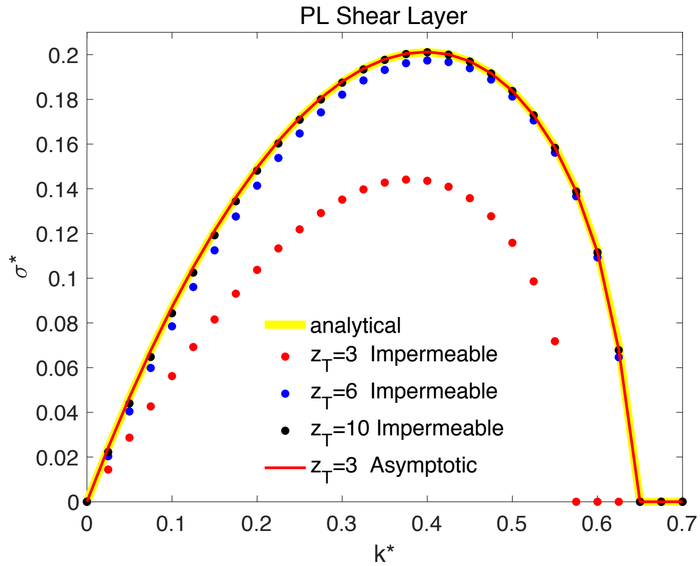

Figure 1: Scaled growth rate versus wavenumber for the piecewise-linear shear layer. Computation was done with impermeable boundaries in three locations (red, blue and black circles), and once with asymptotic boundary conditions (red curve). The analytical result (3.34) is shown in yellow.

8: The piecewise-linear shear layer: numerical solution

(a) Reproduce analytical solution as in the text with algebraic details included.

(b,c) Figure 1 shows results for four numerical solutions of the piecewise linear shear layer, along with the analytical solution (yellow). By using the asymptotic boundary conditions (red dashes) you reproduce the analytical solution very closely. With impermeable boundary conditions, you get terrible results (red) unless you move the boundaries far away from the shear layer (blue, black).

Figure 2 is a plot of eigenfunctions. The red solid curve is the analytical result; the blue dots are the numerical result. Note that both are normalized so that $\hat{w}(0) = 1$. Remember that eigenfunctions are only defined up to a multiplicative constant, so two results can both be right but look totally different. Normalizing solves that.

Here is the code we used:

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Project 2: PL shear layer

% wave vector range
ks=[0:.025:.7]; l=0;

figure
% plot analytical formula
s_an=sqrt(-( (ks-.5).^2-.25*exp(-4*ks)) );
plot(ks,s_an,'y-','linewidth',6);hold on
xlabel('k*','fontsize',fs+2)
ylabel('\sigma*','fontsize',fs+2)
title('PL Shear Layer','fontsize',fs,'fontweight','normal')
```

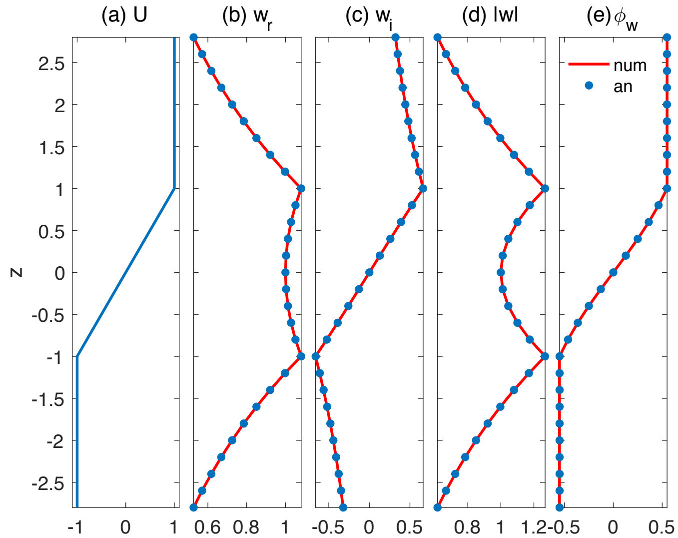

Figure 2: Eigenfunction of $\hat{w}$ for the fastest-growing mode of the piecewise-linear shear layer, numerical (red curve) and analytical (blue dots). At left is the velocity profile followed by the real part, imaginary part, magnitude and phase of $\hat{w}$.

```
%%%%%%%%%
% case 1: ztop=3; impermeable bcs
% set up z grid, BCs, U
ztop=3; zbot=-ztop;
dz=.1;
z=[zbot+dz:dz:ztop-dz]'; % exclude boundary points
iBC=[1 1]; % impermeable boundaries

% background velocity profile
U=z; U(U>1)=1; U(U<-1)=-1;

% stability analysis over range of k
clear sig
for i=1:length(ks)
    [sig(i)]=Ray(z,U,ks(i),l,iBC);
end
plot(ks,real(sig),'r.','markersize',ms);hold on

%%%%%%%%%
% case 2: ztop=6; impermeable bcs
% set up z grid, BCs, U
ztop=6; zbot=-ztop;
dz=.2;
z=[zbot+dz:dz:ztop-dz]'; % exclude boundary points
iBC=[1 1]; % impermeable boundaries

% background velocity profile
```

```
U=z; U(U>1)=1; U(U<-1)=-1;

% stability analysis over range of k
clear sig
for i=1:length(ks)
    [sig(i)]=Ray(z,U,ks(i),l,iBC);
end
plot(ks,real(sig),'b.','markersize',ms);hold on

%%%%%%%%%
% case 3: ztop=10; impermeable bcs
% set up z grid, BCs, U
ztop=10; zbot=-ztop;
dz=.2;
z=[zbot+dz:dz:ztop-dz]'; % exclude boundary points
iBC=[1 1]; % impermeable boundaries

% background velocity profile
U=z; U(U>1)=1; U(U<-1)=-1;

% stability analysis over range of k
clear sig
for i=1:length(ks)
    k=ks(i);
    [sig(i)]=Ray(z,U,k,l,iBC);
end
plot(ks,real(sig),'k.','markersize',ms);

%%%%%%%%%
% case 4: ztop=3; asymptotic bcs
ztop=3;
zbot=-ztop;
dz=.2;
z=[zbot+dz:dz:ztop-dz]'; % exclude boundary points
iBC=[2 2]; % asymptotic boundaries

% background velocity profile
U=z; U(U>1)=1; U(U<-1)=-1;

% stability analysis over range of k
clear sig
for i=1:length(ks)
    [sig(i)]=Ray(z,U,ks(i),l,iBC);
end
plot(ks,real(sig),'r-','linewidth',2);hold on
ylim([0 .21])
set(gca,'fontsize',fs-2)
```

```
h=legend('analytical','z_T=3  Impermeable','z_T=6  Impermeable',...
    'z_T=10 Impermeable','z_T=3  Asymptotic','location','south')
legend boxoff
set(h,'fontsize',fs-2)


print('-dpdf',[direc 'hmwk3_2'])

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% compare eigenfunctions

% Numerical eigfn from asymptotic case
% choose wavenumber of FGM
kt_mx=ks(real(sig)==max(real(sig))); % FGM
k=kt_mx;l=0;

iBC=[2 2];
[~,w]=Ray(z,U,k,l,iBC);

% normalize numerical eigenfunctions so that w(0)=1.
wnorm=w(z==0);
w=w/wnorm;
%u=1i*ddz(z)*w/k;

% Theoretical eigfn (I cheated and solved numerically for the
% coefficients B,C,D,E).
kt_mx=0.40;
k=kt_mx;

s_an=sqrt(-( (k-.5).^2-.25*exp(-4*k)) );
cmx_an=1i*s_an/k;
%
% option 1
x=exp(k);xi=1/x;c=cmx_an
M=[xi -x -xi 0; ...
    0 xi x -xi;...
    (1/(k*(1-c))-1)*xi -x xi 0;...
    0 xi -x (1/(k*(1+c))-1)*xi]
V=null(M);
w_an=V(2)*exp(k*z)+V(3)*exp(-k*z);
w_an(z>1)=V(1)*exp(-k*z(z>1));
w_an(z<-1)=V(4)*exp(k*z(z<-1));

% option 2
x=exp(-2*k);
MM=[-2*(1-c)+1/k  x/k; ...
    -x/k    2*(1+c)-1/k];
```

```
V=null(MM);
w_an=V(1)*exp(-k*abs(z-1))+V(2)*exp(-k*abs(z+1));

% normalize analytical eigenfunctions so that w(0)=1.
wnorm=w_an(z==0);
w_an=w_an/wnorm;

% plot w

figure
subplot(1,10,1:2)
plot(U,z,'linewidth',lw);
axis tight;xlim([-1.1 1.1])
title('(a) U_ ','fontsize',fs,'fontweight','normal')
ylabel('z','fontsize',fs)
set(gca,'fontsize',fs-4)

subplot(1,10,3:4)
plot(real(w_an),z,'r','linewidth',lw)
hold on
plot(real(w),z,'.','markersize',ms);
axis tight;
title('(b) w_r','fontsize',fs','fontweight','normal')
set(gca,'yticklabel','')
set(gca,'fontsize',fs-4)

subplot(1,10,5:6)
plot(imag(w_an),z,'r','linewidth',lw)
hold on
plot(imag(w),z,'.','markersize',ms);
axis tight;
title('(c) w_i','fontsize',fs','fontweight','normal')
set(gca,'yticklabel','')
set(gca,'fontsize',fs-4)

subplot(1,10,7:8)
[Phase, Mag]=cart2pol(real(w),imag(w));
[Phase_an, Mag_an]=cart2pol(real(w_an),imag(w_an));
plot(Mag_an,z,'r','linewidth',lw)
hold on
plot(Mag,z,'.','markersize',ms);
axis tight;
title('(d) |w|_ ','fontsize',fs','fontweight','normal')
set(gca,'yticklabel','')
set(gca,'fontsize',fs-4)

subplot(1,10,9:10)
plot(Phase_an,z,'r','linewidth',lw)
```

```
hold on
plot(Phase,z,'.','markersize',ms);
axis tight;
title('(e)_ \phi_w','fontsize',fs','fontweight','normal')
set(gca,'yticklabel','')
set(gca,'fontsize',fs-4)

h=legend('num','an','location','northeast');legend boxoff
set(h,'orientation','vertical')

print('-dpdf',[direc 'hmwk3_3'])
return
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```