ji}^{FU} U_i^{(1)}$ and $F_j^{h(2)} = -R_{ji}^{FU} U_i^{(2)}$
- Substituting: $R_{ji}^{FU} U_i^{(1)} U_j^{(2)} = R_{ji}^{FU} U_j^{(1)} U_i^{(2)} = R_{ij}^{FU} U_i^{(1)} U_j^{(2)}$
- True for arbitrary $\mathbf{U}^{(1)}$ and $\mathbf{U}^{(2)}$: $R_{ij}^{FU} = R_{ji}^{FU}$

$\mathbf{R}^{\mathbf{FU}}$ symmetric and similarly $\mathbf{R}^{\mathbf{T}\boldsymbol{\omega}}$ symmetric

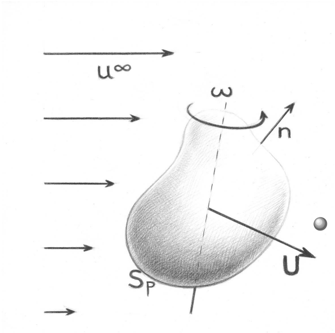
Symmetry properties (2)

- Particle translating at velocity $U_i^{(1)}$
- Same body rotating at $\omega_i^{(2)}$
- Reciprocal theorem:

$$\Rightarrow \int_{S_p} \epsilon_{jkl}\omega_k^{(2)} x_l \sigma_{ij}^{(1)} n_i dS = \int_{S_p} U_j^{(1)} \sigma_{ij}^{(2)} n_i dS$$

- Rotating the indices of the Levi-Civita tensor in a cyclic way: $\omega_j^{(2)} T_j^{h(1)} = U_j^{(1)} F_j^{h(2)}$
- Linearity: $T_j^{h(1)} = -R_{ji}^{TU} U_i^{(1)}$ and $F_j^{h(2)} = -R_{ji}^{F\omega}\omega_i^{(2)}$
- Substituting: $\omega_j^{(2)} R_{ji}^{TU} U_i^{(1)} = \omega_i^{(2)} R_{ij}^{TU} U_j^{(1)} = U_j^{(1)} R_{ji}^{F\omega}\omega_i^{(2)}$
- True for arbitrary $\mathbf{U}^{(1)}$ and $\boldsymbol{\omega}^{(2)}$: $R_{ij}^{TU} = R_{ji}^{F\omega}$

Particle of general geometry in a general linear flow



- Resistance problem: the full "grand resistance matrix"

$$\begin{pmatrix} \mathbf{F^h} \\ \mathbf{T^h} \\ \mathbf{S^h} \end{pmatrix} = -\begin{pmatrix} \mathbf{R^{FU}} & \mathbf{R^{F\omega}} & \mathbf{R^{FE}} \\ \mathbf{R^{TU}} & \mathbf{R^{T\omega}} & \mathbf{R^{TE}} \\ \mathbf{R^{SU}} & \mathbf{R^{S\omega}} & \mathbf{R^{SE}} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{U} - \mathbf{U}^\infty \\ \boldsymbol{\omega} - \boldsymbol{\omega}^\infty \\ -\mathbf{E}^\infty \end{pmatrix}$$

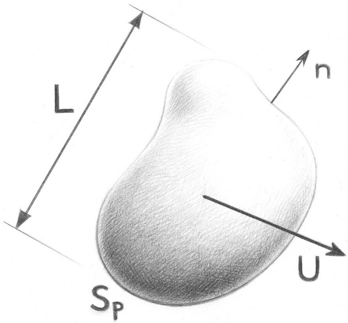
$$= -\boldsymbol{\mathcal{R}} \cdot \begin{pmatrix} \mathbf{U} - \mathbf{U}^\infty \\ \boldsymbol{\omega} - \boldsymbol{\omega}^\infty \\ -\mathbf{E}^\infty \end{pmatrix}$$

- Mobility problem:

$$\begin{pmatrix} \mathbf{U} - \mathbf{U}^\infty \\ \boldsymbol{\omega} - \boldsymbol{\omega}^\infty \\ -\mathbf{E}^\infty \end{pmatrix} = -\boldsymbol{\mathcal{M}} \cdot \begin{pmatrix} \mathbf{F^h} \\ \mathbf{T^h} \\ \mathbf{S^h} \end{pmatrix}.$$

where $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{R}}^{-1}$

Translating particle



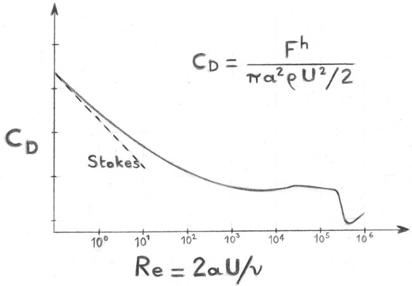
- Drag force

$$\mathbf{F}^{\mathbf{h}} = -\mathbf{R}^{\mathbf{FU}} \cdot \mathbf{U} \quad \text{linear in } \mathbf{U}$$

- Rate of energy dissipation

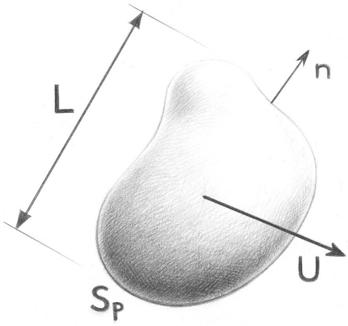
$$\Phi = -\mathbf{U} \cdot \mathbf{F}^{\mathbf{h}} = \mathbf{U} \cdot \mathbf{R}^{\mathbf{FU}} \cdot \mathbf{U} \quad \text{quadratic in } \mathbf{U}$$

Drag coefficient on a sphere



Since Stokes flow dissipates the least energy of all Navier-Stokes flows, the Stokes drag law lies below the actual drag at finite Reynolds number

Translating particle



- Since symmetric, $\mathbf{R}^{\mathbf{FU}}$ may be diagonalized

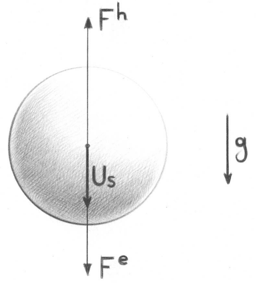
$$\mathbf{R}^{\mathbf{FU}} = \mu L \begin{pmatrix} \lambda_1 & . & . \\ . & \lambda_2 & . \\ . & . & \lambda_3 \end{pmatrix}$$

- Rate of energy dissipation

$$\Phi = \mu L(\lambda_1 U_1^2 + \lambda_2 U_2^2 + \lambda_3 U_3^2) \geq 0$$

with eigenvalues λ_1, λ_2, λ_3 all positive

Settling sphere



$$\mathbf{F^h} = -6\pi\mu a \mathbf{U}$$
$$\mathbf{F^e} = \frac{4}{3}\pi a^3(\rho_p - \rho)\mathbf{g}$$

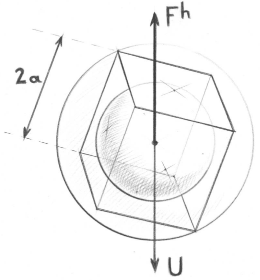
- $\mathbf{R^{FU}}$ isotropic

$$\mathbf{R^{FU}} = \mu a \begin{pmatrix} \lambda & . & . \\ . & \lambda & . \\ . & . & \lambda \end{pmatrix} = 6\pi\mu a\,\mathbf{I}$$

- Force Balance $\mathbf{F^h} + \mathbf{F^e} = 0$

$$\mathbf{U} = \mathbf{U}_S = \frac{2}{9}\frac{a^2}{\mu}(\rho_p - \rho)$$

Settling cube



- $\mathbf{R}^{\mathbf{FU}}$ isotropic

$$\mathbf{R}^{\mathbf{FU}} = \mu a \begin{pmatrix} \lambda & . & . \\ . & \lambda & . \\ . & . & \lambda \end{pmatrix} = \lambda \mu a \, \mathbf{I}$$

- Force Balance $\mathbf{F}^{\mathbf{h}} + \mathbf{F}^{\mathbf{e}} = 0$

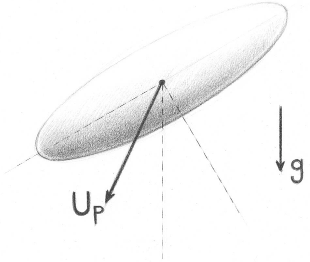
$$\mathbf{U} = \mathbf{U}_{\text{cube}} = \frac{a^2(\rho_p - \rho)\mathbf{g}}{\lambda\mu}$$

- Bounds for drag force

$$6\pi\mu a U \le |\mathbf{F}^{\mathbf{h}}{}_{\text{cube}}| \le 6\pi\mu\sqrt{3}aU$$

by using the minimum dissipation theorem for the inscribing sphere S_i of radius a and the sphere S_o enclosing the cube of radius $\sqrt{3}a$

Settling ellipsoid



- Resistance matrix (x_1 axis as the axis of revolution)

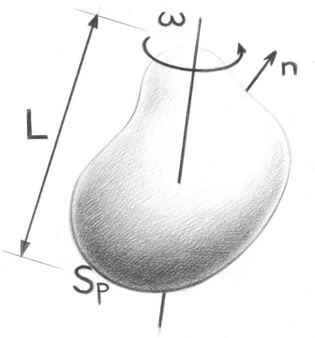
$$\mathbf{R}^{\mathbf{FU}} = -\mu a_1 \begin{pmatrix} \lambda_1 & . & . \\ . & \lambda_2 & . \\ . & . & \lambda_2 \end{pmatrix}$$

- Long ellipsoid: $a_1 \gg a_2 = a_3$

$$\lambda_1 = \frac{4\pi}{\ln \frac{2a_1}{a_2} - \frac{1}{2}}$$
$$\lambda_2 = \lambda_3 = \frac{8\pi}{\ln \frac{2a_1}{a_2} + \frac{1}{2}}$$

- Very long ellipsoid: $\lambda_1 \approx \lambda_2/2$
  factor two between drag for perpendicular and parallel motions at the same velocity

Rotating particle



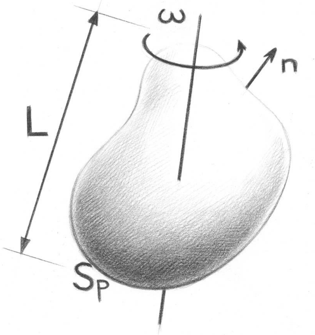
- Hydrodynamic torque

$$\mathbf{T^h} = -\mathbf{R^{T\omega}} \cdot \boldsymbol{\omega} \quad \text{linear in } \boldsymbol{\omega}$$

- Rate of energy dissipation

$$\Phi = -\boldsymbol{\omega} \cdot \mathbf{T^h} = \boldsymbol{\omega} \cdot \mathbf{R^{T\omega}} \cdot \boldsymbol{\omega} \quad \text{quadratic in } \boldsymbol{\omega}$$

Rotating particle



- Since symmetric, $\mathbf{R}^{\mathbf{T}\boldsymbol{\omega}}$ may be diagonalized

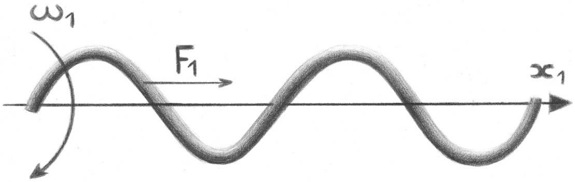
$$\mathbf{R}^{\mathbf{T}\boldsymbol{\omega}} = -\mu L^3 \begin{pmatrix} \zeta_1 & \cdot & \cdot \\ \cdot & \zeta_2 & \cdot \\ \cdot & \cdot & \zeta_3 \end{pmatrix}$$

- Rate of energy dissipation

$$\Phi = \mu L^3(\zeta_1\omega_1^2 + \zeta_2\omega_2^2 + \zeta_3\omega_3^2) \geq 0$$

with eigenvalues ζ_1, ζ_2, ζ_3 all positive

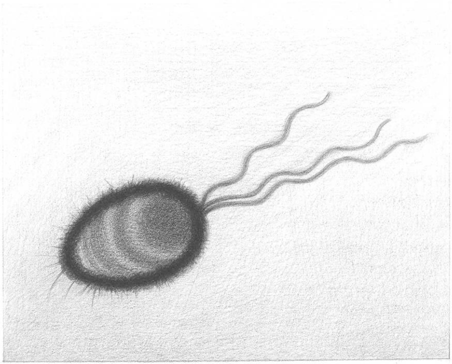
Helicoidal particle



Coupling between translation and rotation

$$F_1 \propto -\mu L^2 \omega_1$$

Escherichia coli



Propulsion accomplished by rotation of the trailing flagella

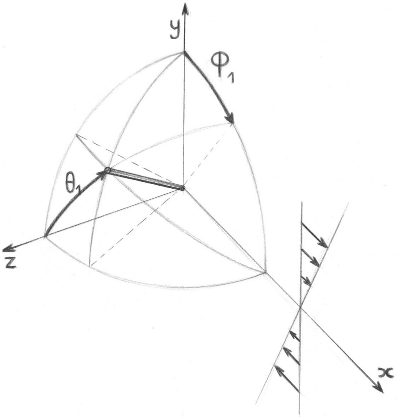
Axisymmetric particle in a simple shear

Equation for the director
(unit vector in the direction of the symmetry axis)

$$\frac{d\mathbf{p}}{dt} = \boldsymbol{\Omega}^{\infty} \cdot \mathbf{p} + \beta \left[\mathbf{E}^{\infty} \cdot \mathbf{p} - \mathbf{p}(\mathbf{p} \cdot \mathbf{E}^{\infty} \cdot \mathbf{p}) \right]$$

The axis of symmetry rotates not only with the rotational portion of the flow, but also with a fraction $\beta = (r^2 - 1)/(r^2 + 1)$ of the straining motion (with $r = a/b$ where a and b are the semi-diameters measured parallel and perpendicular, respectively, to the axis of revolution)

Rigid ellipsoidal particle in simple shear $\mathbf{u}^\infty = (\dot\gamma y, 0, 0)$



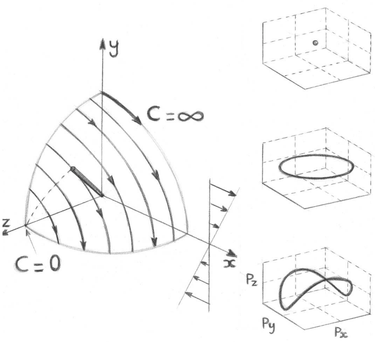
- Equations for the director

$$\begin{aligned}\dot p_x &= \dot\gamma[(\beta+1)p_y/2 - \beta p_x^2 p_y]\\ \dot p_y &= \dot\gamma[(\beta-1)p_x/2 - \beta p_y^2 p_x]\\ \dot p_z &= -\dot\gamma\beta p_x p_y p_z\end{aligned}$$

- Equations for polar angles θ_1 and ϕ_1

$$\begin{aligned}\dot\theta_1 &= \frac{\dot\gamma(r^2-1)}{4(r^2+1)}\sin 2\theta_1 \sin 2\phi_1\\ \dot\phi_1 &= \frac{\dot\gamma}{r^2+1}(\sin^2\phi_1 + r^2\cos^2\phi_1)\end{aligned}$$

Jeffery orbits

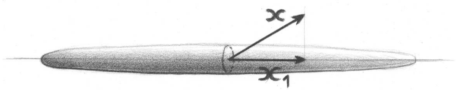


$$\tan\phi_1 = r\tan\left[\frac{\dot\gamma t}{r+(1/r)}\right] \text{ with } \phi_1(0)=0$$

$$\tan\theta_1 = \frac{Cr}{(\sin^2\phi_1 + r^2\cos^2\phi_1)^{1/2}}$$

- Periodic motion with a period $2\pi(r+1/r)/\dot\gamma$
- Orbit constant C defines the eccentricity of this elliptical orbit

Replacement of the body by a line of point forces



- Disturbance created by a long body ≈ Disturbance due to a line density of applied point forces

$$u_i^D(\mathbf{x}) = u_i(\mathbf{x}) - u_i^\infty(\mathbf{x}) = \int_{-a}^{a} \frac{\mathcal{G}_{ij}(\mathbf{x}-\mathbf{x}_1')}{8\pi\mu} f_j^{\rm PF}(x_1')\, dx_1'$$

- Boundary conditions for $\mathbf{x} \to \mathbf{x}_{\rm surface}$: for translation $U_i = u_i^D(\mathbf{x}_{\rm surface})$

First approximation to the uniform point force distribution

- Uniform point force density

$$u_i^D(\mathbf{x}) = f_j^{\mathrm{PF}} \int_{-a}^{a} \frac{\mathcal{G}_{ij}(\mathbf{x}-\mathbf{x}_1')}{8\pi\mu}\, dx_1'$$

- Leading order with $\epsilon = [\ln(2a/b)]^{-1}$ small parameter

$$u_i^D(\mathbf{x}) = \frac{\ln(2a/b)}{4\pi\mu}[f_i^{\mathrm{PF}} + \delta_{i1} f_1^{\mathrm{PF}} + |\mathbf{f}^{\mathrm{PF}}|O(\epsilon)] \quad \text{constant}$$

- Drag force for slender body translating at $\mathbf{U}$

$$\begin{aligned} F_1^h &\sim 4\pi\mu a\epsilon U_1 \quad \text{parallel translation} \\ F_{2,3}^h &\sim 8\pi\mu a\epsilon U_{2,3} \quad \text{perpendicular translation} \end{aligned}$$

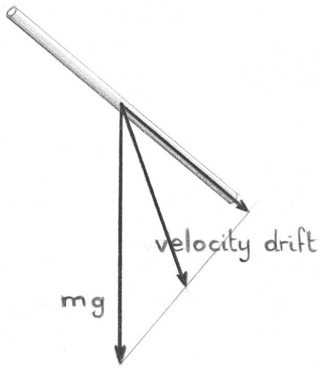
General features

- Drag on an arbitrary object is not much less than that on the enclosing sphere which gives an upper bound by a maximum dissipation argument
- For a long slender body, drag for perpendicular motion $\approx$ twice drag for parallel motion at the same velocity $\Leftarrow$ the induced velocity due to an isolated Stokeslet is twice as large at a point on the axis of symmetry as at a point at an equal distance in the transverse direction
  - Resistance to motion parallel to the long body $\approx$ half that in the perpendicular direction $\Rightarrow$ a fiber parallel to gravity settles twice as fast as a fiber perpendicular to gravity
  - Velocity perpendicular to the axis of revolution reduced by a factor two $\Rightarrow$ a fiber inclined at an angle to the vertical will not settle vertically but will have a sideway drift

Drift of a settling fiber

$$\mathbf{F^h} \sim -\mu a \begin{pmatrix} 4\pi\epsilon & 0 & 0 \\ 0 & 8\pi\epsilon & 0 \\ 0 & 0 & 8\pi\epsilon \end{pmatrix} \cdot \mathbf{U}$$

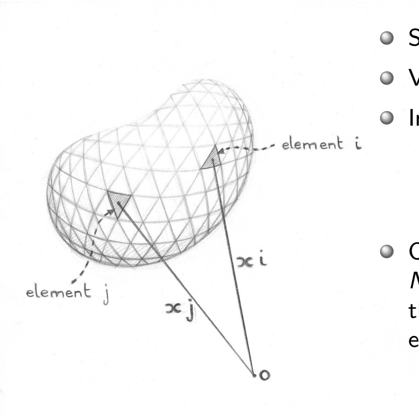
Force balance:
Perpendicular velocity reduced by a factor 2



Boundary integral method [Pozrikidis, 1992]

- Direct application of the integral representation of Stokes flow
- Reduction of dimensionality: 3D PDE $\rightarrow$ 2D (boundary) integral equation in which the unknowns are densities of the Stokes singularities distributed over the boundary of the fluid domains
- Methods particularly well-suited for calculation of flows associated with complex geometries which does not easily fit a finite difference grid
- Methods primarily used for deformable boundary problems or for such geometries as fibers

Sketch of the method for a solid particle



- Surface discretized into small area elements δS_j
- Velocity field at the surface considered
- Integral equation written as a matrix equation

$$\mathbf{u}(\mathbf{x_i}) = \mathbf{u}^{\infty}(\mathbf{x_i}) + \sum_j \boldsymbol{\mathcal{G}}(\mathbf{x_i} - \mathbf{x_j}) \cdot \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x_j}) \delta S_j$$

- Oseen-Burgers tensor integrated over each of the N elements: $3N \times 3N$ square matrix $\boldsymbol{\mathcal{G}}$ relating the velocity at element i to the traction at element j

Resistance and mobility problems

Resistance problem: velocity known

- Tractions computed by a matrix inversion
- Flow field away from the particle surface computed from the full boundary integral

Mobility problem: force and torque known

- $6N$ unknown quantities but at present $3N$ equations
- Velocities must satisfy solid body motion (additional $3N$ equations):

$$\mathbf{u}(\mathbf{x_i}) = \mathbf{U} + \boldsymbol{\omega} \times (\mathbf{x_i} - \mathbf{x_p})$$

$+$ additional constraints on the tractions (6 equations):

$$\mathbf{F^h} = \sum_j \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x_j})\delta S_j \quad \text{and} \quad \mathbf{T^h} = \sum_j (\mathbf{x_j} - \mathbf{x_p}) \times \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x_j})\delta S_j$$

Movie reference

Taylor, G. I.
Low Reynolds Number Flows
1966 National Committee for Fluid Mechanics Films
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General reference on boundary integral method

Pozrikidis, C. 1992. *Boundary integral and singularity methods for linearized viscous flow*. Cambridge University Press.