Chapter 6: Mean Value Theorems

Part B: Mensuration, Numerical Integration

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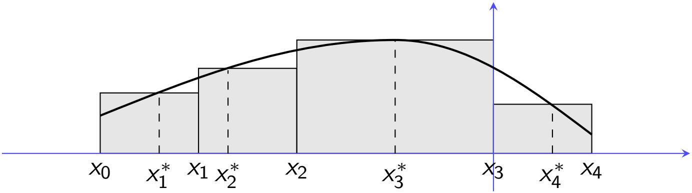
Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_1, \ldots, x_n\}$ of $[a, b]$ be a partition of $[a, b]$.
A **tag** is a choice of points $x_i^* \in (x_{i-1}, x_i)$ for each $i = 1, \ldots, n$.
A **tagged partition** P^* is a partition P along with a tag.
Each tagged partition creates a **Riemann sum**

$$R(f, P^*) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}),$$

which we view as an approximation to the integral of f over $[a, b]$.



A Riemann Sum

Riemann Sums and Integrals

Theorem 1

Let f be integrable on $[a, b]$. Then for every $\epsilon > 0$ there is a partition P_ϵ such that for every refinement Q of P_ϵ and for every tag of Q we have $|\int_a^b f(x)\,dx - R(f, Q^*)| < \epsilon$.

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Proof. By the Riemann Condition, we have step functions s, t which satisfy $s \leq f \leq t$ on $[a, b]$ and $\int_a^b t(x)\,dx - \int_a^b s(x)\,dx < \epsilon$.

Riemann Sums and Integrals

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Proof. By the Riemann Condition, we have step functions s, t which satisfy $s \le f \le t$ on $[a,b]$ and $\int_a^b t(x)\,dx - \int_a^b s(x)\,dx < \epsilon$.
Let P_ϵ be a partition which is adapted to s and t. $\square$

Riemann Sums and Integrals

Theorem 2

Suppose $f : [a, b] \to \mathbb{R}$ is integrable, $I \in \mathbb{R}$, and every partition P of $[a, b]$ has been allotted a tag such that the tagged partitions P^* satisfy the following: For every $\epsilon > 0$ there is a partition P'_ϵ such that for every refinement P of P'_ϵ we have $|I - R(f, P^*)| < \epsilon$. Then $I = \int_a^b f(x)\, dx$.

Riemann Sums and Integrals

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Suppose $f : [a, b] \to \mathbb{R}$ *is integrable,* $I \in \mathbb{R}$*, and every partition* P *of* $[a, b]$ *has been allotted a tag such that the tagged partitions* P^* *satisfy the following: For every* $\epsilon > 0$ *there is a partition* P'_ϵ *such that for every refinement* P *of* P'_ϵ *we have* $|I - R(f, P^*)| < \epsilon$*. Then* $I = \int_a^b f(x)\,dx$*.*

Proof. Fix $\epsilon > 0$. Let P_ϵ be the corresponding partition obtained from the previous theorem.

Riemann Sums and Integrals

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Proof. Fix $\epsilon > 0$. Let P_ϵ be the corresponding partition obtained from the previous theorem.
Let $P = P_\epsilon \cup P'_\epsilon$.

Riemann Sums and Integrals

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Proof. Fix $\epsilon > 0$. Let P_ϵ be the corresponding partition obtained from the previous theorem.
Let $P = P_\epsilon \cup P'_\epsilon$.
Let P^* be the tagged partition obtained from the current hypothesis.
Then we have $|I - R(f, P^*)| < \epsilon$ and $|\int_a^b f(x)\,dx - R(f, P^*)| < \epsilon$.

Riemann Sums and Integrals

Theorem 2

Suppose $f : [a, b] \to \mathbb{R}$ is integrable, $I \in \mathbb{R}$, and every partition P of $[a, b]$ has been allotted a tag such that the tagged partitions P^* satisfy the following: For every $\epsilon > 0$ there is a partition P'_ϵ such that for every refinement P of P'_ϵ we have $|I - R(f, P^*)| < \epsilon$. Then $I = \int_a^b f(x)\,dx$.

Proof. Fix $\epsilon > 0$. Let P_ϵ be the corresponding partition obtained from the previous theorem.
Let $P = P_\epsilon \cup P'_\epsilon$.
Let P^* be the tagged partition obtained from the current hypothesis.
Then we have $|I - R(f, P^*)| < \epsilon$ and $|\int_a^b f(x)\,dx - R(f, P^*)| < \epsilon$.
It follows that $|I - \int_a^b f(x)\,dx| < 2\epsilon$. Since this is true for every $\epsilon > 0$, we must have $I = \int_a^b f(x)\,dx$. □

Riemann Sums and Products

The following result is useful in dealing with products. We'll need it when we discuss surface area.

Theorem 3

Consider continuous functions $f, g\colon [a,b] \to \mathbb{R}$. For any $\epsilon > 0$ we can choose a partition $P = \{x_1, \ldots, x_n\}$ of $[a,b]$ such that for every choice of points $u_i, v_i \in [x_{i-1}, x_i]$ we have

$$\left| \int_a^b f(x)g(x)\,dx - \sum_{i=1}^{n} f(u_i)g(v_i)(x_i - x_{i-1}) \right| < \epsilon.$$

(Proof on next slide)

Riemann Sums and Products

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Proof. Choose a partition $P_{\epsilon/2}$ such that every tag of every refinement Q satisfies $\left|\int_a^b f(x)g(x)\,dx - R(fg, Q^*)\right| < \epsilon/2$.

Riemann Sums and Products

Proof. Choose a partition $P_{\epsilon/2}$ such that every tag of every refinement Q satisfies $\left| \int_a^b f(x)g(x)\,dx - R(fg, Q^*) \right| < \epsilon/2$.

Let M be an upper bound of $|g|$ over $[a, b]$. By the Small Span Theorem, there is a refinement $P = \{x_1, \ldots, x_n\}$ of $P_{\epsilon/2}$ such that $|f(u_i) - f(v_i)| < \epsilon/2M(b-a)$ whenever $u_i, v_i \in [x_{i-1}, x_i]$.

Riemann Sums and Products

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Proof. Choose a partition $P_{\epsilon/2}$ such that every tag of every refinement Q satisfies $\left| \int_a^b f(x)g(x)\,dx - R(fg, Q^*) \right| < \epsilon/2$.

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Let the v_i be considered as a tag of P. Then

$$\begin{aligned}
&\left| R(fg, P^*) - \sum_{i=1}^n f(u_i)g(v_i)(x_i - x_{i-1}) \right| \\
&\qquad = \left| \sum_{i=1}^n f(v_i)g(v_i)(x_i - x_{i-1}) - \sum_{i=1}^n f(u_i)g(v_i)(x_i - x_{i-1}) \right| \\
&\qquad = \left| \sum_{i=1}^n (f(v_i) - f(u_i))g(v_i)(x_i - x_{i-1}) \right| \\
&\qquad \le \sum_{i=1}^n \frac{\epsilon}{2M(b-a)} M(x_i - x_{i-1}) = \epsilon/2.
\end{aligned}$$

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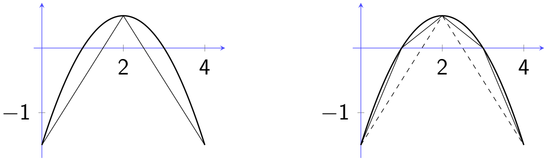
Arc Length

We shall define the length of the curve formed by the graph of a function $f : [a, b] \to \mathbb{R}$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$, and set $y_i = f(x_i)$. The total length of the line segments between successive points (x_i, y_i) on the graph is

$$S(f, P) = \sum_{i=1}^{n} ((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2)^{1/2}.$$

If Q is a refinement of P, the triangle inequality in the plane implies that $S(f, P) \leq S(f, Q)$. Therefore it is reasonable to define the **arc length** of the graph of f by

$$S(f) = \sup\{\, S(f, P) \mid P \text{ is a partition of } [a, b] \,\}.$$



Arc Length

The existence of the arc length $S(f)$ is not guaranteed. It is possible, even with f continuous, that the collection $S(f, P)$ is unbounded.

An example of non-existence of $S(f)$ is provided by the function $x\cos(\pi/2x)$.

We call f **rectifiable** if $S(f)$ exists.

The following observation is useful in arranging calculations:

Theorem 4 (Additivity of Arc Length)

Suppose $f\colon [a,b] \to \mathbb{R}$ and $g\colon [b,c] \to \mathbb{R}$ are rectifiable and $f(b) = g(b)$. Define $h\colon [a,c] \to \mathbb{R}$ by $h(x) = f(x)$ if $x \leq b$ and $h(x) = g(x)$ if $x \geq b$. Then h is rectifiable and $S(h) = S(f) + S(g)$.

Proof. Similar to the additivity of integrals over intervals. □

Rectifiability of Differentiable Functions

Theorem 5

Let $f : [a, b] \to \mathbb{R}$ be continuously differentiable. Then f is rectifiable and the arc length of its graph is given by

$$S(f) = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Rectifiability of Differentiable Functions

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Proof. Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, let us write $y_i = f(x_i)$, $\triangle x_i = x_i - x_{i-1}$ and $\triangle y_i = y_i - y_{i-1}$. Then we have

$$S(f, P) = \sum_{i=1}^{n} ((\triangle x_i)^2 + (\triangle y_i)^2)^{1/2} = \sum_{i=1}^{n} \left(1 + \left(\frac{\triangle y_i}{\triangle x_i}\right)^2\right)^{1/2} \triangle x_i.$$

Rectifiability of Differentiable Functions

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The Mean Value Theorem gives $x_i^* \in (x_{i-1}, x_i)$ such that
$f'(x_i^*) = \triangle y_i / \triangle x_i$.
(continued . . .)

Rectifiability of Differentiable Functions

(. . . continued)
Taking these x_i^* as tags, the $S(f, P)$ become Riemann sums:

$$S(f, P) = \sum_{i=1}^{n} \sqrt{1 + f'(x_i^*)^2}\, \triangle x_i = R(\sqrt{1 + (f')^2}, P^*).$$

Rectifiability of Differentiable Functions

(...continued)
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The function $\sqrt{1 + (f')^2}$ is continuous and hence integrable. By Theorem 1 there is a partition P_1 such that every refinement Q of P_1 satisfies $|\int_a^b \sqrt{1 + f'(x)^2}\, dx - S(f, Q)| =$
$|\int_a^b \sqrt{1 + f'(x)^2}\, dx - R(\sqrt{1 + (f')^2}, Q^*)| < 1.$

Rectifiability of Differentiable Functions

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(…continued)
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The function $\sqrt{1+(f')^2}$ is continuous and hence integrable. By Theorem 1 there is a partition P_1 such that every refinement Q of P_1 satisfies $|\int_a^b \sqrt{1+f'(x)^2}\,dx - S(f,Q)| = |\int_a^b \sqrt{1+f'(x)^2}\,dx - R(\sqrt{1+(f')^2},Q^*)| < 1$. Hence, for an arbitrary partition P,

$$S(f,P)\le S(f,P\cup P_1) < \int_a^b \sqrt{1+f'(x)^2}\,dx + 1.$$

It follows that the collection of all $S(f,P)$ is bounded above and f is rectifiable.
(continued …)

Rectifiability of Differentiable Functions

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(...continued)

Given any $\epsilon > 0$ there is a partition P'_ϵ such that $S(f) - S(f, P'_\epsilon) < \epsilon$.

If P is a refinement of P'_ϵ, then

$$S(f) - R(\sqrt{1 + (f')^2}, P^*) = S(f) - S(f, P) \leq S(f) - S(f, P'_\epsilon) < \epsilon.$$

By Theorem 2, $S(f) = \int_a^b \sqrt{1 + f'(x)^2}\, dx$. □

Example: Catenary

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Example 6

Consider the hyperbolic cosine function $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

We have $1 + (\cosh' x)^2 = 1 + (\sinh x)^2 = \cosh^2 x$.

Therefore the length of the graph over an interval $[0, a]$ is

$$\int_0^a \sqrt{1 + (\cosh' x)^2}\, dx = \int_0^a \cosh x\, dx = \sinh a.$$

Example: Parabola

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Example 7

Let $f(x) = x^2$. The length of the graph over $[0, a]$ is

$$S = \int_0^a \sqrt{1 + (2x)^2}\, dx = \frac{1}{2}\int_0^{2a} \sqrt{1 + x^2}\, dx.$$

One option is to substitute $x = \tan\theta$. This leads to the integral of $\sec^3\theta$, which we have carried out earlier and is quite complicated. More pleasant results are obtained by substituting $x = \sinh t$. Then we have $\sqrt{1 + x^2} = \cosh t$ and $dx = \cosh t\, dt$. This gives

$$S = \frac{1}{2}\int_0^b \cosh^2 t\, dt = \frac{1}{2}\int_0^b \frac{1 + \cosh 2t}{2}\, dt \qquad (b = \sinh^{-1} 2a)$$

$$= \frac{1}{4}\left(b + \frac{\sinh 2b}{2}\right) = \frac{b + \sinh b \cosh b}{4} = \frac{\sinh^{-1} 2a + 2a\sqrt{1 + 4a^2}}{4}.$$

Example: Circular Arcs

Example 8

A semicircle of radius R is obtained as the graph of $f(x) = \sqrt{R^2 - x^2}$ with $x \in [-R, R]$. We have

$$f'(x) = \frac{-x}{\sqrt{R^2 - x^2}} \quad \text{and} \quad \sqrt{1 + f'(x)^2} = \frac{R}{\sqrt{R^2 - x^2}}.$$

Note that f is not differentiable at $x = \pm R$ so we cannot apply the integral formula in one go for the length of the semicircle.
Let us begin with the length of an arc whose central angle θ is less than $\pi/2$. The arc is obtained as the graph of f restricted to $[0, R\sin\theta]$.
Hence its length is

$$\int_0^{R\sin\theta} \sqrt{1 + f'(x)^2}\,dx = \int_0^{R\sin\theta} \frac{R}{\sqrt{R^2 - x^2}}\,dx = R\int_0^{\sin\theta} \frac{1}{\sqrt{1 - x^2}}\,dx$$
$$= R\arcsin x\Big|_0^{\sin\theta} = R\theta.$$

Example: Circular Arcs

Any circular arc can be cut into congruent pieces each of which has central angle less than $\pi/2$.

Combined with the additivity of arc length, this extends the formula $R\theta$ to arbitrary θ.

In particular, we recover the description of π as the ratio of a circle's circumference to its diameter. (Recall that we *defined* π as the ratio of a circle's area to its squared radius.)

Example: Ellipse

Consider an ellipse, given by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (a > b > 0)$. An arc of the ellipse can be viewed as the graph of the function

$$f(x) = \frac{b}{a}\sqrt{a^2 - x^2}.$$

This leads to the integral

$$\int \sqrt{1 + f'(x)^2}\, dx = \int \sqrt{\frac{a^2 + (b/ a^2 - 1)x^2}{a^2 - x^2}}\, dx$$

$$= a \int \sqrt{\frac{1 + e^2 w^2}{1 - w^2}}\, dw \qquad (w = x/a)$$

where $e = (a^2 - b^2)^{1/2}/a$ is called the **eccentricity** of the ellipse and measures its deviation from a circle. This integral turns out to be inexpressible in terms of any combination of our standard functions. We have to treat it as a new function!

Surface of Revolution

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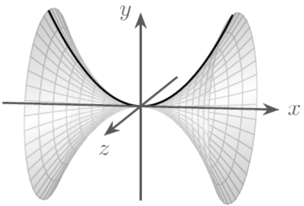
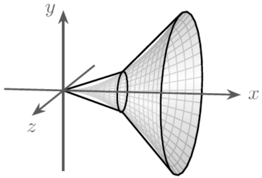
A **surface of revolution** is obtained by taking a curve in the xy-plane and rotating it about the x-axis in three dimensions. The diagram given below shows the result of rotating the graph of $y = x^2$.



The surfaces of revolution include many of our familiar shapes such as cylinders, cones and spheres.

Surface Area

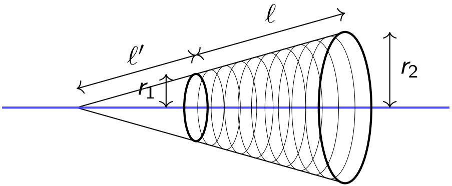
We shall use frustums of cones to describe the surface area of a surface of revolution. Given the graph of a function $y = f(x)$ we first approximate it by line segments. Then we rotate these line segments around the x-axis to create frustums of cones that approximate the surface of revolution.



You may recall that the surface area of a right circular cone with radius r and slant height ℓ is $\pi r\ell$. This formula is obtained by cutting the cone along a generator and flattening it into a sector of a circle.

Surface Area of Frustum of a Cone

Consider the frustum of a cone with end radii r_1, r_2 and slant height ℓ. Visualise the full cone of which the frustum is a part.



Let ℓ' be the slant height of the conical cap that completes the frustum. Using similar triangles, we get

$$\frac{\ell + \ell'}{\ell'} = \frac{r_2}{r_1}, \text{ hence } \ell' = \frac{r_1}{r_2 - r_1}\ell.$$

And now we compute the surface area F of the frustum:

$$F = \pi r_2(\ell + \ell') - \pi r_1 \ell' = \pi(r_1 + r_2)\ell.$$

This includes the limiting case of a cylinder, when $r_1 = r_2$.

Area of Surface of Revolution

Consider a continuously differentiable $f : [a, b] \to [0, \infty)$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$. On each $[x_{i-1}, x_i]$ we approximate the graph of $y = f(x)$ by the line segment joining its endpoints, and the surface of revolution by the frustum of a cone with end radii $f(x_{i-1})$ and $f(x_i)$. We add up the areas of these frustums to get

$$A(f, P) = \sum_{i=1}^{n} \pi(f(x_{i-1}) + f(x_i))\sqrt{(\triangle x_i)^2 + (\triangle y_i)^2},$$

where $\triangle x_i = x_i - x_{i-1}$ and $\triangle y_i = f(x_i) - f(x_{i-1})$. For each $[x_{i-1}, x_i]$:

- Use IVT to get a $u_i \in (x_{i-1}, x_i)$ such that $f(u_i) = (f(x_{i-1}) + f(x_i))/2$.
- Use MVT to get a $v_i \in (x_{i-1}, x_i)$ such that $f'(v_i) = \triangle y_i / \triangle x_i$.

This gives $A(f, P) = 2\pi \sum_{i=1}^{n} f(u_i)\sqrt{1 + f'(v_i)^2}\, \triangle x_i$. By Theorem 3 the numbers $A(f, P)$ approach $A(f) = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$.

Example: Sphere

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Example 9

A sphere of radius R can be obtained by rotating the graph of the function $f\colon [-R, R] \to [0, R]$ defined by $f(x) = \sqrt{R^2 - x^2}$. We have

$$1 + f'(x)^2 = 1 + \frac{x^2}{R^2 - x^2} = \frac{R^2}{R^2 - x^2}.$$

We see that $\sqrt{1 + f'(x)^2}$ is unbounded on $[-R, R]$ and so we cannot use the integral formula to find the area of the entire sphere directly. But we can use it to find the area of the part lying over any interval $[a, b]$ with $-R < a < b < R$:

$$2\pi \int_a^b \sqrt{R^2 - x^2}\,\frac{R}{\sqrt{R^2 - x^2}}\,dx = 2\pi(b - a)R.$$

To get the area of the entire sphere we now let $a \to -R$ and $b \to R$, giving $4\pi R^2$.

Example: Catenoid

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Example 10

The surface created by rotating a catenary is called a **catenoid**. We can compute its surface area:

$$2\pi \int_0^a \cosh x \sqrt{1 + (\cosh' x)^2}\, dx = 2\pi \int_0^a \cosh^2 x\, dx$$
$$= \pi \int_0^a (1 + \cosh 2x)\, dx = \pi(a + \frac{\sinh 2a}{2}).$$

The catenoid turns out to be the surface of minimal area between two rings centred on the same axis. Let us at least see an example where a catenoid has less area than the frustum of a cone between two rings. Consider the catenoid obtained by rotating the catenary over $[0, 1]$. Its area is $\pi(1 + \sinh(2)/2) \approx 8.8$. Now consider the frustum of a cone between the same circles. Its area is

$$\pi(1 + \cosh 1)\sqrt{1 + (\cosh 1 - 1)^2} \approx 9.1.$$

Solid of Revolution

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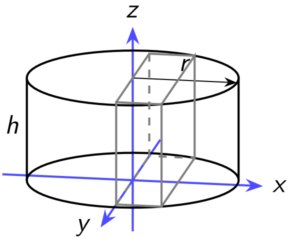
The region enclosed by a surface of revolution is called a **solid of revolution**. We shall obtain its volume by approximating it by a bunch of cylinders.

We take define the volume of a cuboid to be the product of its three dimensions, i.e., base area times height. Let us fit a cuboid inside a cylinder whose height is h and base radius is r.



If we stack several cuboids in this way inside the cylinder, their total volume is their total base area times the height h. As we fill the cylinder with thinner cuboids, the total volume of the cuboids approaches $\pi r^2 h$. This gives $\pi r^2 h$ as the volume of the cylinder.

Volume of a Solid of Revolution

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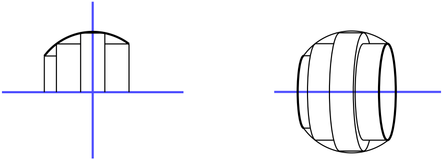
Consider a continuous $f : [a, b] \to \mathbb{R}$ with $f \geq 0$. Let s be a step function such that $0 \leq s(x) \leq f(x)$ on $[a, b]$. Let $P = \{x_0, \ldots, x_n\}$ be a partition adapted to s. Rotating the graph of f around the x-axis creates a solid of revolution. Rotating the graph of s creates a collection of coaxial cylinders. If s_i is the value of s on (x_{i-1}, x_i), then the total volume of the cylinders, $\sum_{i=1}^{n} \pi s_i^2 \triangle x_i$, is an underestimate for the volume.



Similarly, a step function t which lies above f, creates cylinders with total volume $\sum_{i=1}^{n} \pi t_i^2 \triangle x_i$ and this is an overestimate for the volume of the solid of revolution. The integral $\int_a^b \pi f(x)^2 \, dx$ is the unique number lying above all the underestimates and below all the overestimates. We take it as the volume of the solid of revolution. This approach is called the **discs method** as it visualizes the solid as made of thin coaxial discs.

Examples: Sphere and Cone

Example 11

A solid sphere of radius R can be obtained as the solid of revolution obtained by rotating the graph of $f(x) = \sqrt{R^2 - x^2}$, $-R \leq x \leq R$. Its volume is especially easy to calculate:

$$\int_{-R}^{R} \pi(R^2 - x^2)\,dx = \left(R^2 x - \frac{x^3}{3}\right)\Big|_{-R}^{R} = \frac{4}{3}\pi R^3.$$

Example 12

A solid cone of base radius R and height h can be obtained as the solid of revolution obtained by rotating the graph of $f(x) = \frac{R}{h}x$, $0 \leq x \leq h$. Its volume is

$$\int_0^h \pi\left(\frac{R}{h}x\right)^2 dx = \pi\frac{R^2 x^3}{3h^2}\Big|_0^h = \frac{1}{3}\pi R^2 h.$$

Example: Torus

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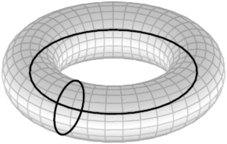
Example 13

A **torus** is obtained by rotating a disc of radius r around a circle of larger radius R:



It can be generated by rotating the region lying between the graphs of $y = R + \sqrt{r^2 - x^2}$ and $y = R - \sqrt{r^2 - x^2}$, $-r \le x \le r$.

$$\begin{aligned} \text{Volume} &= \int_{-r}^{r} \pi(R + \sqrt{r^2 - x^2})^2 \, dx - \int_{-r}^{r} \pi(R - \sqrt{r^2 - x^2})^2 \, dx \\ &= \pi \int_{-r}^{r} \left((R + \sqrt{r^2 - x^2})^2 - (R - \sqrt{r^2 - x^2})^2\right) dx \\ &= 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx = 2\pi^2 R r^2. \end{aligned}$$

Gabriel's Horn

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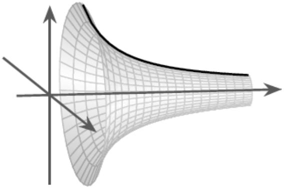
Consider the function $f(x) = 1/x$ with $x \geq 1$. Rotate its graph about the x-axis.



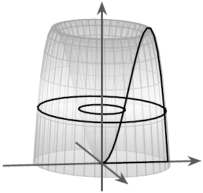
The volume enclosed by the horn is $\int_1^\infty \pi \frac{1}{x^2}\, dx = -\pi \lim_{b\to\infty} \frac{1}{x}\Big|_1^b = \pi$.

$$S = \int_1^\infty 2\pi \frac{1}{x}\sqrt{1 + (-1/x^2)^2} dx \geq \int_1^\infty 2\pi \frac{1}{x} dx = 2\pi \lim_{b\to\infty} \log x\Big|_1^b = \infty.$$

The surface area is infinite. This is often presented as a paradox. Since the horn has infinite area, we need an infinite amount of paint to paint it. But its volume is finite! Do you agree?

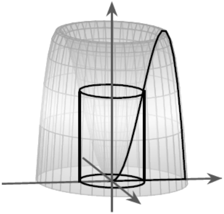
Shells Method

A solid of revolution can also be created by rotating a region in the xy-plane about the y-axis.



If we attempt to find the volume of such a solid by the discs method, we have to use horizontal discs, and finding their radii involves inverting the function $y = f(x)$ whose graph bounds the rotated region.

It is usually easier to view the solid as made of concentric cylindrical shells:



Shells Method

Suppose we rotate the region lying under the graph of the function $y = f(x)$ over the interval $[a, b]$ with $0 \leq a < b$. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$ with a tag $x_1*, \ldots, x_n^*$. Consider the cylindrical shell with height $f(x_i^*)$, inner radius x_{i-1} and outer radius x_i. The volume of this shell is

$$\pi x_i^2 f(x_i^*) - \pi x_{i-1}^2 f(x_i^*) = 2\pi x_i^{**} f(x_i^*) \triangle x_i, \quad \text{where } x_i^{**} = \frac{x_{i-1} + x_i}{2}.$$

The total volume of the cylindrical shells is

$$\sum_{i=1}^{n} 2\pi x_i^{**} f(x_i^*) \triangle x_i.$$

By Theorem 3, these approach $\int_a^b 2\pi x f(x)\, dx$ as we take finer partitions.

Example

Example 14

The volcano shaped solid we have been using to illustrate the shell method is generated by the region lying under the graph of $y = x^2 - x^4$ with $0 \leq x \leq 1$.

Therefore, its volume is

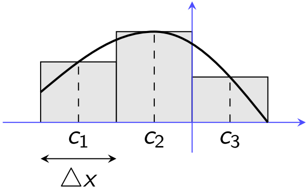
$$\int_0^1 2\pi x(x^2 - x^4)\,dx = 2\pi\left(\frac{x^4}{4} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{\pi}{6}.$$

Table of Contents

Midpoint Rule

To approximate $\int_a^b f(x)\,dx$ we shall partition $[a,b]$ into n equal parts and let c_i be the midpoint of the ith subinterval. The **Midpoint rule** approximates $\int_a^b f(x)\,dx$ by the Riemann sum

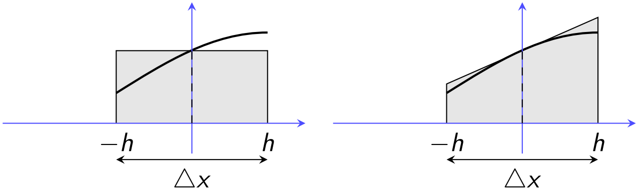
$$M_a^b(f) = \sum_{i=1}^{n} f(c_i)\triangle x, \quad \text{where } \triangle x = \frac{b-a}{n}.$$



Any estimate needs error bounds if it is to be useful. If we have some control over how much f deviates from being straight, we may hope to bound the error from approximating it by straight lines.

Midpoint Rule: Error Estimates

Let f be twice continuously differentiable on $[a, b]$. Let us consider the accuracy of each rectangle as an approximation to the corresponding portion of the integral. For simplicity, let the rectangle have base $[-h, h]$.



Rotating the top edge of the rectangle through its midpoint gives a trapezium with the same area. So we rotate it until it becomes the tangent line ℓ to the graph of f. By the Remainder Theorem, the gap between f and ℓ at any point is

$$f(x) - \ell(x) = \frac{f''(t_x)}{2}x^2,$$

for some $t_x \in (-h, h)$.

Midpoint Rule: Error Estimates

Note that the value $f''(t_x)$ must change continuously with x. By the Mean Value Theorem for Weighted Integration, we get

$$\int_{-h}^{h} f(x)\,dx - \int_{-h}^{h} \ell(x)\,dx = \int_{-h}^{h} \frac{f''(t_x)}{2} x^2\,dx$$
$$= f''(t)\int_{-h}^{h} \frac{x^2}{2}\,dx = \frac{h^3}{3} f''(t) = \frac{(\triangle x)^3}{24} f''(t),$$

for some $t \in [-h, h]$. We apply this result to each rectangle in the midpoint method to obtain:

$$\int_a^b f(x)\,dx - M_a^b(f) = \frac{(\triangle x)^3}{24}\sum_{i=1}^{n} f''(t_i)$$
$$= \frac{(\triangle x)^3}{24} n f''(t) = \frac{(\triangle x)^2}{24}(b-a) f''(t),$$

for some $t \in [a, b]$.

Example

Example 15

We have used the Midpoint rule earlier, without naming it, to estimate $\int_0^3 e^{-x^2}\,dx$. We set $n = 6$ and obtained

$$\int_0^3 e^{-x^2}\,dx \approx 0.886213\ldots$$

while the precise value of this integral is $0.88620734\ldots$. Thus, we had accuracy to 4 decimal places.

Let us gauge the accuracy through our own calculations. We have $f'(x) = -2xe^{-x^2}$ and $f''(x) = (-2 + 4x^2)e^{-x^2}$. This gives the bound $M = 2$, and hence

$$\left| \int_0^3 e^{-x^2}\,dx - 0.886213\ldots \right| \leq \frac{3 \times 2}{24}(0.5)^2 = 0.06.$$

This is a correct bound, but it clearly plays too safe and fails to convey how well the calculation has actually worked.

Simpson's Rule

The Midpoint rule is based on linear approximations to the integrand. We can hope to get better results by using a quadratic approximation.

We first fit a polynomial of degree at most 2 to any three data points. To simplify calculations, suppose the values y_0, y_1 and y_2 of a function f are known at $x = -h, 0, h$ respectively.

We wish to find a quadratic function $q(x)$ such that $q(-h) = y_0$, $q(0) = y_1$ and $q(h) = y_2$. It must have the form $q(x) = y_1 + xp(x)$ where $p(x)$ is linear. Then p has to satisfy $y_0 = y_1 - hp(-h)$ and $y_2 = y_1 + hp(h)$. This implies

$$p(-h) = \frac{y_1 - y_0}{h},\ p(h) = \frac{y_2 - y_1}{h} \text{ and } p(0) = \frac{y_2 - y_0}{2h}.$$

From these values we obtain the following expressions for p and q:

$$p(x) = \frac{y_2 - y_0}{2h} + \frac{y_0 - 2y_1 + y_2}{2h^2}x,$$
$$q(x) = y_1 + \frac{y_2 - y_0}{2h}x + \frac{y_0 - 2y_1 + y_2}{2h^2}x^2.$$

Simpson's Rule

This gives the following approximation for the integral of f:

$$\int_{-h}^{h} f(x)\,dx \approx \int_{-h}^{h} q(x)\,dx = (y_0 + 4y_1 + y_2)\,\frac{h}{3}.$$

Now, if f has domain $[a, b]$, we take a partition $P = \{x_0, \ldots, x_{2n}\}$ where the subintervals have equal width $\triangle x = (b - a)/2n$. Let $y_i = f(x_i)$. On each interval $[x_{2i-2}, x_{2i}]$ we apply the above approximation. This gives **Simpson's rule**:

$$\int_a^b f(x)\,dx \approx S_a^b(f)$$

where

$$\begin{aligned} S_a^b(f) &= (y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{2n-2} + 4y_{2n-1} + y_{2n})\,\frac{\triangle x}{3} \\ &= \left(y_0 + 2\sum_{i=1}^{n-1} y_{2i} + 4\sum_{i=1}^{n} y_{2i-1} + y_{2n}\right)\frac{\triangle x}{3}. \end{aligned}$$

Simpson's Rule: Convergence

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Simpson's rule can be expressed in a different way:

$$S_a^b(f) = \frac{1}{6}\left(2\triangle x\sum_{i=0}^{n-1} y_{2i}\right) + \frac{1}{6}\left(2\triangle x\sum_{i=1}^{n} y_{2i}\right) + \frac{2}{3}\left(2\triangle x\sum_{i=1}^{n} y_{2i-1}\right).$$

Each bracketed term is a Riemann sum and converges to $\int_a^b f(x)\,dx$, hence so does their combination.

Although Simpson's rule uses a quadratic approximation, it is exact even for cubics. This is easy to see over $[-h, h]$, since $\int_{-h}^{h} x^3\,dx = S_{-h}^{h}(x^3) = 0$.

Let $q(x)$ be a cubic which matches $f(x)$ at $x = -h, 0, h$ and also satisfies $q'(0) = f'(0)$. Set $y_0 = f(-h)$, $y_1 = f(0)$, $y_1' = f'(0)$, $y_2 = f(h)$. We have $q(x) = y_1 + xy_1' + x^2 p(x)$, where $p(x)$ is linear. We can solve for $p(x)$ as we did earlier, and the result is

$$q(x) = y_1 + y_1' x + \frac{y_0 - 2y_1 + y_2}{2h^2}x^2 + \frac{y_2 - y_0 - 2hy_1'}{2h^3}x^3.$$

Integrating q gives us the original Simpson's formula again.

Error for Cubic Interpolation

Cambridge University Press

Theorem 16

Let $f : [a, b] \to \mathbb{R}$ be four times continuously differentiable. Let $a = x_0 < x_1 < x_2 = b$. Let $q(x)$ be a polynomial of degree three or less such that $q(x_0) = f(x_0)$, $q(x_1) = f(x_1)$, $q'(x_1) = f'(x_1)$ and $q(x_2) = f(x_2)$. Then for every $x \in [a, b]$ there is a $\xi_x \in (a, b)$ such that

$$f(x) - q(x) = \frac{f^{(4)}(\xi_x)}{4!}(x - x_0)(x - x_1)^2(x - x_2).$$

Proof. We can assume $x \neq x_i$ for $i = 0, 1, 2$. Define a function g by

$$g(t) = f(t) - q(t) - M(t - x_0)(t - x_1)^2(t - x_2),$$

where M is chosen such that $g(x) = 0$. Then we have four distinct zeroes of g: $g(x_0) = g(x_1) = g(x_2) = g(x) = 0$. Rolle's Theorem gives us three distinct zeroes of g' in the open intervals created by x_0, x_1, x_2 and x. We also have $g'(x_1) = 0$. So we again have four distinct zeroes of g'. (continued...)

Error for Cubic Interpolation

(...continued)

Applying Rolle's Theorem repeatedly, we get three distinct zeroes of g'', then two of g''', and finally one of $g^{(4)}$ which we shall call ξ_x.

If we now differentiate g four times using its definition, we obtain $M = f^{(4)}(\xi_x)/4!$. Substituting this in $g(x) = 0$ gives the result. □

Simpson's Rule: Error Formula

Applying this error formula to our cubic approximation to f over $[-h, h]$ gives

$$f(x) - q(x) = \frac{f^{(4)}(\xi_x)}{4!} x^2(x^2 - h^2).$$

Therefore the error for the integral is

$$\int_{-h}^{h} f(x)\,dx - \int_{-h}^{h} q(x)\,dx = \int_{-h}^{h} \frac{f^{(4)}(\xi_x)}{4!} x^2(x^2 - h^2)\,dx$$
$$= \frac{f^{(4)}(\xi)}{4!} \int_{-h}^{h} x^2(x^2 - h^2)\,dx = \frac{f^{(4)}(\xi)}{90} h^5.$$

Applying this to the rule for a partition of $[a, b]$ into $2n$ equal subintervals, we get the error

$$\int_a^b f(x)\,dx - S_a^b(f) = \sum_{i=1}^{n} \frac{f^{(4)}(\xi_i)}{90} (\triangle x)^5$$
$$= \frac{(\triangle x)^5 n}{90} f^{(4)}(\xi) = \frac{(\triangle x)^4}{180} (b - a) f^{(4)}(\xi).$$

Approximation by Polynomials

Example 17

If we apply Simpson's rule to $\int_0^3 e^{-x^2}\,dx$ by cutting $[0,3]$ into 6 equal subintervals, we get the estimate $0.886172\ldots$ which has similar accuracy to the midpoint rule with the same number of intervals.

The difference becomes visible when we increase the number of intervals by a factor of 10. Simpson's rule now gives $0.88620734\ldots$ which is accurate to 8 decimal places. With the same number of function evaluations the Midpoint rule was accurate to 6 decimal places.

Gaussian Quadrature

The Midpoint and Simpson's rules are weighted averages of certain function values. We can create new rules by varying the points where we sample the function as well as the weight attached to each sampling location.

In the Gaussian quadrature method we attempt to fix the sampling locations and the weights so as to get exact results for polynomials of as high a degree as possible.

If we use n sampling points, we also have n weights to play with, and can hope to get exact results for polynomials up to degree $2n - 1$.

Gaussian Quadrature: $n = 1, 2$

Let us see how this works for small n and over an interval $[-h, h]$.

First consider $n = 1$. We hope to get exact results for linear polynomials. This is already achieved by the Midpoint rule.

Now let $n = 2$. Let the sample points be a, b with corresponding weights w_a, w_b. We get four equations by matching with 1, x, x^2 and x^3:

$$w_a + w_b = 2h \qquad w_a a + w_b b = 0 \qquad (1)$$

$$w_a a^2 + w_b b^2 = \frac{2}{3}h^2 \qquad w_a a^3 + w_b b^3 = 0 \qquad (2)$$

The equations in the second column give $w_a a(b^2 - a^2) = 0$ and hence $a = 0$ or $-b$. Now $a = 0$ is ruled out by the second equation, so we have $a = -b \neq 0$. But then the second equation gives $w_a = w_b$ and the first one gives $w_a = w_b = h$. Substituting all this in the third equation gives $a = -h/\sqrt{3}$ and $b = h/\sqrt{3}$. Thus, the rule is

$$\int_{-h}^{h} f(x)\,dx \approx G_2(f) = h(f(-h/\sqrt{3}) + f(h/\sqrt{3})).$$

Example

Example 18

Let us apply the basic integration rules to $\int_0^{\pi/2} \sqrt{x} \cos x \, dx = 0.704$. Write $g(x) = \sqrt{x} \cos x$.

The Midpoint rule uses one function value and gives the value

$$g(\pi/4) \cdot \pi/2 = 0.984.$$

Simpson's rule uses three function values and gives

$$(g(0) + 4g(\pi/4) + g(\pi/2)) \cdot \pi/12 = 0.656.$$

Gaussian quadrature uses two function values and gives

$$\left(g((1 - 1/\sqrt{3})\pi/4) + g((1 + 1/\sqrt{3})\pi/4)\right) \cdot \pi/4 = 0.712.$$

Gaussian Quadrature: Error Formula for $n = 2$

Suppose $q(x)$ is a cubic polynomial such that $q(h/\sqrt{3}) = f(h/\sqrt{3})$, $q(-h/\sqrt{3}) = f(-h/\sqrt{3})$, $q'(h/\sqrt{3}) = f'(h/\sqrt{3})$ and $q'(-h/\sqrt{3}) = f'(-h/\sqrt{3})$.

Task 1

Suppose $a \neq b$ and y_a, y_b, y_a', y_b' are given real numbers. Show that there is a unique polynomial of degree at most three such that $q(a) = y_a$, $q(b) = y_b$, $q'(a) = y_a$ and $q'(b) = y_b$.

Since q is cubic, and matches f on $\pm h/\sqrt{3}$, we have $\int_{-h}^{h} q(x)\,dx = G(q) = G(f)$. Therefore,

$$\int_{-h}^{h} f(x)\,dx - G_2(f) = \int_{-h}^{h} f(x)\,dx - \int_{-h}^{h} q(x)\,dx = \int_{-h}^{h} (f(x) - q(x))\,dx.$$

Error for Interpolation in Gaussian Quadrature

Theorem 19

Let $f : [a, b] \to \mathbb{R}$ be four times continuously differentiable. Let $a \leq x_1 < x_2 \leq b$. Let $q(x)$ be a polynomial of degree three or less such that $q(x_1) = f(x_1)$, $q'(x_1) = f'(x_1)$, $q(x_2) = f(x_2)$ and $q'(x_2) = f'(x_2)$. Then for every $x \in [a, b]$ there is a $\xi_x \in (a, b)$ such that

$$f(x) - q(x) = \frac{f^{(4)}(\xi_x)}{4!}(x - x_1)^2(x - x_2)^2.$$

Proof. Exercise. Similar to Theorem 16. □

Gaussian Quadrature: Error Formula for $n = 2$

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Integrate this error formula to get

$$\begin{aligned}\int_{-h}^{h} f(x)\,dx - G_2(f) &= \int_{-h}^{h} (f(x) - q(x))\,dx \\ &= \frac{f^{(4)}(\xi)}{4!} \int_{-h}^{h} (x + h/\sqrt{3})^2 (x - h/\sqrt{3})^2\,dx \\ &= \frac{f^{(4)}(\xi)}{4!} h^5 \int_{-1}^{1} (x^2 - 1/3)^2\,dx = \frac{f^{(4)}(\xi)}{135} h^5.\end{aligned}$$

So Gaussian quadrature achieves similar accuracy to Simpson's rule, while using two function values instead of three.