7: Numerical analysis of shear instability

Here is a function that solves the Rayleigh equation in matrix form. You can choose impermeable or asymptotic boundary conditions. The function also outputs growth rate, vertical velocity eigenfunction, and various functions from the energy budget (needed later). These outputs pertain to either the fastest-growing mode or some other mode as selected via the input parameter nmode.

```
function [sig1,w1,uw1,SP1,EF1,K1,p1]=Ray(z,U,k,l,iBC,nmode)
%
% MINIMAL USAGE: [sig1]=Ray(z,U,k)
%    FULL USAGE: [sig1,w1,uw1,SP1,EF1,K1]=Ray(z,U,k,l,iBC,nmode)
%
% Stability analysis for inviscid, homogeneous, parallel shear flow
% (Rayleigh equation)
%
% INPUTS:
% z = vertical coordinate vector (evenly spaced)
% U = velocity profile
% k,l = wave vector components [default l=0]
% iBC = [1BC(1) iBC(2)]: boundary condition choice at z(0) and z(N+1), resp.
%   1 for impermeable boundary; 2 for asymptotic boundary; [default iBC=[1 1]]
% nmode = mode selection in terms of growth rate:
%       1 = fastest-growing mode [default nmode=1]
%       2 = second-fastest mode, etc.
%
% OUTPUTS:
% sig1 = complex growth rate
% w1 = vertical velocity eigfn
% uw1, SP1 = Reynolds stress and shear production
% EF1 = energy flux
% K1 = perturbation kinetic energy
%
% W. Smyth, Sep 2004, Jan 2012, Feb 2016

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stage 1: Preliminaries
%

% check for equal spacing
if abs(std(diff(z))/mean(diff(z)))>.000001
    disp(['ddz2: values not evenly spaced!'])
    sig1=NaN;
    return
end

% defaults
if nargin<4; l=0;end        % l=0
```

```
if nargin<5; iBC=[1 1];end  % impermeable BCs
if nargin<6; nmode=1;end    % choose FGM

% define constants
ii=complex(0.,1.);
del=z(2)-z(1);
N=length(z);
kt=sqrt(k^2+l^2);


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stage 2: Set up derivative matrices
%
D2=ddz2(z); % 2nd derivative matrix with 1-sided boundary terms
Uzz=D2*U;   % Uzz is computed BEFORE BCs are applied.

% Boundary conditions
% First assume impermeable boundaries
D2(1,:)=0;D2(1,1)=-2/del^2;D2(1,2)=1/del^2;
D2(N,:)=0;D2(N,N)=-2/del^2;D2(N,N-1)=1/del^2;
% Change to asymptotic boundaries if requested
if iBC(1)==2
    D2(1,:)=0;D2(1,1)=-2*(1+del*kt)/del^2;D2(1,2)=2/del^2;
end
if iBC(2)==2
    D2(N,:)=0;D2(N,N)=-2*(1+del*kt)/del^2;D2(N,N-1)=2/del^2;
end
% 2nd derivative matrix complete

% Laplacian matrix
L=D2-kt^2*eye(N);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stage 3: Set up stability matrices,
%          solve eigval problem, sort results

% Set up arrays for eigenvalue analysis.
A=L;
B=-ii*k*diag(U)*A+ii*k*diag(Uzz);

% Solve eigenvalue problem.
[w,S]=eig(B,A);
s=diag(S);

% Sort eigvals and eigvecs by real growth rate
[sr,ind]=sort(real(s),1,'descend');
sigma=s(ind);
w=w(:,ind);
```

```
% Save the mode selected via nmode
sig1=sigma(nmode);w1=w(:,nmode);

% Normalize by the value at the max of abs(w). This works better than using
% the value at z=0, which can turn out to be zero.
cnorm=w1(abs(w1)==max(abs(w1)));
w1=w1/cnorm;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Stage 4 (optional): Compute extra profiles if requested (2D modes only)
%

if nargout<3;return;end
% Compute horizontal velocity eigenfunction and perturbation kinetic energy
wz1=ddz(z)*w1;
u1=(ii/k)*wz1;
K1=(u1.*conj(u1)+w1.*conj(w1))/4;

% Compute shear production terms
uw1=real(u1.*conj(w1))/2;
Uz=ddz(z)*U;
SP1=-uw1.*Uz;

% Compute pressure eigfn and energy flux
p1=-(sig1+ii*k*U).*wz1/k^2 +(ii/k)*Uz.*w1;
EF1=real(p1.*conj(w1))/2;
return
end
```

Here is the “wrapper” that defines the problem and calls the Rayleigh subroutine.

```
% Test Rayleigh equation solver in 2nd order FD form
clear
close all

% plotting parameters
fs=18;
lw=2;
ms=18;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% problem 7b, tanh test
% define wave vector
k=.47;l=0;

% define grid
```

```
dz=0.2;
ztop=4;
zbot=-ztop;
z=[zbot+dz:dz:ztop-dz]'; % NOTE: BOUNDARIES ARE EXCLUDED!

% define background flow
U=tanh(z);

% compute growth rate, eigenfunction
[sg,w]=Ray(z,U,k,1);

% plot results
figure
subplot(1,3,1)
plot(real(w),z,'linewidth',lw)
hold on
plot(imag(w),z,'linewidth',lw)
legend('w_r','w_i')
legend boxoff
xlabel('w','fontsize',fs)
ylabel('z/h','fontsize',fs)
title(...
   sprintf('FGM: \\sigma = %.4f u_0/h',sg),...
   'fontsize',fs-2,'fontweight','normal')
set(gca,'fontsize',fs-2)

subplot(1,3,2)
plot(abs(w),z,'linewidth',lw)
xlabel('|w|','fontsize',fs)
set(gca,'yticklabel','','fontsize',fs-2)

subplot(1,3,3)
plot(phase(w),z,'linewidth',lw)
xlabel('\phi [rad]','fontsize',fs)
set(gca,'yticklabel','','fontsize',fs-2)
```

Figure 1 shows the growth rate and eigenfunction for the given parameter values.

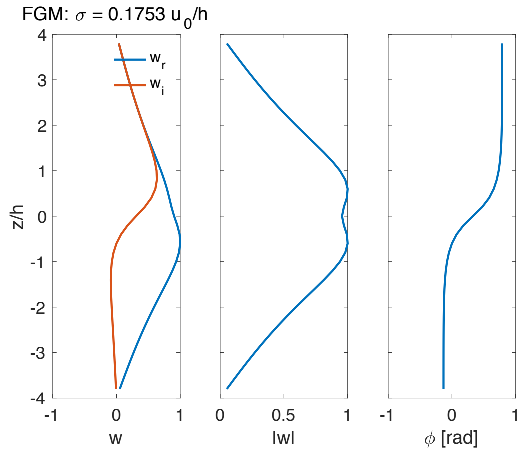

Figure 1: Growth rate and eigenfunction for a shear instability. (a) Real and imaginary parts. (b) Magnitude. (c) Phase.