Circular plates with displacement variation in the θ-direction

In the main treatment of circular plates we ignored the variation in the θ - direction. For an actuator equipped with axially symmetrical electrodes, it may seem that these are the only modes of interest, but to be general, the variation in the θ-direction cannot be ignored. Moreover, fabrication imperfections often give rise to real asymmetries.

A revised model is based on the following equations from (Roark et al. 2002)(Table 11.2, Case 19). The displacement is given by

$$u = \frac{1}{16\pi D}\left[\frac{r_i^2 r_2^2}{r_a^2} - r_1^2\left(1 + 2\ln\left(\frac{r_i r_2}{r_a r_1}\right)\right)\right] \tag{6.1}$$

where

$$r_1^2 = r_i^2 + r_j^2 - 2r_i r_j \cos\left(2\left(\theta_j - \theta_i\right)\right) \tag{6.2}$$

$$r_x = \frac{r_a}{\cos\left(2\left(\theta_j - \theta_i\right)\right)} = \frac{r_a^2}{r_i} \tag{6.3}$$

$$r_2^2 = r_x^2 + r_j^2 - 2r_x r_j \cos\left(2\left(\theta_j - \theta_i\right)\right) \tag{6.4}$$

Figure. 6.1 defines geometry parameters r_i, r_j, r_1, r_2, r_x, and r_a. The listing of the implementation of this model is given in Figure. 6.2. The listing shows that the implementation that takes θ into account is approximately at the same level of complexity as the previous model that ignores θ-asymmetries. Notice, however, that the number of elements has dramatically increased, because this is a true 2D model, not a parameterized 1D model. Before we had any desirable resolution in the θ-direction. Here, it does not come for free, but it is far more informative, as shown in Figure. 6.3, where we see that there are three more modes between the first two radial modes.

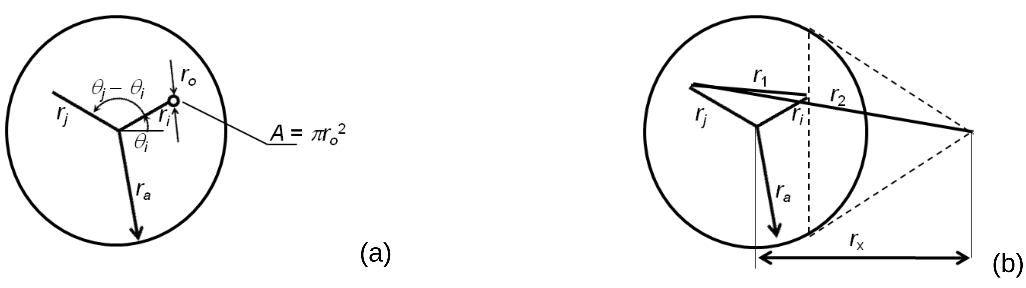

Figure. 6.1: A sketch of the plate introduces the geometric parameters r_i, r_j, r_1, r_2, r_x, and r_a used in Eqs. (6.1)-(6.4).

```
from numpy as np
from enthought.mayavi import mlab
n  = 1.                          # current mode
N1 = 20.                         # Number of elements in the r-direction
N2 = 20.                         # Number of elements in the theta-direction
N  = N1*N2                      # Total number of elements – compare to 1D (radial only model) !
r_a= 200e-6                     # Radius [m]
h = 1e-6                        # Height [m]
nu = 0.3                        # Poisson ratio
Y = 150e9                       # Young's modulus [N/m**2]
rho = 2330.                     # Mass density [kg/m**3]
D = Y*h**3/12/(1-nu**2)    # Plate modulus
a = np.zeros((N,N))              # Initialize mass and stiffness matrices
m = np.zeros((N,N))
dq = 2*np.pi/N2; dr = r_a/N1  # define the element size
for i in np.arange(0,N):
    i1 = floor(i*1.0/N1)+.5
    i2 = i % N1
    ri = r_a*i1/N1
    qi = 2*np.pi*i2/N2
    m[i,i] = ri*dq*dr*rho*h                 # mass matrix is diagonal as before
    for j in np.arange(i,N):
        j1 = floor(j*1/N1)+.5
        j2 = j % N1
        rj = r_a*j1/N1
        qj = 2*np.pi*j2/N2
        r1 = np.sqrt(ri**2+rj**2-2*ri*rj*np.cos(qj-qi))
        rx = r_a**2/ri
        r2 = np.sqrt(rx**2 + rj**2 -2*rx*rj*np.cos(qj-qi))
        if i == j:
            a[i,j] = 1/(16*pi*D)*(r_a**2-ri**2)**2/r_a**2
        else:
            a[i,j] = 1/(16*pi*D)*(ri**2*r2**2/r_a**2-r1**2*(1+2*np.log(ri*r2/r_a/r1)))
            a[j,i] = a[i,j]
        if np.isnan(a[i,j]):
            print i,j,r1
w,v = np.linalg.eig(dot(a,m))                # Note that matrix multiplication in Python uses dot() fcn
freq_kHz = 1/np.sqrt(w)/2/np.pi/1e3          # Note also that the first output of eig are eigenvalues, not eigenvectors
```

Figure. 6.2: A Python implementation of a plate based on Eqs. (6.1)-(6.4).

See also FEA analysis of a circular plate in Ansys.

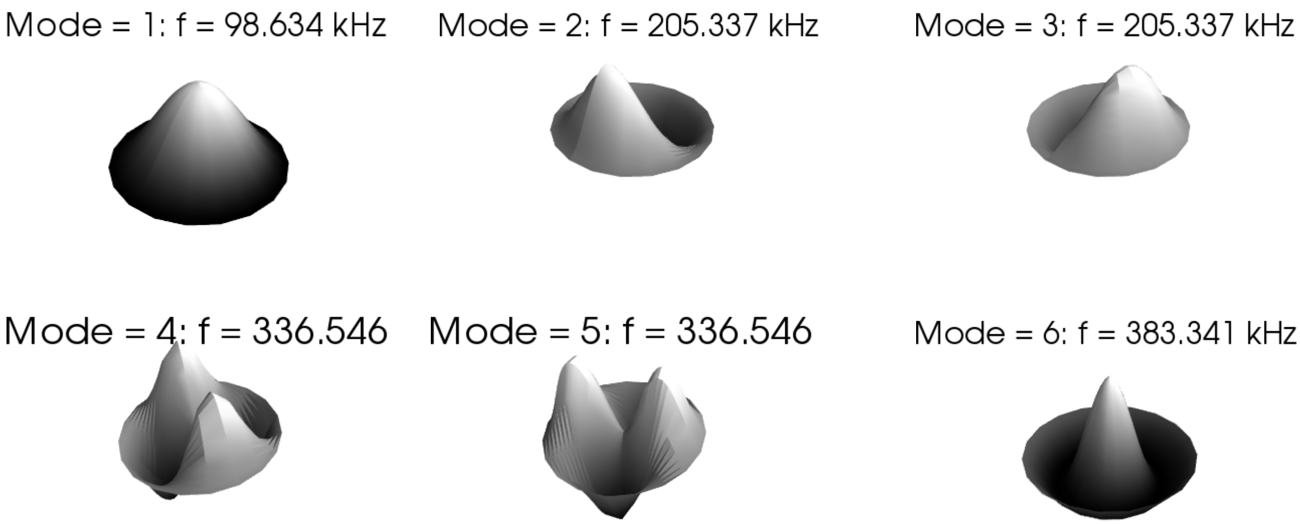

Figure. 6.3: Modal analysis – the first six modes. Silicon plate with radius L = 200 μm and height h = 1 μm. The plate is discretized with N_1 = 20 degrees of freedom in the r-direction and N_2 = 20 degrees of freedom in the θ-direction (total of 400 DOF!). Note that modes n = 2 and n = 3 are degenerate modes (also modes n = 4 and n = 5). Due to symmetry in the θ-direction, there are infinite mode shapes that correspond to these modes – we can imagine any arbitrary rotation of these two as a valid mode shapes.