Hints and Solutions to "Semigroups of Linear Operators with Applications to Analysis, Probability and Physics", by David Applebaum

Chapter 1

1. (a) For all $t \in [0, T], M, N \in \mathbb{N}, M < N$

$$\left|\left|\sum_{n=M}^{N} \frac{t^n}{n!} A^n\right|\right| \leq \sum_{n=M}^{N} \frac{t^n}{n!} ||A||^n.$$

From this we see that the series of interest is uniformly Cauchy, hence uniformly convergent on $[0, T]$. Absolute convergence follows from the fact that $\sum_{n=0}^{\infty} \frac{t^n}{n!} ||A||^n = e^{t||A||}$.

(b) (S2) is obvious and (S1) is proved in the same way as $e^{a+b} = e^a e^b$ for $a, b \in \mathbb{R}$. For norm continuity

$$||T_t - I|| \leq \sum_{n=1}^{\infty} \frac{t^n}{n!} ||A||^n = e^{t||A|} - 1 \to 0 \text{ as } t \to 0.$$

(c) Spectral theory defines $e^{tA} = \int_{\mathbb{R}} e^{t\lambda} dE(\lambda)$, where $(E(\lambda), \lambda \in \mathbb{R})$ is a resolution of the identity. Then for all $\phi, \psi \in E$, by Fubini's theorem,

$$\begin{aligned} \langle e^{tA}\phi, \psi\rangle &= \int_{\mathbb{R}} e^{t\lambda} d\langle E(\lambda)\phi, \psi\rangle \\ &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} d\langle E(\lambda)\phi, \psi\rangle \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}} \lambda^n d\langle E(\lambda)\phi, \psi\rangle \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \langle A^n\phi, \psi\rangle \\ &= \left\langle \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \phi, \psi \right\rangle, \end{aligned}$$

and the result follows. With suitable domain restrictions, a similar argument holds in the case where A is unbounded and self–adjoint.

2. (RI1) In the definition of the integral, just observe that if $a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$ is a partition of $[a, b]$, then $a + c = t_0 + c < t_1 + c < \cdots < t_n + c < t_{n+1} + c = b + c$ is a partition of $[a + c, b + c]$.

   (RI2) Choose a partition in which c is one of the points.

   (RI3) For any $\epsilon > 0$ there exists a partition $a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$ so that

$$\left|\left| \int_a^b \Phi(s)ds - \sum_{j=1}^{n+1} \Phi(u_j)(t_j - t_{j-1}) \right|\right| < \epsilon/2,$$

   where $t_{j-1} < u_j < t_j$, for all $j = 1, \ldots, n+1$. Since the mapping $t \to ||\Phi(t)||$ is continuous, there exists a partition $a = s_0 < s_1 < \cdots < s_m < s_{m+1} = b$ so that

$$\left| \int_a^b ||\Phi(s)||ds - \sum_{j=1}^{m+1} ||\Phi(v_j)||(s_j - s_{j-1}) \right| < \epsilon/2,$$

   where $s_{j-1} < v_j < s_j$, for all $j = 1, \ldots, m+1$. Taking the common partition, we can then assert that there exists a partition $a = r_0 < r_1 < \cdots < r_n < r_{N+1} = b$ so that

$$\begin{aligned} \left|\left| \int_a^b \Phi(s)ds \right|\right| &\leq \left|\left| \sum_{j=1}^{N+1} \Phi(u_j)(r_j - r_{j-1}) \right|\right| + \epsilon/2 \\ &\leq \sum_{j=1}^{N+1} ||\Phi(u_j)||(r_j - r_{j-1}) + \epsilon/2 \\ &\leq \int_a^b ||\Phi(s)||ds + \epsilon, \end{aligned}$$

   and the result follows.

   (RI4) By continuity, given any $\epsilon > 0$ there exists $\delta > 0$ so that if $t < s < t + \delta$, then $||\Phi(s) - \Phi(t)|| < \epsilon$. Define $F(t) := \int_0^t \Phi(s)ds$, then if $0 < h < \delta$, $F(t + h) - F(t) = \int_t^{t+h} \Phi(s)ds$,

$$\left|\left| \frac{F(t+h) - F(t)}{h} - \Phi(t) \right|\right| \leq \frac{1}{h} \int_t^{t+h} ||\Phi(s) - \Phi(t)||ds < \epsilon,$$

   and the result follows.

(1.3.6) follows by using partitions, and the fact that

$$X\sum_{j=1}^{n+1}\Phi(u_j)(t_j - t_{j-1}) = \sum_{j=1}^{n+1}X\Phi(u_j)(t_j - t_{j-1}).$$

3. Suppose that D is a dense subspace of E, and $\lim_{t\to 0}||T_t\psi - \psi|| = 0$ for all $\psi \in D$. Now given any $\phi \in E$, and $\epsilon > 0$ there exists $\psi \in D$ so that $||\phi - \psi|| < \epsilon$. Then for $t \leq 1$ (using (1.2.4)

$$\begin{aligned} ||T_t\phi - \phi|| &\leq ||T_t\phi - T_t\psi|| + ||T_t\psi - \psi|| + ||\psi - \phi|| \\ &\leq (||T_t|| + 1)||\psi - \phi|| + ||T_t\psi - \psi|| \\ &\leq C||\psi - \phi|| + ||T_t\psi - \psi||, \end{aligned}$$

where $C := M(e^a \vee 1) + 1$, and the result follows.

4. In $C_0(\mathbb{R})$ this follows immediately by the fact that all functions therein are uniformly continuous. In L^p, we choose $D = C_c(\mathbb{R})$ and $f \in C_c(\mathbb{R})$ with $\mathrm{supp}(f) = K$. Then K is bounded so there exists $A \geq 0$ such that $|x| \leq A$ for all $x \in K$. Hence if $0 \leq t \leq 1, |x + t| \leq A + 1$ for all $x \in K$. Since all functions in $C_c(\mathbb{R})$ are uniformly continuous we have

$$\begin{aligned} ||T_t f - f||_p^p &= \int_{\mathbb{R}} |f(x+t) - f(x)|^p dx \\ &\leq 2\sup_{x\in\mathbb{R}} |f(x+t) - f(x)|(A+1) \to 0 \text{ as } t \to 0. \end{aligned}$$

5. For the case of $S + T$, let $(\psi_n, n \in \mathbb{N})$ be a sequence in D_T which converges to $\psi \in E$, and such that the sequence $((S+T)\psi_n, n \in \mathbb{N})$ converges to $\phi \in E$. Now S is bounded and so $S\psi_n \to S\psi$ as $n \to \infty$. Hence $T\psi_n \to \phi - S\psi$, But T is closed, and so $\psi \in D_{S+T} = D_T$ and $\phi = (S+T)\psi$. The result follows.

   For the case of ST, let $(\psi_n, n \in \mathbb{N})$ converge to ψ, as before and assume that $(ST\psi_n, n \in \mathbb{N})$ converges to $\chi \in E$. If $S = 0$ the result is obvious, so assume this is not the case. The sequence $(ST\psi_n, n \in \mathbb{N})$ is Cauchy, hence given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n > N, ||ST(\psi_m - \psi_n)|| < \epsilon$. But then $||T(\psi_m - \psi_n)|| < \epsilon/||S||$. So $(T\psi_n, n \in \mathbb{N})$ is Cauchy, hence convergent to $\xi \in E$ (say). Now T is closed, hence $\psi \in D_T$ and $\xi = T\psi$. But S is bounded and so $(ST\psi_n, n \in \mathbb{N})$ converges to $ST\psi$, and we are done.

   If T is closable, then so are $S + T$ and ST. TS only has meaning if $\mathrm{Ran}(S) \subseteq \mathrm{Dom}(T)$. In that case, if T is closed, then TS is closed, by a similar argument to that given above. But Dom(TS) =Dom$(T) = E$, and so TS is bounded by the closed graph theorem. A variation on the same argument establishes that TS is bounded, even if T is only closable.

6. (a) (S1) and (S2) are easy. (S3)$'$ follows from writing, for $\psi \in E$,

$$||T_t^{(1)}T_t^{(2)}\psi - \psi|| \leq ||T_t^{(1)}||||T_t^{(2)}\psi - \psi|| + ||T_t^{(1)}\psi - \psi||,$$

   and using $||T_t^{(1)}|| \leq M_1 e^{tc_1}$ within the first term.

   (b) For $\psi \in D_{A_1+A_2}$, use

$$\begin{aligned} & \left|\left| \frac{T_t^{(1)}T_t^{(2)}\psi - \psi}{t} - A_1\psi - A_2\psi \right|\right| \\ = & \left|\left| T_t^{(1)} \left(\frac{T_t^{(2)}\psi - \psi}{t} - A_2\psi \right) \right|\right| \\ + & \left|\left| \frac{T_t^{(1)}\psi - \psi}{t} - A_1\psi \right|\right| + ||T_t^{(1)}A_2\psi - A_2\psi||, \end{aligned}$$

   and again apply $||T_t^{(1)}|| \leq M_1 e^{tc_1}$ within the first term.

7. As $(e^{-ct}, t \geq 0)$ is a C_0–semigroup in $\mathbb{R}$, then $(e^{-ct}I, t \geq 0)$ is a C_0–semigroup in E and the fact that $(T_t, t \geq 0)$ is a C_0–semigroup is a direct consequence of Problem 6. The generator is $A - cI$ with domain D_A.

8. (a) Let (f_n) is any sequence in D converging to $f \in E$. For $m, n \in \mathbb{N}$,

$$||Af_m - Af_m|| \leq K||f_n - f_m||,$$

   from which we deduce that the sequence (Af_n) is Cauchy, hence convergent. Let $h = \lim_{n\to\infty} Af_n$. We define the required extension by $\tilde{A}f = h$. It is clearly well–defined, for if (g_n) is any other sequence in D converging to f then

$$||Af_n - Ag_n|| \leq K||f_n - g_n|| \to 0 \text{ as } n \to \infty.$$

   It is bounded as

$$||\tilde{A}f|| = \lim_{n\to\infty} ||Af_n|| \leq K \lim_{n\to\infty} ||f_n|| = K||f||.$$

   Finally it is straightforward to check that $\tilde{A}$ is linear.

(b) If A is densely defined, then $\overline{D_A} = E$, but if D_A is closed then $\overline{D_A} = D_A$.

(c) If the semigroup is norm continuous, then its generator A is bounded and hence $D_A = E$ which is closed. Conversely if D_A is closed, then $D_A = E$ by (b). But A is closed, so its graph $G_A = \{(\psi, A\psi); \psi \in E\}$ is closed. Hence by the closed graph theorem, A is bounded, and so generates a norm continuous semigroup.

9. Suppose that $(\psi_n, X\psi_n)$ is a Cauchy sequence under the graph norm where $\psi_n \in D_X$ for all $n \in \mathbb{N}$. Then it is easy to see that (ψ_n) and $(X\psi_n)$ are both Cauchy sequences in E. But then $\psi_n \to \psi \in E$ and $X\psi_n \to \phi \in E$ as $n \to \infty$. However X is closed and so $\psi \in D_X$ and $\phi = X\psi$. But then $(\psi_n, X\psi_n)$ converges to $(\psi, X\psi) \in G_X$ as $n \to \infty$ (with respect to the graph norm), and the result follows.

10. The result holds for $n = 1$ by Theorem 1.5.1 (2). Suppose it is true for some $n \in \mathbb{N}$, then for $f \in E$, by (1.3.6) and Fubini's theorem

$$\begin{aligned} R_z^{n+1} f &= \int_0^\infty e^{-zs} T_s \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} T_t f dt ds \\ &= \int_0^\infty \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-z(s+t)} T_{s+t} f dt ds \\ &= \int_0^\infty \frac{t^{n-1}}{(n-1)!} \int_t^\infty e^{-zu} T_u f du dt \\ &= \int_0^\infty \frac{t^n}{n!} e^{-zt} T_t f dt, \end{aligned}$$

where we used integration by parts to obtain the final line.

Chapter 2

1. (a) By Lemmas 2.1.1 and 2.1.2, we have

$$||\lambda R_\lambda x - x|| = ||A R_\lambda x|| \to 0 \text{ as } \lambda \to \infty.$$

(b) This is true for $n = 1$. Now assume it holds for some $n \in \mathbb{N}$. Now if $x \in D_{A^{n-1}}, R_\lambda x \in D_{A^n}$ for

$$A^n R_\lambda x = A R_\lambda A^{n-1} x = A^{n-1} x + \lambda R_\lambda A^{n-1} x.$$

Now given any $y \in E$ and $\epsilon > 0$, by hypothesis there exists $x \in D_{A^{n-1}}$ such that $||x - y|| < \epsilon/2$, and by (a), for sufficiently large λ, $||\lambda R_\lambda x - x|| < \epsilon/2$. Hence by the triangle inequality, $||\lambda R_\lambda x - y|| < \epsilon$, and the result follows.

2. Recall the definition of the dissipativity set $\mathcal{E}_X$. We will show that if $(f, \phi) \in \mathcal{E}_X$ then $\phi = f$. Let $\phi \in H = V \oplus V^\perp$, where $V :=$linspan$\{f\}$, so $\phi = \lambda f + \psi$, where $\psi \in V^\perp$. We must have $\langle f, \phi \rangle = 1$, which holds if and only if $\lambda = 1$. We must also have $||\phi||^2 = ||f||^2 + ||\psi||^2 = 1$, which holds if and only if $\psi = 0$. The result follows.

3. (a) $\Rightarrow$ (b). First observe that

$$\Re\langle(\lambda I - X)f, f\rangle = -\Re\langle Xf, f\rangle + \Re(\lambda)||f||^2 \geq \Re(\lambda)||f||^2.$$

Then by the Cauchy–Schwarz inequality,

$$\begin{aligned} ||(\lambda I - X)f||\,||f|| &\geq \Re\langle(\lambda I - X)f, f\rangle \\ &\geq \Re(\lambda)||f||^2. \end{aligned}$$

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a).

$$\begin{aligned} \lambda^2||f||^2 &\leq ||(\lambda I - X)f||^2 \\ &= \lambda^2||f||^2 - 2\lambda\Re\langle Xf, f\rangle + ||Xf||^2 \end{aligned}$$

and so for all $\lambda > 0, ||Xf||^2 \geq 2\lambda\Re\langle Xf, f\rangle$. If $\Re\langle Xf, f\rangle > 0$, we can just take $\lambda > ||Xf||^2/2\lambda\Re\langle Xf, f\rangle$ to obtain a contradiction.

4. $||(X+I)f|| \leq ||(X-I)f||$ if and only if $||(X+I)f||^2 \leq ||(X-I)f||^2$ if and only if (after expanding) $2\Re\langle Xf, f\rangle \leq -2\Re\langle Xf, f\rangle$, and the result follows.

5. (a) This follows from $V_tV_{-t} = V_{-t}V_t = I$.

   (b) The fact that $(T_t^+, t \geq 0)$ and $(T_t^-, t \geq 0)$ are C_0–semigroups is easy. We only compute the generator of $(T_t^-, t \geq 0)$, as the argument for $(T_t^+, t \geq 0)$ is similar. For all $\psi \in D_B$,

$$\begin{aligned} \lim_{t\to 0} \frac{T_t^-\psi - \psi}{t} &= \lim_{t\to 0} \frac{V_{-t}\psi - \psi}{t} \\ &= -\lim_{t\to 0} \frac{V_{-t}\psi - \psi}{-t} = -B\psi. \end{aligned}$$

   For all $s, t \geq 0$,

$$T_s^+T_t^- = V_sV_{-t} = V_{-t}V_s = T_t^-T_s^+.$$

(c) Applying the Feller–Miyadera–Phillips theorem to $(T_t^+, t \geq 0)$, we have $\{\lambda \in \mathbb{C}; \Re(\lambda > a\} \subseteq \rho(B)$ and for all $\lambda > a$,

$$||R_\lambda(B)||^n \leq \frac{M}{(\lambda - a)^n}.$$

We have $\lambda \in -\rho(B)$ if and only if $-\lambda \in \rho(B)$ since

$$R_{-\lambda}(B) = (-\lambda I - B)^{-1} = -(\lambda I + B)^{-1} = -R_\lambda(B),$$

so $\{\lambda \in \mathbb{C} : \Re(\lambda) < -a\} \subseteq \rho(-B) = -\rho(B)$ and for $\lambda < -a$,

$$||R_\lambda(B)||^n = ||R_{-\lambda}(-B)||^n \leq \frac{M}{(-\lambda - a)^n} = \frac{M}{(|\lambda| - a)^n}.$$

6. (a) As stated in the question, this is just the proof of Theorem 2.2.1.

(b) This follows by a (careful) limiting argument from the easily checked fact that for all $s, t \geq 0$,

$$e^{sB_\lambda} e^{t(-B)_\lambda} = e^{t(-B)_\lambda} e^{sB_\lambda}.$$

(c) We obtain a C_0 semigroup by Problem 1.6, and the action of the generator on D_B is $B + -B = 0$. It then follows that for all $t \geq 0$,

$$T_t^+ T_t^- = T_t^- T_t^+ = I,$$

as required.

(d) This is now straightforward, for example to establish the group property in the case where $t \geq 0, s < 0, t + s > 0$, observe that

$$V_t V_s = T_t^+ T_{-s}^- = T_{t+s}^+ T_{-s}^+ T_{-s}^- = T_{t+s}^+ = V_{t+s}.$$

7. Using the Cauchy–Schwarz inequality for both sums and integrals, we

get

$$\begin{aligned}
& |B[u,v]| \\
\leq \;& \int_U \sum_{i,j=1}^d |a_{ij}(x)||\partial_i u(x)||\partial_j v(x)|dx + \\
+ \;& \int_U \sum_{i=1}^d |b_i'(x)||\partial_i u(x)||v(x)|dx + \int_U |c(x)|u(x)||v(x)|dx \\
\leq \;& \int_U \left(\sum_{i,j=1}^d |a_{ij}(x)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^d |\partial_i u(x)|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^d |\partial_j v(x)\right)^{\frac{1}{2}} dx + \\
+ \;& \int_U \left(\sum_{i=1}^d |b_i'(x)|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^d |\partial_i u(x)|^2\right)^{\frac{1}{2}} |v(x)|dx \\
+ \;& \int_U |c(x)|u(x)||v(x)|dx \\
\leq \;& \left(\sum_{i,j=1}^d ||a_{ij}||_\infty^2\right)^{\frac{1}{2}} \left(\int_U \sum_{i=1}^d |\partial_i u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_U \sum_{j=1}^d |\partial_j v(x)|^2 dx\right)^{\frac{1}{2}} + \\
+ \;& \left(\sum_{i=1}^d ||b_i'||_\infty^2\right)^{\frac{1}{2}} \left(\int_U \sum_{i=1}^d |\partial_i u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_U |v(x)|^2 dx\right)^{\frac{1}{2}} \\
+ \;& ||c||_\infty \left(\int_U |u(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_U |v(x)|^2 dx\right)^{\frac{1}{2}},
\end{aligned}$$

and the result follows from here, essentially by replacing each term involving u and v with its Sobolev norm.

8. I am not going to write out the solution to this. There are plenty of hints in the question, and of course, the proof can be found in full details in Evans' book, which is widely available. I might change my mind if there are howls of anguish from readers!

Chapter 3

1. For $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$, we have

$$\begin{aligned}
\Phi_t(y) &= \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot y} e^{-\frac{|x|^2}{4t}} dx \\
&= \phi_t(y_1) \cdots \phi_t(y_d),
\end{aligned}$$

where $\phi_t(y_j) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{ix_j y_j} e^{-\frac{x_j^2}{4t}} dx$. Writing $x = x_j$ and $y = y_j$ for convenience we have, on differentiating with respect to y and using dominated convergence,

$$\phi_t'(y) = \frac{2it}{(4\pi t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} e^{ixy} \frac{x}{2t} e^{-\frac{x^2}{4t}} dx,$$

and after integration by parts, we get the differential equation,

$$\phi_t'(y) = -2ty, \text{ with initial condition } \phi_t(0) = 1.$$

The unique solution is $\phi_t(y) = e^{-ty^2}$, and the result follows.

2. If $y < 0$, we integrate over the contour $(-R, R) \cup \Delta_R$, where Δ_R is a circle of radius R in the lower half plane. We need to reverse orientation to proceed in a clockwise direction, and this introduces a factor of -1 to keep track of. Bearing that in mind, we must take account of the residue at $z = -it$, and so the required result is obtained by calculating

$$-2\pi i \lim_{z \to -it} \frac{t}{\pi} \frac{e^{iyz}}{z^2 + t^2} = e^{ty} = e^{-t|y|}.$$

By continuity, we find that

$$\Phi_t(0) = \lim_{y \to 0} \Phi_t(y) = 1,$$

and so $\int_{\mathbb{R}} c_t(x)dx = \Phi_t(0) = 1$. Of course, this last fact is also easy to verify directly using elementary calculus.

3. For associativity, for all $f \in B_b(\mathbb{R}^d)$ it is an easy calculation to see that

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)((\mu_1 * \mu_2) * \mu_3)(dx) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y + z)\mu_1(dx)\mu_2(dy)\mu_3(dz) \\ &= \int_{\mathbb{R}^d} f(x)(\mu_1 * (\mu_2 * \mu_3))(dx). \end{aligned}$$

For commutativity, by Fubini's theorem

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)(\mu_1 * \mu_2)(dx) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x + y)\mu_1(dx)\mu_2(dy) \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y + x)\mu_2(dy)\mu_1(dx) \\ &= \int_{\mathbb{R}^d} f(x)(\mu_2 * \mu_1)(dx). \end{aligned}$$

4. For all $g \in B_b(\mathbb{R}^d)$, by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^d} g(x)(\mu_1 * \mu_2)(dx) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x+y)\mu_1(dx)\mu_2(dy) \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x+y)f_1(x)f_2(y)dxdy \\ &= \int_{\mathbb{R}^d} g(x)\left(\int_{\mathbb{R}^d} f_1(x-y)f_2(y)dy\right)dx \\ &= \int_{\mathbb{R}^d} g(x)(f_1 * f_2)(x)dx,\end{aligned}$$

and the result follows.

If μ_2 is not absolutely continuous, but μ_1 remains so, a variant on the above argument yields

$$\int_{\mathbb{R}^d} g(x)(\mu_1 * \mu_2)(dx) = \int_{\mathbb{R}^d} g(x)\int_{\mathbb{R}^d} f_1(x-y)\mu_2(dy)dx,$$

and so $\mu_1 * \mu_2$ is absolutely continuouus with respect to Lebesgue measure, with density

$$(f_1 * \mu_2)(x) := \int_{\mathbb{R}^d} f_1(x-y)\mu_2(dy).$$

5. Write $\eta = \eta_1 + \eta_2 + \eta_3$, where for all $y \in \mathbb{R}^d$,

$$\eta_1(y) = ib \cdot y - ay \cdot y$$

,

$$\begin{aligned}\eta_2(y) &= \int_{B_1}(e^{ix\cdot y} - 1 - ix\cdot y)\nu(dx), \\ \eta_3(y) &= \int_{B_1^c}(e^{ix\cdot y} - 1)\nu(dx).\end{aligned}$$

We have

$$|\eta_1(y)| \leq |b\cdot y| + |ay\cdot y|,$$

and $|b\cdot y| \leq |b||y| \leq \frac{|b|}{2}(1+|y|^2)$. By repeated use of Cauchy's inequality for sums, we get

$$|ay\cdot y| \leq \sum_{i,j=1}^{d}|a_{ij}||y_i||y_j| \leq \left(\sum_{i,j=1}^{d}|a_{ij}|^2\right)^{\frac{1}{2}}|y|^2.$$

Using the hint we obtain

$$|e^{ix\cdot y} - 1 - ix\cdot y| = \frac{\theta}{2}|x\cdot y|^2 \leq \frac{1}{2}|x|^2|y|^2,$$

and so

$$|\eta_2(y)| \leq \left(\frac{1}{2}\int_{B_1}|x|^2\nu(dx)\right)|y|^2.$$

Finally, an easy estimate yields

$$|\eta_3(y)| \leq 2\nu(B_1^c),$$

and the required result follows from combining the estimates for η_1, η_2 and η_3.

6.

$$\begin{aligned} Af(x) &= -\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{ix\cdot y}(\sqrt{a^2+y^2}-a)\widehat{f}(y)dy \\ &= -(\sqrt{a-\Delta}-aI). \end{aligned}$$

When $a = 0$, we return to the Cauchy process/Poisson kernel world.

7. Since $x \to (1+|x|)|f(x)|$ is bounded, there exists $K \geq 0$ such that for all $x \in \mathbb{R}^d$,

$$|f(x)| \leq \frac{K}{1+|x|}.$$

Hence $\limsup_{|x|\to\infty}|f(x)| = 0$, and so $f \in C_0(\mathbb{R}^d)$.

For the L^p case, observe that for all $m \in \mathbb{Z}_+, \sup_{x\in\mathbb{R}^d}(1+|x|^m)|f(x)| < \infty$, and

$$||f||_p^p \leq \left(\sup_{x\in\mathbb{R}^d}(1+|x|^m)|f(x)|\right)^p \int_{\mathbb{R}^d}\frac{1}{(1+|x|^m)^p}dx.$$

The integral is finite. Indeed, when $d = 1$, choose $m = 2$ and the result is straightforward. When $d > 1$, standard results on integrals of radial functions yield

$$\int_{\mathbb{R}^d}\frac{1}{(1+|x|^m)^p}dx = \omega_{d-1}\int_0^\infty \frac{r^{d-1}}{(1+r^m)^p}dr,$$

where $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ is the surface area of the $(d-1)$ sphere. As the mapping $r \to \frac{r^{d-1}}{(1+r^m)^p}$ is continuous on $[0,1]$, its sufficient to consider

$$\int_1^\infty \frac{r^{d-1}}{(1+r^m)^p}dr \leq \int_1^\infty r^{d-1-mp}dr,$$

which is finite provided we choose $m > d/p$.

8. (a)

$$U_\lambda(\mathbb{R}^d) = \int_0^\infty e^{-\lambda t} dt = \frac{1}{\lambda}.$$

(b) By Fubini's theorem,

$$\begin{aligned} R_\lambda f(x) &= \int_0^\infty e^{-\lambda t} T_t f(x) dt \\ &= \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^d} f(x+y) \mu_t(dy) dt \\ &= \int_{\mathbb{R}^d} f(x+y) \left(\int_0^\infty e^{-\lambda t} \mu_t(dy) dt \right) \\ &= \int_{\mathbb{R}^d} f(x+y) U_\lambda(dy). \end{aligned}$$

(c) By Fubini's theorem again,

$$\begin{aligned} U_\lambda(A) &= \int_0^\infty e^{-\lambda t} \int_A \rho_t(x) dx dt \\ &= \int_A \int_0^\infty e^{-\lambda t} \rho_t(x) dt dx \\ &= \int_A u_\lambda(x) dx. \end{aligned}$$

(d) $u_0(x) = \int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}} dt$. Substitute $s = |x|^2/2t$ to obtain, after some cancellation,

$$u_0(x) = \frac{1}{2(\pi)^{\frac{d}{2}} |x|^{d-2}} \int_0^\infty s^{d/2-2} e^{-s} ds,$$

and the result follows from the definition of the gamma function.

Chapter 4

1. (a) Let x be the corresponding eigenvector, and normalise it so that $||x|| = 1$. Then

$$\lambda = \lambda ||x||^2 = \langle Tx, x \rangle == \langle x, T^* x \rangle = \overline{\lambda} ||x||^2 = \overline{\lambda},$$

and so $\lambda \in \mathbb{R}$.

(b) We show that there exists no non–zero $\phi \in D_T$ such that $T\phi = \pm i\phi$. For convenience, we just work with the case $T\phi = i\phi$, as the argument in the other case is the same. Assume that such a ϕ exists. Arguing as in (a) we have

$$i\langle \phi, \phi \rangle = \langle T\phi, \phi \rangle = \langle \phi, T\phi \rangle = -i\langle \phi, \phi \rangle.$$

Hence $2i||\phi||^2 = 0$, which implies $\phi = 0$, and we have the desired contradiction.

(c) Let $x = (T - \lambda I)u$ and $y = (T - \lambda I)v$, then a straightforward calculation shows that

$$\langle Ux, Uy \rangle = \langle (T - \overline{\lambda} I)u, (T - \overline{\lambda} I)v \rangle = \langle (T - \lambda I)u, (T - \lambda I)v \rangle = \langle x, y \rangle.$$

2. (a) $\Rightarrow$ (b) is proved in Problem 4 1(b) above (noting that every self–adjoint operator is closed, as the adjoint of a densely defined operator in a Hilbert space is always closed).

For (b) $\Rightarrow$ (c), we begin by assuming that $\mathrm{Ran}(T - iI)$ is not dense in H. Then there exists a non–zero $\psi \in \mathrm{Ran}(T - iI)^{\perp}$. Hence for all $\phi \in D_T$ we have $\langle \psi, T\phi - i\phi \rangle = 0$. But then $\psi \in D_{T^*}$, and $\langle T^*\psi + i\psi, \phi \rangle = 0$. But since D_T is dense in H, it follows that $\psi \in \mathrm{Ker}(T^* + iI)$ and we have our desired contradiction. Next we show that $\mathrm{Ran}(T - iI)$ is closed. First observe that for all $\phi \in D_T$, by an easy calculation

$$||(T - iI)\phi||^2 = ||T\phi||^2 + ||\phi||^2.$$

Now let (ϕ_n) be a sequence in D_T such that the sequence $((T - iI)\phi_n)$ converges to $f \in H$. From the identity in the last display, it follows that (ϕ_n) converges to some $g \in H$ and $(T\phi_n)$ converges to some $h \in H$. But T is closed, and so $g \in D_T$ and $h = Tg$. It follows $f = (T - iI)g$, and so $\mathrm{Ran}(T - iI)$ is closed. Since it is also dense, we must have $\mathrm{Ran}(T - iI) = H$. The result for $\mathrm{Ran}(T + iI)$ is proved in the same way.

For (c) $\Rightarrow$ (a). Let $\phi \in D_{T^*}$. Since $\mathrm{Ran}(T - iI) = H$, there exists $\psi \in D_T$ such that

$$T\psi - i\psi = T^*\phi - i\phi.$$

Now T is symmetric and so $D_T \subseteq D_{T^*}$. Then it follows from the last display that $(T^* - iI)(\phi - \psi) = 0$. But $\mathrm{Ran}(T + iI) = H$ implies that $\mathrm{Ker}(T^* - iI) = \{0\}$ and so $\phi = \psi \in D_T$. Hence $D_{T^*} \subseteq D_T$ and so T is self–adjoint.

3. By the essential criterion for self–adjointness $\mathrm{Ran}(T \pm iI) = H$ and so by the result of Problem 1(c), U is an isometry mapping H onto H. Hence it is bijective, and so unitary. The same holds for all pure imaginary λ, as a straightforward generalisation of the last problem shows that T is self–adjoint if and only if $\mathrm{Ran}(T \pm \lambda I) = H$ for all such λ. The converse is valid, i.e. if the Cauchy transform of T is unitary for $\lambda = i$, then T is self–adjoint. For this, see Akheizer and Glazman, pp.266–9. It is possible to go further, and show that T is self–adjoint if and only if $\mathrm{Ran}(T \pm \lambda I) = H$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, but we will not pursue that here (see e.g. Akheizer and Glazman, Volume II, pp. 351–4).

4. We must show that the graph G_T is closed. Let $((\phi_n, T\phi_n)), n \in \mathbb{N})$ be a sequence in G_T converging to $(\phi, \psi) \in H \times H$. Fix $f \in H$.

$$\begin{aligned} \langle \psi, f \rangle &= \lim_{n \to \infty} \langle T\phi_n, f \rangle \\ &= \lim_{n \to \infty} \langle \phi_n, Tf \rangle \\ &= \langle \phi, Tf \rangle \\ &= \langle T\phi, f \rangle. \end{aligned}$$

Since f is arbitrary, we deduce that $\psi = T\phi$. Hence G_T is closed, and the result follows.

5. It is clearly symmetric and its domain is the whole of H. Hence by the Hellinger–Toeplitz theorem, it is bounded and self–adjoint. By Theorems 4.1.2 and 4.1.3, M_f will generate a self–adjoint contraction semigroup if and only if $M_f = -B$ where $M \geq 0$. This holds if and only if for all $\psi \in H, \langle B\psi, \psi \rangle \geq 0$, i.e $-\int_{\mathbb{R}^d} f(x)|\psi(x)|^2 dx \geq 0$, i.e. if and only if $f = -g$ where $g \geq 0$.

6. This follows directly from Theorem 4.2.1 (iv).

7. We have for all $n \in \mathbb{N}, X_1, \ldots, X_n, Y_1, \ldots, Y_n \in \mathcal{L}(H)$,

$$\sum_{i,j=1}^{n} Y_i^* U_t X_i X_j^* U_t^* Y_j = \left(\sum_{i=1}^{n} Y_i^* U_t X_i \right) \left(\sum_{i=1}^{n} X_j^* U_t^* Y_j \right).$$

The term on the right hand of the display is of the form A^*A and so is positive, as required.

Chapter 5

1. Let $X \in \mathcal{K}(H), T \in \mathcal{L}(H)$. Let (f_n) be a bounded sequence in H. Then $(Xf_n, n \in \mathbb{N})$ has a convergent subsequence, and so $(TXf_n, n \in \mathbb{N})$ also has a convergent subsequence, hence $TX \in \mathcal{K}(H)$. Furthermore (Tf_n) is a bounded sequence in H. So (XTf_n) has a convergent subsequence. Thus $XT \in \mathcal{K}(H)$.

2. (a) Let $t = t_0 + h$ where $h > 0$. Then since T_h is bounded, we have $T_t = T_{t_0}T_h$ is compact by Problem 5.1.

   (b) Let C be any closed interval in $[0, \infty)$ and let B be the unit ball in E. then since T_{t_0} is compact, the set $\overline{T_{t_0}(B)}$ is compact. Applying the hint, for all $t \in C$ we get

$$\lim_{s \to t} ||T_t f - T_s f|| = \lim_{s \to t} ||(T_{t-t_0} f - T_{s-t_0})T_{t_0} f|| = 0,$$

   uniformly for $f \in B$. Hence

$$\lim_{s \to t} ||T_t - T_s|| = \lim_{s \to t} \sup_{f \in B} ||T_t f - T_s f|| = 0.$$

3. (a) Let (f_n) be another such basis. Since T is a positive self–adjoint operator, it has a positive self–adjoint square–root S. Then by Parseval's identity, and Fubini's theorem

$$\begin{aligned} \sum_{n \in \mathbb{N}} \langle Tf_n, f_n \rangle &= \sum_{n \in \mathbb{N}} ||Sf_n||^2 \\ &= \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} |\langle Sf_n, e_m \rangle|^2 \\ &= \sum_{m \in \mathbb{N}} \sum_{n \in \mathbb{N}} |\langle f_n, Se_m \rangle|^2 \\ &= \sum_{m \in \mathbb{N}} ||Se_m||^2 \\ &= \sum_{m \in \mathbb{N}} \langle Te_m, e_m. \rangle \end{aligned}$$

(b) If (f_n) is another basis, as above, then by Fourier expansion,

$$\begin{aligned}\sum_{n\in\mathbb{N}}\langle Te_n, e_n\rangle &= \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}}\langle Te_n, f_m\rangle\langle f_m, e_n\rangle \\ &= \sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}\langle f_m, e_n\rangle\langle e_n, T^* f_m\rangle \\ &= \sum_{m\in\mathbb{N}}\langle f_m, T^* f_m\rangle \\ &= \sum_{m\in\mathbb{N}}\langle Tf_m, f_m\rangle.\end{aligned}$$

The interchange of the two infinite series is a consequence of Fubini's theorem, using the facts that any trace–class operator may be written as the product of two Hilbert–Schmidt operators, and that the adjoint of a Hilbert–Schmidt operator is itself Hilbert–Schmidt.

4. (a) If $A \in \mathcal{L}(H)$, write $A = A_1 + iA_2$, where $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2i}(A - A^*)$.

(b) If $U = A \pm i\sqrt{I - A^2}$ then $U^* = A \pm i\sqrt{I - A^2}$ and it is easy to check that $UU^* = U^*U = I$.

(c) Write

$$\begin{aligned}A &= \frac{||A_1||}{2}\left(\frac{A_1}{||A_1||} + i\sqrt{\left(I - \frac{A_1^2}{||A_1^2||}\right)}\right) + \frac{||A_1||}{2}\left(\frac{A_1}{||A_1||} - i\sqrt{\left(I - \frac{A_1^2}{||A_1^2||}\right)}\right) \\ &+ i\frac{||A_2||}{2}\left(\frac{A_2}{||A_2||} + i\sqrt{\left(I - \frac{A_2^2}{||A_2^2||}\right)}\right) + i\frac{||A_2||}{2}\left(\frac{A_2}{||A_2||} - i\sqrt{\left(I - \frac{A_2^2}{||A_2^2||}\right)}\right).\end{aligned}$$

(d) First assume that A is trace–class and B is unitary, We have

$$\mathrm{tr(AB)} = \sum_{\mathrm{n}\in\mathbb{N}}\langle \mathrm{ABe_n, e_n}\rangle.$$

We obtain another complete orthonormal basis by defining $f_n = Be_n$ for all $n \in \mathbb{N}$, and then we have

$$
\begin{aligned}
\mathrm{tr(AB)} &= \sum_{n\in\mathbb{N}} \langle ABB^* f_n, B^* f_n \rangle \\
&= \sum_{n\in\mathbb{N}} \langle A f_n, B^* f_n \rangle \\
&= \sum_{n\in\mathbb{N}} \langle BA f_n, f_n \rangle \\
&= \mathrm{tr(BA)}.
\end{aligned}
$$

The result extends to arbitrary bounded B by using (c).

5. (a) Let $\mu \neq \lambda \in \rho(X)$. The the resolvent identity yields

$$R_\mu = R_\lambda + (\lambda - \mu) R_\lambda R_\mu,$$

and this is compact by the ideal property (see Problem 5.1).

(b) If X is bounded, then so is $\lambda I - X$. But then $I = (\lambda I - X)(\lambda I - X)^{-1} = (\lambda I - X)R_\lambda$ is compact, by the ideal property, and this is only possible if $\dim(E) < \infty$.

6. The resolvent $(\lambda I - A)^{-1}$ is compact and self–adjoint, so it has a discrete spectrum by the Hilbert–Schmidt theorem (Theorem 5.1.1). But then $\lambda I - A$ has a discrete spectrum (indeed its eigenvalues are the inverses of those of the resolvent), and it follows that A has a discrete spectrum too.

7. For all $f, g \in L^2(S^1), t \geq 0$,

$$
\begin{aligned}
\langle \widetilde{T}_t f, g \rangle &= \int_{S^1} \widetilde{T}_t f([x]) \overline{g([x])} d[x] \\
&= \int_{\mathbb{R}} T_t (f \circ \natural)(x), \overline{(g \circ \natural)(x)} dx \\
&= \int_{\mathbb{R}} (f \circ \natural)(x), \overline{T_t (g \circ \natural)(x)} dx \\
&= \int_{S^1} f([x]) \overline{\widetilde{T}_t g([x])} d[x] \\
&= \langle f, \widetilde{T}_t g \rangle.
\end{aligned}
$$

Chapter 6

1. Since $||R_{1/t}|| \leq t$ for all $t > 0$, it is clear that $F(t)$ is a contraction. For all $\psi \in D_A$ we have, using the fact that $R_{1/t}\psi = \int_0^\infty e^{-s/t}T_s\psi ds$, and dominated convergence,

$$\begin{aligned}
\lim_{t\to 0}\frac{\frac{1}{t}R_{1/t}\psi - \psi}{t} &= \lim_{t\to 0}\frac{1}{t^2}\int_0^\infty e^{-s/t}(T_s\psi - \psi)ds \\
&= \lim_{t\to 0}\frac{1}{t}\int_0^\infty e^{u}(T_{tu}\psi - \psi)du \\
&= \int_0^\infty ue^{u}\lim_{t\to 0}\frac{1}{tu}(T_{tu}\psi - \psi)du \\
&= \left(\int_0^\infty ue^{u}du\right)A\psi = A\psi,
\end{aligned}$$

and the result follows when Chernoff's product formula is applied.

2. We have $T_t g = \lim_{n\to\infty} F(t/n)^n g$ for all $g \in D$, where D is dense in E. Let $f \in E$, then given any $\epsilon > 0$, there exists $g \in D$ so that $||f - g|| < \epsilon/3$. Then for sufficiently large n,

$$\begin{aligned}
& ||T_t f - F(t/n)^n f|| \\
\leq\ & ||T_t f - T_t g|| + ||T_t g - F(t/n)^n g|| + ||F(t/n)^n g - F(t/n)^n f|| \\
\leq\ & 2||f - g|| + ||T_t g - F(t/n)^n g|| \\
\leq\ & 2\epsilon/3 + \epsilon/3 = \epsilon,
\end{aligned}$$

where we have used the fact that T_t and $F(t/n)^n$ are contractions.

3. Since B is relatively bounded with respect to A, we have for all $f \in D_A$,

$$\begin{aligned}
||Af|| &= ||(A+B)f - Bf|| \\
&\leq ||(A+B)f|| + ||Bf|| \\
&\leq ||(A+B)f|| + a||Af|| + b||f||
\end{aligned}$$

It follows that

$$-b||f|| + (1-a)||Af|| \leq ||(A+B)f|| \leq ||Af|| + ||Bf|| \leq (a+1)||Af|| + b||f||,$$

and so

$$b||f|| + (1-a)||Af|| \leq ||(A+B)f|| + 2b||f|| \leq ||Af|| + ||Bf|| \leq (a+1)||Af|| + 3b||f||.$$

The result follows since the graph norm of $A + B$ is itself equivalent to that given by $f \to ||(A+B)f|| + 2b||f||$.

4. (a) First note that since A is self–adjoint, $-i\lambda \in \rho(A)$ and so $A + i\lambda I$ is invertible. We compute

$$||(A + i\lambda I)\phi||^2 = ||A\phi||^2 + \lambda^2||\phi||^2.$$

Now take $\phi = (A + i\lambda I)^{-1}\psi$, to get

$$||\psi||^2 = ||A(A + i\lambda I)^{-1}\psi||^2 + \lambda^2||(A + i\lambda I)^{-1}\psi||^2,$$

and the required estimates follow easily from here.

(b) Since B is relatively bounded with respect to A, we have

$$||B\phi|| \leq a||A\phi|| + b||\phi||,$$

and taking $\phi = (A + i\lambda I)^{-1}\psi$ as before we get

$$\begin{aligned} ||X\psi|| &\leq a||A(A + i\lambda I)^{-1}\psi|| + b||(A + i\lambda I)^{-1}\psi|| \\ &\leq (a + b/\lambda)||\psi||, \end{aligned}$$

using the estimates of (a).

(c) Take $\lambda > 2b/(1 - 2a)$ and then we get $||X\psi|| < \frac{1}{2}||\psi||$ for all $\psi \in H$ and so $||X|| < 1$. Then the spectral radius $r(X) = \lim_{n\to\infty} ||X^n||^{1/n} \leq ||X|| < 1$, so $-1 \notin \sigma(X)$.

(d) As $-1 \in \rho(X)$, then $I + X$ is invertible and so $\mathrm{Ran}(I + X) = H$.

(e) This follows from the identity, as both $I + X$ and $A + i\lambda I$ are invertible. Hence $\mathrm{Ran}((I + X)(A + i\lambda I)) = H$.