Chapter 15

```
In[1]:= Needs["Graphics`PlotField`"]
```

à Question 1

In[2]:= D[s (1 - E^{-a e}), e]

Out[2]= $a\, e^{-a e}\, s$

In[3]:= D[s (1 - E^{-a e}), {e, 2}]

Out[3]= $-a^2\, e^{-a e}\, s$

Since the second derivative is negative, then there is diminishing returns to effort.

In[4]:= D[s (1 - E^{-a e}), s]

Out[4]= $1 - e^{-a e}$

In[5]:= D[s (1 - E^{-a e}), {s, 2}]

Out[5]= 0

Since the second derivative is zero, there is constant returns to stock size.

à Question 2

In[6]:= Solve$\left[D\left[r\, s\, \text{Log}\left[\frac{k}{s}\right], s\right] == 0, s\right]$

Out[6]= $\left\{\left\{s \to \frac{k}{e}\right\}\right\}$

Hence, $s_m = k/E$, i.e., the stock size which achieves the maximum sustainable yield.

In[7]:= Simplify$\left[r \left(\frac{k}{E}\right) \text{Log}\left[k\, \frac{E}{k}\right]\right]$

Out[7]= $\frac{k\, r}{e}$

i.e., the growth rate at s_m is $\frac{k r}{E}$.

à Question 3

(i)

In[8]:= **sdot = r s $\left(1 - \frac{s}{k}\right)$ - a e s**

Out[8]= $-a\,e\,s + r\,s\left(1 - \frac{s}{k}\right)$

In[9]:= **Solve[sdot == 0, s]**

Out[9]= $\left\{\{s \to 0\}, \left\{s \to -\frac{k\,(a\,e - r)}{r}\right\}\right\}$

Given

$$\frac{h}{e} = a\,s$$

then $\frac{h}{e}$ is equal to

In[10]:= **Apart[Simplify[a $\frac{k\,(-a\,e + r)}{r}$]]**

Out[10]= $a\,k - \frac{a^2\,e\,k}{r}$

which is linear in effort

In[11]:= **sdot2 = r s Log[$\frac{k}{s}$] - a e s**

Out[11]= $-a\,e\,s + r\,s\,\mathrm{Log}\left[\frac{k}{s}\right]$

In[12]:= **Solve[sdot2 == 0, s]**

Solve::verif : Potential solution {s → 0} (possibly discarded by verifier) should be checked by hand. May require use of limits.

Out[12]= $\{\{s \to e^{-\frac{a\,e}{r}}\,k\}\}$

In[13]:= **Apart[Simplify[a $E^{-\frac{a\,e}{r}}$ k]]**

Out[13]= $a\,e^{-\frac{a\,e}{r}}\,k$

Then Log[$\frac{h}{e}$] is equal to

In[14]:= **PowerExpand[Log[a $E^{-\frac{a\,e}{r}}$ k]]**

Out[14]= $-\frac{a\,e}{r} + \mathrm{Log}[a] + \mathrm{Log}[k]$

Which is log-linear in effort.

(ii)

In each case let α and β denote the ordinary least squares estimates (OLS estimates). Then in the first case,

$$\alpha = a\,k \qquad \text{and} \qquad \beta = \frac{a^2\,k}{r} = \alpha\left(\frac{a}{r}\right)$$

Accordingly it is not possible to identify the parameters from the OLS estimates.

In the second case $\alpha = \mathrm{Log}[a\,k]$ and $\beta = -\frac{a}{r}$. Then

In[15]:= **Solve[α == Log[-β r k], β]**

Out[15]= $\left\{\left\{\beta \to -\frac{e^{\alpha}}{k\,r}\right\}\right\}$

and again it is not possible to identify the parmeters from the OLS estimates.

à Question 4

If the yield is to be at the maximum sustainable, then $s_m = \frac{k}{2} = \frac{1000}{2} = 500$, and $f(s_m) = \frac{r\,k}{4} = \frac{(0.2)(1000)}{4} = 50$. It must be necessary then for the harvest to be 50 and the stock size to be 500.

Since supply is $q^s = 1.2\,p + 0.05\,s$ and $q^s = h = 50$ and $s = 500$, then

```
In[16]:= Solve[50 == 1.2 p + 0.05 (500), p]

Out[16]= {{p → 20.8333}}
```

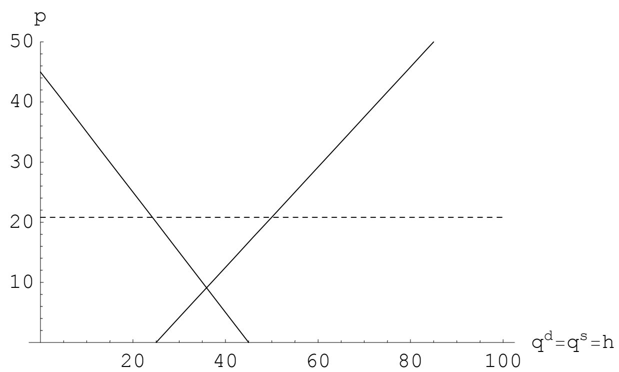
But at $p = 20.8333$ q^d is equal to

```
In[17]:= 45 - 20.8333

Out[17]= 24.1667
```

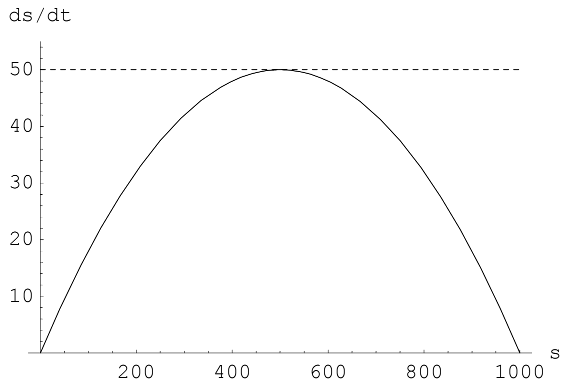
which is not equal to h. Accordingly it is not possible to set a price to clear the market at a stable equilibrium with yield equal to the maximum sustainable yield.

The situation can be shown in the following figures.

```
In[18]:= Simplify[Solve[q == 1.2 p + 0.05 (500), p]]

Out[18]= {{p → -20.8333 + 0.833333 q}}

In[19]:= Simplify[Solve[q == 45 - p, p]]

Out[19]= {{p → 45 - q}}

In[20]:= lines = Plot[{20.8333, -20.8333 + 0.833333 h, 45 - h},
           {h, 0, 100}, PlotRange -> {0, 50}, AxesLabel -> {"q^d=q^s=h", "p"},
           PlotStyle -> {{Dashing[{.01}]}, {}, {}}];
```



```
In[21]:= growth = Plot[{50, 0.2 s (1 - s/1000)}, {s, 0, 1000}, PlotRange -> {0, 55},
    AxesLabel -> {"s", "ds/dt"}, PlotStyle -> {{Dashing[{.01}]}, {}}];
```



à Question 5

(i)

The equation $g = k\,s\,(s_u - s)$ denotes the growth function fish stock as a function of the stock. It is quadratic, with growth of zero when $s = 0$ and when $s = s_u$. Since

```
In[22]:= Solve[D[k s (su - s), s] == 0, s]
```

$$\text{Out[22]= } \left\{\left\{s \to \frac{su}{2}\right\}\right\}$$

and

```
In[23]:= D[k s (su - s), {s, 2}]

Out[23]= -2 k
```

with $k > 0$, the growth curve must reach a maximum when $s = s_u$.

The harvesting function, $h = a\,e\,s$, is linear in s for given effort, e. A rise in effort increases the slope of the line; while a fall in effort reduces the slope.

The profit function $\pi = p\,h - w\,e = p\,a\,e\,s\;-\;w\,e$. But in equilibrium stock size is a function of effort, because in equilibrium

```
In[24]:= Solve[k s (su - s) - a e s == 0, s]
```

$$\text{Out[24]= } \left\{\{s \to 0\},\ \left\{s \to \frac{-a\,e + k\,su}{k}\right\}\right\}$$

$$\text{In[25]:= Expand}\left[p\,a\,e\left(\frac{-a\,e + k\,su}{k}\right) - w\,e\right]$$

$$\text{Out[25]= } -\frac{a^2\,e^2\,p}{k} + a\,e\,p\,su - e\,w$$

i.e., $\pi = \left(-\frac{a^2\,p}{k}\right)e^2 + (a\,p\,s_u - w)\,e$, which is quadratic in e. Since

```
In[26]:= Solve[-(a^2 e^2 p)/k + a e p su - e w == 0, e]
Out[26]= {{e → 0}, {e → (k (a p su - w))/(a^2 p)}}

In[27]:= Solve[D[-(a^2 e^2 p)/k + a e p su - e w, e] == 0, e]
Out[27]= {{e → (k (a p su - w))/(2 a^2 p)}}

In[28]:= D[-(a^2 e^2 p)/k + a e p su - e w, {e, 2}]
Out[28]= -(2 a^2 p)/k
```

Profit begins at zero when effort is zero, rises to a maximum when effort is $\frac{k(a\,p\,s_u - w)}{2\,a^2\,p}$ and returns to zero at twice this value.

(ii)

(a)

```
In[29]:= g[s_] := 0.5 s (25 - s)

In[30]:= h[s_] := 2.5 e s

In[31]:= Solve[D[g[s], s] == 0, s]
Out[31]= {{s → 12.5}}

In[32]:= g[12.5]
Out[32]= 78.125
```

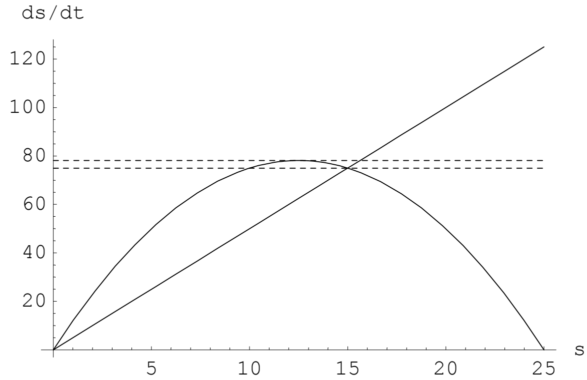
Hence, the maximum sustainable yield is 78.125 and the stock size at this level is $s = 12.5$.

(b)

```
In[33]:= sdot = g[s] - h[s]
Out[33]= -2.5 e s + 0.5 (25 - s) s

In[34]:= Solve[sdot == 0, s] /. {e -> 2}
Out[34]= {{s → 0.}, {s → 15.}}

In[35]:= h[s] /. {e -> 2}
Out[35]= 5. s

In[36]:= g[15]
Out[36]= 75.
```

Hence, when the stock size is 15 there is a steady-state. This can be seen in the following diagram as the solution where the harvest function cuts the growth curve.

```
In[37]:= Plot[{78.125, 75, g[s], 5 s}, {s, 0, 25},
           PlotStyle -> {{Dashing[{.01}]}, {Dashing[{.01}]}, {}, {}},
           AxesLabel -> {"s", "ds/dt"}];
```



(c)

It can be seen from the diagram that at the stock size where the growth curve is a maximum (the value of *s* which achieves the maximum sustainable yield), natural growth of fish stocks is in excess of harvesting. This means that the stock size will rise. It will rise until, given the value of effort implied in the harvest function ($e = 2$), natural growth equals the rate of harvesting. Hence, the solution for *s* must exceed the stock size at the maximum sustainable yield.

(iii) (a)

```
In[38]:= Simplify[Solve[sdot == 0, e]]

Out[38]= {{e → 5. - 0.2 s}}

In[39]:= profit = 0.6 h[s] - 12 e /. {e -> 5 - 0.2 s}

Out[39]= -12 (5 - 0.2 s) + 1.5 (5 - 0.2 s) s

In[40]:= Solve[D[profit, s] == 0, s]

Out[40]= {{s → 16.5}}

In[41]:= estar = 5 - 0.2 s /. {s -> 16.5}

Out[41]= 1.7

In[42]:= profitstar = profit /. {e -> 5 - 0.2 s, s -> 16.5}

Out[42]= 21.675

In[43]:= g[16.5]

Out[43]= 70.125
```

Summary:

$s^* = 16.5$
$e^* = 1.7$
$\pi^* = 21.675$
$g^* = 70.125$

(iii) (b)

```
In[44]:= hb = h[s] /. {e -> 2}

Out[44]= 5. s

In[45]:= profitb = 0.6 hb - 12 (2)

    General::spell1 :
     Possible spelling error: new symbol name "profitb" is similar to existing symbol "profit".

Out[45]= -24 + 3. s

In[46]:= sdotb = g[s] - hb

    General::spell1 :
     Possible spelling error: new symbol name "sdotb" is similar to existing symbol "sdot".

Out[46]= -5. s + 0.5 (25 - s) s

In[47]:= Solve[sdotb == 0, s]

Out[47]= {{s → 0.}, {s → 15.}}

In[48]:= g[15]

Out[48]= 75.

In[49]:= h[15] /. {e -> 2}

Out[49]= 75.

In[50]:= profitb /. {s -> 15}

Out[50]= 21.
```

Summary:

$s^* = 15$
$e^* = 2$ (by assumption)
$\pi^* = 21$
$g^* = 75$

(iii)(c)

With effort $e = 2$ profits are 21, but are not at their maximum value. In fact too much effort is being expended to maximise profits. Reducing effort to $e = 1.7$ raises profits to a maximum of 21.675.

(iv)

(a) $p = 0.8$, $w = 12$ and $a = 2.5$

```
In[51]:= Clear[g, h]

In[52]:= g[s_] := 0.5 s (25 - s)

In[53]:= h[s_] := a e s
```

```
In[54]:= profit = p h[s] - w e

Out[54]= a e p s - e w

In[55]:= sdot = g[s] - h[s]

Out[55]= -a e s + 0.5 (25 - s) s

In[56]:= Simplify[Solve[sdot == 0, e]]
```

Out[56]= $\left\{\left\{e \to \frac{12.5 - 0.5\,s}{a}\right\}\right\}$

In[57]:= newprofit = profit /. $\left\{e \text{ -> } \frac{12.5 - 0.5\,s}{a}\right\}$

Out[57]= $p\,(12.5 - 0.5\,s)\,s - \frac{(12.5 - 0.5\,s)\,w}{a}$

```
In[58]:= Solve[D[newprofit, s] == 0, s]
```

Out[58]= $\left\{\left\{s \to \frac{1.\,(12.5\,p + \frac{0.5\,w}{a})}{p}\right\}\right\}$

In[59]:= sstar1 = $\frac{1.\grave{}\,(12.5\grave{}\,p + \frac{0.5\grave{}\,w}{a})}{p}$ /. {p -> 0.8, w -> 12, a -> 2.5}

```
Out[59]= 15.5
```

In[60]:= estar1 = $\frac{12.5 - 0.5\,s}{a}$ /. {s -> 15.5, a -> 2.5}

```
General::spell1 :
 Possible spelling error: new symbol name "estar1" is similar to existing symbol "sstar1".

Out[60]= 1.9

In[61]:= profit1 = newprofit /. {s -> 15.5, p -> 0.8, w -> 12, a -> 2.5}

Out[61]= 36.1

In[62]:= g1 = g[15.5]

Out[62]= 73.625
```

(b) $p = 0.6$, $w = 15$ and $a = 2.5$

In[63]:= sstar2 = $\frac{1.\,(12.5\,p + \frac{0.5\,w}{a})}{p}$ /. {p -> 0.6, w -> 15, a -> 2.5}

```
Out[63]= 17.5
```

In[64]:= estar2 = $\frac{12.5 - 0.5\,s}{a}$ /. {s -> 17.5, a -> 2.5}

```
General::spell1 :
 Possible spelling error: new symbol name "estar2" is similar to existing symbol "sstar2".

Out[64]= 1.5

In[65]:= profit2 = newprofit /. {s -> 17.5, p -> 0.6, w -> 15, a -> 2.5}

Out[65]= 16.875
```

```
In[66]:= g2 = g[17.5]

Out[66]= 65.625
```

(c) $p = 0.6$, $w = 12$ and $a = 3$

In[67]:= sstar3 = $\frac{1.\ (12.5\,p + \frac{0.5\,w}{a})}{p}$ /. {p -> 0.6, w -> 12, a -> 3}

Out[67]= 15.8333

In[68]:= estar3 = $\frac{12.5 - 0.5\,s}{a}$ /. {s -> 15.8333, a -> 3}

```
General::spell1 :
 Possible spelling error: new symbol name "estar3" is similar to existing symbol "sstar3".

Out[68]= 1.52778

In[69]:= profit3 = newprofit /. {s -> 15.8333, p -> 0.6, w -> 12, a -> 3}

Out[69]= 25.2083

In[70]:= g1 = g[15.8333]

Out[70]= 72.5696
```

The results can be summarised in the following table :

	$p = 0.8$ $w = 12$ $a = 2.5$	$p = 0.6$ $w = 15$ $a = 2.5$	$p = 0.6$ $w = 12$ $a = 3$
s	15.5	17.5	15.8333
e	1.9	1.5	1.52778
π	36.1	16.875	25.2083
g	73.625	65.625	72.5696

The implications are as follows:
A rise in price, *ceteris paribus*, leads to a rise in all variables, i.e., to a rise in s, e, π and g.
A rise in wages, *ceteris paribus*, leads to a rise in s, but to a fall in the other three variables.
A rise in technology, *ceteris paribus*, leads to a fall in effort but to a rise in the other three variables.

(v)-(vi)

```
In[71]:= Clear[g, h, profit, sdot]

In[72]:= g[s_] := 0.5 s (25 - s)

In[73]:= h[s_] := 2.5 e s

In[74]:= profit = 0.6 h[s] - 12 e

Out[74]= -12 e + 1.5 e s

In[75]:= sdot = g[s] - h[s]

Out[75]= -2.5 e s + 0.5 (25 - s) s
```

```
In[76]:= edot = v (profit) /. {v -> 5}

    General::spell1 :
     Possible spelling error: new symbol name "edot" is similar to existing symbol "sdot".

Out[76]= 5 (-12 e + 1.5 e s)

In[77]:= Solve[edot == 0, s]

Out[77]= {{s → 8.}}
```

i.e., the no entry/no exit stock level is $s = 8$.
The equation for the isocline $\dot{s} = 0$ is given by

```
In[78]:= Solve[-2.5 e s + 0.5 (25 - s) s == 0, e]

Out[78]= {{e → 0.2 (25. - 1. s)}}
```

or

$$e = 5 - \frac{s}{5}$$

which is linear. Furthermore, when $\dot{s} > 0$, then

$$5 - \frac{s}{5} > e$$

and so below (to the left of) the isocline stock size is rising; while above (to the right of) the isocline stock size is falling.

(vii)

```
In[79]:= Solve[sdot == 0, e] /. {s -> 8}

Out[79]= {{e → 3.4}}

In[80]:= g[8]

Out[80]= 68.

In[81]:= h[8] /. {e -> 3.4}

Out[81]= 68.

In[82]:= profit /. {s -> 8, e -> 3.4}

Out[82]= 0.

In[83]:= Simplify[Solve[sdot == 0, e]]

Out[83]= {{e → 5. - 0.2 s}}

In[84]:= sdotline = Plot[5 - 0.2 s, {s, 0, 10},
            PlotRange -> {0, 5.5}, DisplayFunction -> Identity];

In[85]:= linept =
          Graphics[{Line[{{8, 0}, {8, 5.5}}], PointSize[0.02], Point[{8, 3.4}]}];

In[86]:= linespoint = Show[sdotline, linept,
            AxesLabel -> {"s", "e"}, DisplayFunction -> Identity];
```

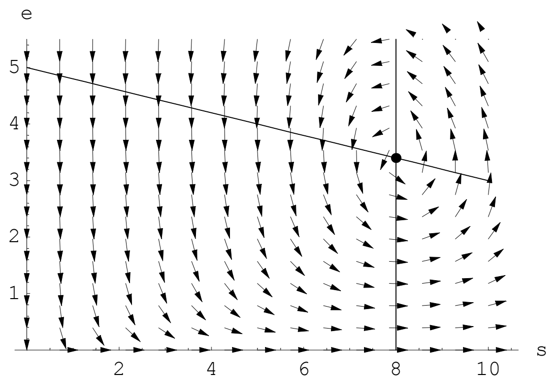
The two differential equations of this system are

$\dot{s} = 0.5\, s(25 - s) - 2.5\, e\, s$

$\dot{e} = 5[0.6\,(2.5)\, e\, s\; - \;12\, e]$

```
In[87]:= field = PlotVectorField[{0.5 s (25 - s) - 2.5 e s, 7.5 e s - 60 e},
           {s, 0, 10}, {e, 0, 5.5}, Axes -> True, ScaleFunction -> (1 &),
           AspectRatio -> 0.8, DisplayFunction -> Identity];
```

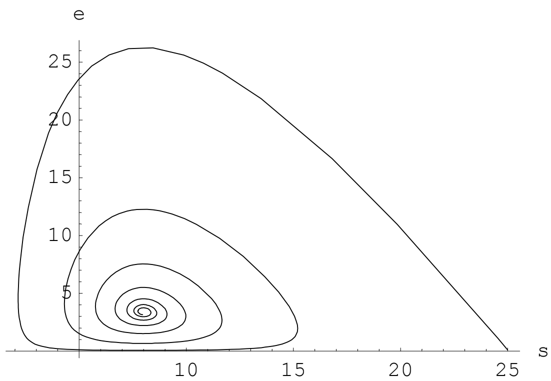
```
In[88]:= Show[linespoint, field, DisplayFunction -> $DisplayFunction];
```



Note in the following that we take the initial point as (*s*, *e*) = (25, 0.1).

```
In[89]:= sol = NDSolve[{s '[t] == 0.5 s[t] (25 - s[t]) - 2.5 e[t] s[t],
           e '[t] == 7.5 e[t] s[t] - 60 e[t], s[0] == 25, e[0] == 0.1}, {s, e}, {t, 0, 10}]

Out[89]= {{s → InterpolatingFunction[{{0., 10.}}, <>],
           e → InterpolatingFunction[{{0., 10.}}, <>]}}
```

```
In[90]:= traj = ParametricPlot[Evaluate[{s[t], e[t]} /. sol],
           {t, 0, 2}, PlotRange -> All, AxesLabel -> {"s", "e"}];
```



If we write the system as percentage changes then

$\dot{s}/s = 0.5\,(25 - s) - 2.5\, e = 12.5 - 0.5\, s - 2.5\, e$

$\dot{e}/e = 5[0.6\,(2.5)\, s - 12 = 60 + 7.5\, s$

which in matrix form can be expressed,

$$\begin{pmatrix} \dot{s}/s \\ \dot{e}/e \end{pmatrix} = \begin{pmatrix} 12.5 \\ 60 \end{pmatrix} + \begin{pmatrix} -0.5 & -2.5 \\ 7.5 & 0 \end{pmatrix}$$

so the matrix of the system is

$$\mathbf{A} = \begin{pmatrix} -0.5 & -2.5 \\ 7.5 & 0 \end{pmatrix}$$

```
In[91]:= matrixA = (-0.5  -2.5
                     7.5    0 )

Out[91]= {{-0.5, -2.5}, {7.5, 0}}

In[92]:= Eigenvalues[matrixA]

Out[92]= {-0.25 + 4.3229 ⅈ, -0.25 - 4.3229 ⅈ}

In[93]:= Sum[matrixA[[i, i]], {i, 1, 2}]

Out[93]= -0.5

In[94]:= Det[matrixA]

Out[94]= 18.75
```

Since tr(**A**) < 0 and det(**A**) > 0, and tr(**A**) < det(**A**), then the steady state is approached by a spiral path. This is also indicated by the fact that the eigenvalues are complex conjugate.

(viii)

As just revealed, the system approaches the steady state in a spiral fashion, approaching with a counter-clockwise movement. The higher the value of the parameter *v* the greater the entry and exit from the industry. This can have two implications. First, if the stock size remains positive, then the speed to the new steady state is quicker. However, with greater entry it is possible for the stock size to diminish to zero. The industry would collapse, and no steady state would be achieved!

à Question 6

(i)

```
In[95]:= f[s_] := 0.2 s (1 - s/1000)

In[96]:= Solve[D[f[s], s] == 0, s]

Out[96]= {{s → 500.}}

In[97]:= sdot = 0.2 s (1 - s/1000) - 0.125 e s

Out[97]= -0.125 e s + 0.2 (1 - s/1000) s

In[98]:= Solve[sdot == 0, e] /. s -> 500

Out[98]= {{e → 0.8}}
```

(ii)

```
In[99]:= g[e_] := 125/0.2 (0.2 - 0.125 e) e

In[100]:= Solve[D[g[e], e] == 0, e]

Out[100]= {{e → 0.8}}
```

which is a maximum since

```
In[101]:= D[g[e], {e, 2}]

Out[101]= -156.25
```

(iii)

```
In[102]:= Clear[sdot, r, a, e, s, p, w]
```

In[103]:= sdot = $r\,s\left(1-\frac{s}{k}\right)-a\,e\,s$

Out[103]= $-a\,e\,s+r\,s\left(1-\frac{s}{k}\right)$

```
In[104]:= Solve[sdot == 0, s]
```

Out[104]= $\left\{\{s\to 0\},\left\{s\to -\frac{k\,(a\,e-r)}{r}\right\}\right\}$

In[105]:= TR = $p\,a\,e\left(\frac{k\,(-a\,e+r)}{r}\right)$

Out[105]= $\frac{a\,e\,k\,p\,(-a\,e+r)}{r}$

```
In[106]:= TC = w e

Out[106]= e w

In[107]:= profit6 = TR - TC
```

Out[107]= $\frac{a\,e\,k\,p\,(-a\,e+r)}{r}-e\,w$

```
In[108]:= Solve[D[profit6, e] == 0, e]
```

Out[108]= $\left\{\left\{e\to \frac{r\,(a\,k\,p-w)}{2\,a^2\,k\,p}\right\}\right\}$

In[109]:= sstar = $\frac{k\,(-a\,e+r)}{r}$ /. $\left\{e \to \frac{r\,(a\,k\,p-w)}{2\,a^2\,k\,p}\right\}$

```
General::spell1 :
 Possible spelling error: new symbol name "sstar" is similar to existing symbol "estar".
```

Out[109]= $\frac{k\left(r-\frac{r\,(a\,k\,p-w)}{2\,a\,k\,p}\right)}{r}$

In[110]:= sols = Apart$\left[\text{Simplify}\left[\frac{k\left(r-\frac{r\,(a\,k\,p-w)}{2\,a\,k\,p}\right)}{r}\right]\right]$

```
General::spell1 :
 Possible spelling error: new symbol name "sols" is similar to existing symbol "sol".
```

Out[110]= $\frac{k}{2}+\frac{w}{2\,a\,p}$

```
In[111]:= values = sols /. {k -> 1000, w -> 10, p -> 0.4, a -> 0.125}

Out[111]= 600.
```

à Question 7

Equation (15.20) and (15.21) indicate that at the optimal stock size s^* we have

$$s^* = \frac{w}{a\,p}$$
$$e = (\frac{r}{a}) - (\frac{r}{k\,a})\,s^*$$

```
In[112]:= D[w / (a p), p]
```

$$Out[112]= -\frac{w}{a\,p^2}$$

so optimal stock size falls for a rise in price. Also

```
In[113]:= D[(r / a) - (r / (k a)) (w / (a p)), p]
```

$$Out[113]= \frac{r\,w}{a^2\,k\,p^2}$$

which is positive, so effort rises with a rise in price.

Similarly, considering a rise in the productivity of the fishing industry (a rise in the parameter a), then

```
In[114]:= D[w / (a p), p]
```

$$Out[114]= -\frac{w}{a\,p^2}$$

which is negative, so once again stock size declines. Also

```
In[115]:= D[(r / a) - (r / (k a)) (w / (a p)), a]
```

$$Out[115]= -\frac{r}{a^2} + \frac{2\,r\,w}{a^3\,k\,p}$$

This can be expressed,

$$\frac{r}{a^2}\,(\frac{2\,s^*}{k} - 1)$$

which is positive so long as $s^* > \frac{k}{2}$, i.e., so long as the optimal stock size is greater than half the maximum carrying capacity. Put another way, the the optimal stock size has to be greater than the stock size which leads to the maximum sustainable yield. This will be the case so long as profits can be made.

à Question 8

```
In[116]:= Clear[p, h, e, s, w, f, h]
```

The Hamiltonian is given by

$$H(e, s, t) = p\,h(e, s) - w\,e + \lambda(f(s) - h(e, s))$$

```
In[117]:= H = p h[e, s] - w e + λ (f[s] - h[e, s])

Out[117]= -e w + λ (f[s] - h[e, s]) + p h[e, s]
```

The first order conditions are:

$$\frac{\partial H}{\partial e} = 0$$

$$\dot{\lambda} = -\frac{\partial H}{\partial s} \qquad \dot{s} = f(s) - h(e, s)$$

```
In[118]:= Solve[D[H, e] == 0, w]
Out[118]= {{w → p h^(1,0)[e, s] - λ h^(1,0)[e, s]}}
```

Since $h^{(1,0)} = \frac{\partial h}{\partial e}$, then we have for the first of the first-order conditions,

$$(p - \lambda)\,\frac{\partial h}{\partial e} = w$$

```
In[119]:= D[H, s]
Out[119]= λ (f′[s] - h^(0,1)[e, s]) + p h^(0,1)[e, s]
```

Since $h^{(0,1)} = \frac{\partial h}{\partial s}$ then we have,

$$\dot{\lambda} = -\lambda\, f'(s) + \lambda\,\frac{\partial h}{\partial s} - p\,\frac{\partial h}{\partial s} = (p - \lambda)\,\frac{\partial h}{\partial s} - \lambda\, f'(s)$$

To summarise, the first-order conditions are:

(i) $(p - \lambda)\,\frac{\partial h}{\partial e} = w$

(ii) $\dot{\lambda} = (p - \lambda)\,\frac{\partial h}{\partial s} - \lambda\, f'(s)$

(iii) $\dot{s} = f(s) - h(e, s)$

à Question 9

```
In[120]:= mL = {{0, 3, 40}, {0.1, 0, 0}, {0, 0.5, 0}}
Out[120]= {{0, 3, 40}, {0.1, 0, 0}, {0, 0.5, 0}}
```

■ (i)

```
In[121]:= Eigenvalues[mL]
Out[121]= {1.33919, -0.669596 + 1.02229 ⅈ, -0.669596 - 1.02229 ⅈ}
```

Hence the growth rate is 34%.

■ (ii)

```
In[122]:= Eigenvectors[mL]
Out[122]= {{-0.996838, -0.0744358, -0.0277913},
           {0.996114 + 0. ⅈ, -0.0446617 - 0.0681862 ⅈ, -0.0133252 + 0.0305719 ⅈ},
           {0.996114 + 0. ⅈ, -0.0446617 + 0.0681862 ⅈ, -0.0133252 - 0.0305719 ⅈ}}

In[123]:= sumeig1 = -0.996838 - 0.0744358 - 0.0277913
Out[123]= -1.09907
```

```
In[124]:= {-0.996838, -0.0744358, -0.0277913} / sumeig1

Out[124]= {0.906987, 0.0677265, 0.0252863}

In[125]:= mH1 = {{1 / 4, 0, 0}, {0, 1 / 4, 0}, {0, 0, 1 / 4}}

Out[125]= {{1/4, 0, 0}, {0, 1/4, 0}, {0, 0, 1/4}}

In[126]:= mI = {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

Out[126]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}

In[127]:= mM1 = (mI - mH1) .mL

Out[127]= {{0., 2.25, 30.}, {0.075, 0., 0.}, {0., 0.375, 0.}}

In[128]:= mM1 // MatrixForm

Out[128]//MatrixForm=
  ( 0.     2.25   30. )
  ( 0.075  0.     0.  )
  ( 0.     0.375  0.  )

In[129]:= Eigenvalues[mM1]

Out[129]= {1.00439, -0.502197 + 0.766718 i, -0.502197 - 0.766718 i}

In[130]:= Eigenvectors[mM1]

Out[130]= {{-0.996838, -0.0744358, -0.0277913},
           {0.996114 + 0. i, -0.0446617 - 0.0681862 i, -0.0133252 + 0.0305719 i},
           {0.996114 + 0. i, -0.0446617 + 0.0681862 i, -0.0133252 - 0.0305719 i}}

In[131]:= sumeigM1 = -0.996838 - 0.0744358 - 0.0277913

    General::spell1 :
     Possible spelling error: new symbol name "sumeigM1" is similar to existing symbol "sumeig1".

Out[131]= -1.09907

In[132]:= {-0.996838, -0.0744358, -0.0277913} / sumeigM1

Out[132]= {0.906987, 0.0677265, 0.0252863}
```

group 1, 90.7%; group 2, 6.8%; group 3, 2.5%.

à Question 10

```
In[133]:= Solve[(1 - h) (3) (0.1) + (1 - h)^2 (40) (0.1) (0.5) == 1, h]

Out[133]= {{h → 0.363927}, {h → 1.78607}}
```

Since h must lie between 0 and 1, then let $h = 0.363927$.

```
In[134]:= mH2 = {{0, 0, 0}, {0, 0.363927, 0}, {0, 0, 0.363927}}

Out[134]= {{0, 0, 0}, {0, 0.363927, 0}, {0, 0, 0.363927}}
```

```
In[135]:= mM2 = (mI - mH2).mL

Out[135]= {{0., 3., 40.}, {0.0636073, 0., 0.}, {0., 0.318037, 0.}}

In[136]:= mM2 // MatrixForm

Out[136]//MatrixForm=
  ( 0.          3.        40. )
  ( 0.0636073   0.        0.  )
  ( 0.          0.318037  0.  )

In[137]:= Eigenvalues[mM2]

Out[137]= {1., -0.5 + 0.747782 i, -0.5 - 0.747782 i}

In[138]:= Eigenvectors[mM2]

Out[138]= {{-0.99778, -0.0634661, -0.0201845},
           {0.997199 + 0. i, -0.0391936 - 0.0586165 i, -0.00952547 + 0.0230384 i},
           {0.997199 + 0. i, -0.0391936 + 0.0586165 i, -0.00952547 - 0.0230384 i}}

In[139]:= sumeigM2 = -0.99778 - 0.0634661 - 0.0201845

Out[139]= -1.08143

In[140]:= {-0.99778, -0.0634661, -0.0201845} / sumeigM2

Out[140]= {0.922648, 0.0586872, 0.0186646}
```

Hence, h = 363927 with group 1, 92.2%; group 2, 5.9%; group 3, 1.9%.