NONLINEAR DYNAMICS: COMPUTER EXERCISES

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to use with **iDMC** software[2] and
NONLINEAR DYNAMICS: A PRIMER[3]

contents

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[2] see www.dss.uniud.it/nonlinear

[3] Medio and Lines, Cambridge University Press, 2001.

NUMERICAL SIMULATIONS

COMMENTS AND CAVEATS

In the formulation of dynamical models the range of parameter values for which the difference or differential system describes the phenomena of interest will be known up to a certain precision. If the mechanism of interest persists, the parameter values in its formal description must be such that the variables eventually settle to an invariant set of values. In applied science much of what is observed is persistent behaviour and the relevant parameter ranges imply bounded trajectories.

However, in the analysis of a (contextless) generic nonlinear system, it is likely that randomly chosen parameter values will lead to unstable solutions which take on very large values quickly. After some experience in simulating such systems one comes to the realisation that the collection of parameter values leading to stable limit sets is indeed a small part of the parameter space. Care must then be taken in selecting parameter values for the analysis and in the following exercises we have provided the relevant parameter ranges or values so as to save time and frustration.

A related problem in the analysis of a generic nonlinear system is that the basin of attraction of the stable limit set is probably unknown (and may be quite small and/or have a complicated shape) so that initial values must also be chosen with care. Also keep in mind that as parameter values are varied, the basin boundaries change.

A more specific problem of interpreting information from simulations is the case of a system for which parameter values and initial conditions have been so chosen as to place it “outside” but “close” to the basin boundary of a stable limit set. Orbits and trajectories may appear to converge to the limit set over long periods of time, only to suddenly diverge.

Other difficulties will be described as they arise, but if the above are kept in mind the information, or confirmation, supplied by numerical simulations will be easier to decipher.

interactive Dynamical Model Calculator

The software programme interactive Dynamical Model Calculator (henceforth, iDMC) is a research tool for studying nonlinear dynamical systems. It is particularly useful for researchers because it allows the user to choose from among well-known models or insert his or her own models. Please refer to the site mentioned in the footnote for installation instructions and a user's guide which includes a complete list of available algorithms and more detailed information on their use, as well as instructions on inserting new models. New features are in the process of being developed and included in the programme, consequently the exercise sets will occasionally be modified or expanded.

The first step in any session is to select a model from the directory models. The directory has a number of well-known models, as well as a directory with economic models and a directory called Primer which contains all the models used in the following computer exercises. Once a model is chosen click on `New plot` and, for example, `Trajectory`. You will be presented with an interface that must be completed with values for initial conditions, parameters, number of transients (number of iterations starting from the initial condition that will be truncated and not considered), number of iterations after the transients (number of iterations starting from the last transient that will be considered). Clicking `Start` the trajectory should appear, to change parameter values or options click `Reset`. A number of graphical options are available besides the default. In the `Options` menu the typle of plotting points can be changed to big dots and/or connect dots (useful for discrete-time systems). In `Plot` menu the `Manual bounds` option allows the user to define the axes and the user may choose the type of plot (either time evolution or state space). A zoom feature is available (right click and select desired area of plot). A number of models can be open at any time, click `New model` to open a different model. Graphic files are saved and printed as .png files and options regarding the look of the graphics can be changed using the `Options` menu. Click `Exit` to leave the program.

The following is a sample session in which trajectories of an economic model of partial equilibrium (in discrete time) are simulated. Further details can be found in Section 1.1 and 1.2[4]

Consider the description of a single market with demand and supply functions of the price of the good and a price adjustment equation which depends

[4] All figure, chapter, section, equation and exercise numbers refer to those in **Nonlinear dynamics: a primer A. Medio e M. Lines, Cambridge University Press, 2001.**

on the previous price and excess demand:

$$D(p_n) = a - bp_n \tag{1.1}$$
$$S(p_n) = -m + sp_n \tag{1.1}$$
$$p_{n+1} = p_n + h\theta\,[D(p_n) - S(p_n)] \tag{1.3}$$

with $a, b, m, s > 0$. Let $h = \theta = 1$, substitute demand and supply functions into the price-adjustment equation and the model equation is

$$p_{n+1} = a + m + (1 - b - s)p_n$$

or, setting $a + m = \alpha$ and $1 - b - s = \beta$

$$p_{n+1} = \alpha + \beta p_n. \tag{1.5}$$

The general solution to (1.5) is

$$p(n) = \bar{p} + [p(0) - \bar{p}]\beta^n. \tag{1.9}$$

Dynamics and simulations

It should be observed from (1.5) and (1.9) that the asymptotic behaviour of the model, that is the qualitative dynamics, are only influenced by the parameters b and s. These determine the slope of the line in (1.5) defined in the (p_n, p_{n+1}) space. Variations in parameter values for a and m only change the value of the equilibrium price $\bar{p}$.

Click on `File` and `New model`, from the Primer directory select the model (`dispar`). Again, from the `File` menu select `New plot`, then `Trajectory`. Values must be provided for initial conditions and parameters. In the following exercises the choice of the initial condition $p(0)$ is left to the student. Recall, however, that if $p(0) = \bar{p}$ there will be no price adjustment. It is necessary to provide values to be used by the algorithm: number of transients, number of iterations after the transients. In order to see all of the trajectory starting from the assigned initial condition, the choice for transients is 0. The number of iterations can be set low, such as 50. The `Auto ranges` feature asks how many iterations should be considered in calculating the ranges for the axes in the default plot, this can be set at the same value as the algorithm iterations. (If desired, the plot axes can be defined in the `Manual bounds` option in the `Plot` menu).

In order to see the plotted points you will probably have to change the style of plotting. The default is pictorially correct as it gives exactly the information calculated. A dot represents the iterated value for a discrete-time system or the integrated value for a continuous-time system. The option `Big dots` makes the points more visible but it may be necessary to use `Connect dots` to get clearer curves and better printed copies. In order to understand the dynamics of the trajectory it may be necessary to slow down the presentation. This is possible by using the arrow on the bar above the plot. The further the arrow is dragged to the right the slower the presentation of the data.

a. Open the `Trajectory` plot and insert these values for the parameters
`a` 10
`m` 2
`b` 0.2
`s` 0.1
Produce the trajectory of p for 100 iterations. Given that $0 < \beta < 1$ $(\beta = 0.7)$, the trajectory converges in monotonically to $\bar{p} = \frac{a+m}{b+s} = 40$.

b. Change the parameter values of a and m to `a` 10, `m` 2. Note that with respect to exercise a., only the equilibrium values $\bar{p} = 10.5$ has changed, the convergence to equilibrium is the same. In fact, as long as $0 < \beta < 1$ monotonic convergence is guaranteed.

c. Change the parameter values of b and s to `b` 0.5, `s` 0.7. The price converges to an equilibrium value of 10 but with oscillations as $-1 < \beta < 0$.

d. Change the parameter values of b and s to `b` 0.9, `s` 1.5, and now $\beta < -1$. The price has an equilibrium values at $\bar{p} = 20$ but trajectories beginning at an arbitrary distance from that value diverge with overshooting. The distance from $\bar{p}$ increases exponentially and even 25 iterations are sufficient. No set of parameter values for model (1.5) will lead to monotonic divergence since, under the hypotheses, $\beta < 1 \; \forall \; b, s > 0$.

e. Multiple trajectories can be displayed in a single plot by using `Variation`, found in the `Plot` menu. This procedure allows the user to increase or decrease the value of a parameter or initial condition any number of times, by specifying the amount to change at each variation. Set parameter values as they were in exercise a. Place a 0 in the second box for all parameters or initial conditions that do not change. For example, keying in 0.1 next to `b` and 0 in all of the other second boxes, and using 10 variations results in a plot for which, at each subsequent simulation, the value of the parameter b is increased by 0.1. The first trajectory is with $b = 0.2$, the second trajectory is for $b = 0.3$, and so on. Click on `Plot` and note how the transient dynamics change as the value of b varies. (These plots should be familiar to those who have worked through Exercises 1.1 and 1.2 at the end of chapter 1.)

f. Use the `Variation` routine to demonstrate that the dynamics and equilibrium of the model are independent of the choice of initial values.

To exit iDMC select `Exit` from the `Files` menu.

Computer exercises chapter 1

SET 1

In this set we further study the discrete partial equilibrium model (1.5) (`dispar`) used above and presented in Section 1.2.

a. From the `Files` menu select `New plot` and `Shifted and cobweb`. Begin this session by using the suggested values for exercise a. in the sample session. For the `Algorithm order` use 1, this will produce a plot of p_{n+1} against p_n (an order of 2 would produce a plot of p_{n+2} against p_n, etc.). Set the `horizontal` axis and `vertical` axis to cover the same positive range which should include the equilibrium value. Equilibrium $\bar{p}$ is at the point that p_{n+1} and p_n have the same value.

Now click on the `Cobweb animation` option, provide an initial value and choose 0 transients. The routine will include the bisector and, beginning from the given initial value, draw the trajectory path towards equilibrium at $\bar{p} = 40$ (see figure 1.2(a)). Use the same routine with a different set of parameter values.

b. Follow the instructions for `Variation` to get the time evolution as b varies. Choose values for other parameters and the variation options so as to produce a plot with the following transient dynamics: monotonic convergence to equilibrium; no adjustment (the inital condition is the equilibrium value); monotonic convergence to equilibrium; oscillatory convergence towards equilibrium; oscillatory divergence. It may be helpful to refer to the analysis in Chapter 1 Section 1.2 (summarised in the parameter space represented in Figure 1.3). Click on `Close` and then, from `Files`, click on `Exit` to quit or select `ctpar` to continue with set 2 below.

SET 2

In this set we use the continuous time partial equilibrium model (`conpar`) presented in Section 1.3. In this version of partial equilibrium the price adjustment again depends on excess demand, but adjustment is in continouous time:

$$\begin{aligned} D(p) &= a - bp \\ S(p) &= -m + sp \\ \dot{p} &= [D(p) - S(p)]. \end{aligned}$$

After substitution we have the differential equation

$$\dot{p} = (a + m) - (b + s)p \tag{1.10}$$

with general solution

$$p(t) = \bar{p} + [p(0) - \bar{p}]e^{-(b+s)t}. \tag{1.13}$$

The nature of the dynamics will be determined by the exponent in (1.13) which, under the hypotheses for the parameter values of b and s is always positive. That is, whatever the particular values choosen for the parameters (satisfying the hypotheses), the trajectories will converge to the equilibrium value. In fact,

$$\lim_{t\to\infty} e^{-(b+s)t} = 0 \quad \text{and} \quad \lim_{t\to\infty} p(t) = \bar{p} + 0 = \bar{p}.$$

The simulation of models in continuous-time differs from that of models in discrete-time in that differential equations like (1.10) must be integrated rather than simply iterated. The default integrator in iDMC is the Runge–Kutta procedure, which uses a fixed step size (other integration procedures are available under `Step function` in the `Plot` menu). In addition to the usual information, the user must also provide the `step size`. For example, a step size of 0.02 means that the interval between time t and time $t + 1$ is divided into 50 equal intervals. The number of `iterates` multiplied by the step size gives the number of time periods. Using 1000 iterations with a step size of 0.02 means 20 time periods have been simulated. Trajectories and orbits will be smoother with smaller steps but the number of iterations must be adjusted accordingly to cover the same time period.

Using the `Variation` routine create and print a plot which represents independence to initial conditions. Annotate the figure by noting the parameter values chosen for the simulations, labeling trajectories with parameter values and discussing the resulting dynamical behaviours (for example, is it possible in this model to have transient behaviour characterised by monotonic divergence or oscillatory convergence).

computer exercises chapter 2

SET 1

In this set we use the discrete partial equilibrium model with lagged supply response presented in section 2.6. This model differs from the model in (1.1), (1.3) in the assumption that producers require time to adjust their supply of the product and therefore base their quantity choice on the price in the previous period, while demand adjusts to the current price. Then

$$\begin{aligned} D(p_n) &= a - bp_n & (1.1)\\ S(p_{n-1}) &= -m + sp_{n-1} & (2.41)\\ p_{n+1} &= p_n + h\theta\left[D(p_n) - S(p_{n-1})\right] \end{aligned}$$

and again it is assumed that a, b, m, $s > 0$. Substituting the demand and supply functions in the price-adjustment equation we have

$$\begin{aligned} p_{n+1} &= p_n + (a - bp_n + m - sp_{n-1})\\ &= (1-b)p_n - sp_{n-1} + (a+m). \end{aligned} \tag{2.42}$$

Recall that to put a difference or differential equation of order m into the canonical form of a first order system in m equations, auxiliary variables are introduced. (See Remark 1.1, Chapter 1 for differential equations and Exercise 1.7 for difference equations.) For the trasformation into a first-order system of 2 equations, let $z_n = p_{n-1}$. To render the new system homogeneous another auxiliary varible is introduced which places the equilibrium at the origin:

$$\tilde{z}_n = \tilde{p}_{n-1} = p_{n-1} - \frac{a+m}{b+s} \tag{2.44}$$

and we have the first–order, homogeneous system

$$\begin{aligned} \tilde{z}_{n+1} &= \tilde{p}_n\\ \tilde{p}_{n+1} &= -s\tilde{z}_n + (1-b)\tilde{p}_n. \end{aligned} \tag{2.45}$$

This system has a unique equilibrium value $(0, 0)$ while the equilibrium for the original variable is, once again,

$$\bar{p} = \frac{a+m}{b+s}.$$

The dynamical behaviour of system (2.45) is characterized by the eigenvalues of the constant matrix

$$B = \begin{pmatrix} 0 & 1\\ -s & 1-b \end{pmatrix} \tag{2.46}$$

which are the roots of the characteristic equation $\kappa^2 - (1-b)\kappa + s = 0$, that is,

$$\kappa_{1,2} = \frac{1}{2}\left[(1-b) \pm \sqrt{(1-b)^2 - 4s}\right].$$

The parameters which determine the dynamics of the system, the stability and type of transient behaviour, are again b and s. In the following exercises it will prove useful to refer to the range of possible dynamics represented in the parameter space (b, s) in Fig. 2.9. We study these dynamical behaviours by choosing values for the parameters that distinguish three cases for the sign of the discriminant $\Delta = (1-b)^2 - 4s$.

From the `Model` menu select the `disparlag` model. Choose reasonable values for the parameters a and m (those in the sample session, for example). For simplicity use the same initial condition for $\tilde{z}$ and $\tilde{p}$. In order to see the entire trajectory again use 0 transients. The number of iterations should be kept small for visual clarity, no more than 200. Simulations of multi-dimensional models can be represented as the time evolution of single variables (referred to as trajectories) or the state space of two variables (referred to as orbits). The default plot is the state space, but it may be helpful to look at the time evolution of the variables by choosing the `Time plot` from the `Plot` menu.

Case 1 $\Delta > 0$.
a. Set $b = 0.5$, $s = 0.05$. The eigenvalues of B are: $\kappa_1 = \frac{1}{4}\left(1 + \frac{1}{5}\sqrt{5}\right)$, $\kappa_2 = \frac{1}{4}\left(1 - \frac{1}{5}\sqrt{5}\right)$, giving $|\kappa_1|, |\kappa_2| < 1$ and $\kappa_1, \kappa_2 > 0$. The system converges monotonically to the equilibrium which is a stable node. Use the `Variation` procedure to verify that the trajectories do not depend on the initial conditions.

b. Set $b = 2$, $s = 0.1$. The eigenvalues are: $\kappa_1 = -\frac{1}{2}\left(1 + \frac{\sqrt{15}}{5}\right)$, $\kappa_2 = \frac{1}{2}\left(1 - \frac{\sqrt{15}}{5}\right)$, therefore $|\kappa_1|, |\kappa_2| < 1$ and $\kappa_1, \kappa_2 < 0$. The equilibrium is a stable node but because of the negative eigenvalues, the trajectory convergences with improper oscillations (see Section 2.4).

c. Set $b = 5$, $s = 6$. The eigenvalues are: $\kappa_1 = -3/2$, $\kappa_2 = -5/2$, therefore $|\kappa_1|, |\kappa_2| > 1$ e $\kappa_1, \kappa_2 < 0$. The equilibrium is an unstable node with improper oscillations. Given that b and s are assumed positive, the equilibirum at the origin cannot be an unstable node with monotonic divergence as $|\kappa_1|, |\kappa_2| > 1$ and $\kappa_1, \kappa_2 > 0$.

d. Set $b = 3$, $s = 0.75$. In this case we have $\kappa_1 = -1/2$, $\kappa_2 = -3/2$, therefore $|\kappa_1| < 1$, $|\kappa_2| > 1$ and $\kappa_1, \kappa_2 < 0$. The equilibrium is a saddle point and orbits from any generic initial condition eventually diverge from the origin with improper oscillations. However, from initial conditions choosen so as to be positioned on the eigenvector associated with eigenvalue κ_1, orbits converge to the equilibrium, overshooting at each iteration. This can be verified by setting $\tilde{z}_0 = 6$ and $\tilde{p}_0 = -3$ (the point (6,-3) lies on the relevant eigenvector.).

Case 2 $\Delta < 0$ and eigenvalues are complex $(\kappa_1, \kappa_2) = \sigma \pm i\theta$.

a. Set $b = 1$, $s = 0.6$. Then, $r = |\sigma \pm i\theta| = \sqrt{\det(B)} = \sqrt{s} < 1$ and the equilibrium is a stable focus.

b. Set $b = 2$, $s = 4$, giving $r > 1$ and (0,0) is an unstable focus.

c. Set $b = 1$, $s = 1$, giving $r = 1$. Recall (see Secton 2.4) that solutions for discrete systems are sequences of points lying on curves. In the present case the solutions lie on closed curves which may be periodic or quasiperiodic, depending on the frequency of the trigonometric oscillation given by $\omega/2\pi$, $\omega = \arccos[\mathrm{tr}(B)/2]$ (see see Case 2, Section 2.5). With the values assigned to parameters we have $\omega = \arccos(0) = \pi/2$, and the ratio $\omega/2\pi$ is a rational number. The solution sequences are k-periodic, in this case $k = 4$. Solutions in the case of such special parameter patterns (giving $r = 1$) are sensitive to initial conditions, which position the oscillation on a particular curve. Use the `Variation` routine to verify that changing the starting point changes also the radius of the closed curve.

d. Set $b = 0.5$, $s = 1$, and again, $r = 1$. In this case it is easy to check that $\omega/2\pi$ is an irrational number. The solution is quasiperiodic in that the oscillation does not returns exactly to a previously visited point and instead orbits continue to oscillate around the curve on which the solution lies, coming close to previous points and filling in the curve. Use a high number of iterates and slow motion (move the bar above the plot to the right) to see how the curve eventually fills in.

Case 3 $\Delta = 0$ and eigenvalues coincide $\kappa_1 = \kappa_2 = \bar{\kappa} = \mathrm{tr}(B)/2$.

a. Set $b = 0.4$, $s = 0.04$, $|\bar{\kappa}| < 1$ and the system converges to a stable node.

b. Set $b = 4$, $s = 4$ giving $\bar{\kappa} = -3/2$ ($|\bar{\kappa}| > 1$). The equilibrium is an unstable node with orbits diverging in improper oscillations.

c. Set $b = 3$, $s = 1$ giving $\bar{\kappa} = -1$. In this case solutions diverge linearly rather than exponentially from the equilibrium (with improper oscillations). To see how these two expansions differ use the `Time plot` for either variable and the `Variation` routine to represent both linear and exponential expansion.

SET 2

In this set we study the generic differential system in the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.29}$$

with a_{ij} real constants. The dynamical behaviour of system (2.29) is characterised by the eigenvalues of the the canstant matrix A, which are roots to the characteristic equation

$$\det(A - \lambda I) = 0 \qquad \text{that is} \qquad \lambda^2 - \text{tr}(A)\lambda + \det(A) = 0$$

that is,

$$\lambda_{1,2} = \frac{1}{2}\left(\text{tr}(A) \pm \sqrt{\Delta}\right) \tag{2.30}$$

where $\Delta \equiv \left([\text{tr}(A)]^2 - 4\det(A)\right) = (a_{11} - a_{22})^2 + 4a_{12}a_{21}$.

Open the model (`con2d`)In the simulations we once again assign parameter values falling into one of three categories regarding the sign of the discriminant. Note also that the trace and determinant of the constant matrix A completely characterise the dynamics of the system. It may be helpful to refer to Figures 2.1 and 2.3 while working with system (2.29). Choose initial conditions as you please. iDMC must integrate system (2.29) and you must provide a step size (say 0.02 or 0.05) for the algorithm. Again use 0 transients so that the entire orbit or trajectory is included in the representation. Recall that the actual number of periods will be the product of the step size and the number of iterations, so consider a high value for this number (10,000 iterations with a step size of 0.02 means 200 periods).

Case 1 $\Delta > 0$

a. Set $a_{11} = -4$, $a_{12} = 6$, $a_{21} = 0.5$, $a_{22} = -6$. These values give $\text{tr}(A) < 0$, $\det(A) > 0$ and the eigenvalues are both real and negative, $\lambda_1 = -1$, $\lambda_2 = -9$. The equilibrium is a stable node. Use the `Variation` routine to verify that orbits converge to the equilibrium independently of initial conditions, use the `Time plot` for one of the variables to see the same convergence in trajectories.

b. Set $a_{11} = 3$, $a_{12} = 2$, $a_{21} = 6$, $a_{22} = 4$. Then $\det(A) = 0$ and the eigenvalues are zero and $\text{tr}(A)$ so that it is the sign of the trace that determines the qualitative dynamics. Moreover, the equilibrium is not a point in this special case but a set of points that lie on the line passing through the origin

$$y/x = -(a_{11}/a_{12}) = -(a_{21}/a_{22}).$$

For this collection of paramter values $\mathrm{tr}(A) = 7 > 0$, $\lambda_1 = 0$, $\lambda_2 = 7$. The equilibrium set consists of the points whose coordinates (x, y) are such that $y = -3/2\ x$. Position the initial conditions of the orbit on a generic point in the state space and verify that the equilibrium set is unstable.

c. Set $a_{11} = -5$, $a_{12} = 2$, $a_{21} = 10$, $a_{22} = -4$. Again we have $\det(A) = 0$, but the trace is negative and the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -9$. The equilibrium set is the points on the line $y = 5/2\ x$.

d. Set $a_{11} = 4$, $a_{12} = -2$, $a_{21} = -4$, $a_{22} = 6$. Then $\mathrm{tr}(A) > 0$ and $\det(A) > 0$. The eigenvalues are real and positive $\lambda_1 = 8$, $\lambda_2 = 2$ and the equilibrium is an unstable node.

e. Set $a_{11} = 3$, $a_{12} = 6$, $a_{21} = 4$, $a_{22} = 5$. Then $\det(A) < 0$ and $\mathrm{tr}(A) > 0$. A negative determinate implies eigenvalues of opposite sign, independent of the sign of the trace. In this case $\lambda_1 = 9, \lambda_2 = -1$ and the one-dimensional eigenspace generated by λ_1 is unstable, while that generated by λ_2 is stable and the equilibrium is a saddle point.

Set the step size to 0.01 with 350 iterations. Use the `Variation` routine to show two orbits, one beginning at $(-6, 4)$, a second beginning at a small distance from the first. Notice how the orbit dynamics change. The initial condition of the first orbit lies on the eigenvector associated with the negative eigenvalue $(A - \lambda_2 I)v = 0$ whose solutions lie on the line $y = -\frac{2}{3}x$. All orbits initiating on the line converge to equilibrium, all others diverge.

Next use the same initial conditions $(-6, 4)$ but increase the iterations to 450. Notice that the orbit first moves towards the equilibrium (the 3.5= 350×0.01 periods of the first run), only to diverge quickly after. In theory the orbit should converge to the saddle point at (0,0) but in practice, given finite precision both in placing the initial condition and in integrating forward in time, the errors compound and the orbit eventually diverges.

Case 2 $\Delta < 0$ and eigenvalues are complex $(\lambda_1, \lambda_2) = \alpha \pm i\beta$, $\alpha = \frac{1}{2}\mathrm{tr}(A)$, $\beta = \frac{1}{2}\sqrt{-\Delta}$.

a. Set $a_{11} = 3$, $a_{12} = 5$, $a_{21} = -5$, $a_{22} = -5$. These values give $\mathrm{tr}(A) < 0$, and the real part of the eigenvalues are negative ($\mathrm{Re}\ \lambda = \alpha = \mathrm{tr}(A)/2 < 0$): $\lambda_1 = -1 + 3i$ and $\lambda_2 = -1 - 3i$. Orbits converge to a stable focus.

b. Set $a_{11} = 5$, $a_{12} = -5$, $a_{21} = 6.1$, $a_{22} = 4$. Then $\mathrm{tr}(A) > 0$, $\mathrm{Re}\ \lambda = \alpha > 0$ and the system diverges and the equilibrium is an unstable focus.

c. Set $a_{11} = 3$, $a_{12} = -5$, $a_{21} = 2$, $a_{22} = -3$. Set the step size to 0.01 and iterations to at least 650. We have $\mathrm{tr}(A) = 0$, $\mathrm{Re}\ \lambda = \alpha = 0$ ($\lambda_1 = i$,

$\lambda_2 = -i$), $\det(A) > 0$. In fact the oscillations are with constant amplitude, neither diverging nor converging and the equilibrium is called a center. Use the `Variation` routine to demonstrate that the amplitude of the oscillations depends on the initial conditions.

Case 3 $\Delta = 0$ and eigenvalues coincide.

a. Set $a_{11} = -6$, $a_{12} = 4$, $a_{21} = -4$, $a_{22} = 2$. The eigenvalues are $\lambda_1 = \lambda_2 = \bar{\lambda} = -2$, to which is associated an eigenvector u of the type $k \cdot (1, 1)$, k real. Notice that $A \neq \lambda I$. In this case the equilibrium is a node, known as a Jordan node (see Figure 2.2). Verify that u is an invariant set and that on the vector u the system converges to equilibrium.

b. Set $a_{11} = 2$, $a_{12} = 0$, $a_{21} = 0$, $a_{22} = 2$. We have two eigenvectors $\lambda_1 = \lambda_2 = \bar{\lambda} = 2$ such that $A = \lambda I$. The equilibrium is an unstable node, called bicritical (see Figure 2.2). Verify that all of the half-lines passing through the origin are solutions to the system.

computer exercises chapter 3

From Chapter 3 we know that the local properties of equilibrium points of a nonlinear dynamical system can be studied by making use of the Jacobian matrix of the first partial derivates calculated at the equilibrium point. As stated in the Hartman-Grobman Theorem (Theorem 3.1) the linear approximation of a nonlinear system preserves the local properties of the fixed points if they are hyperbolic. The fixed point of a differential system is hyperbolic if no eigenvalue has real part equal to zero. The fixed point of a system of difference equations is hyperbolic if no eigenvalue has modulus equal to one.

SET 1

In this set we study the local behaviour in the neighbourhood of the fixed points of the system found in exercise 3.4(a) at the end of chapter 3 (which should be worked through before beginning the computer analysis):

$$\begin{aligned} \dot{x} &= y^2 - 3x + a \\ \dot{y} &= x^2 - y^2 \end{aligned} \qquad 3.4(a)$$

where $a = 2$. Select the model `conlocal` (continuous time model for local dynamics). Use the state space plots and choose initial conditions so as to illustrate each of the local dynamical behaviour of the four types of fixed points. Recommended values for integration data are step size of 0.02 with 500 iterations and `Manual bounds` for the horizontal axis (0,5) and the vertical axis (-5,5). Plot sample state spaces using the `Variation` procedure to get a few orbits, including one initiating on the fixed point.

SET 2

In this set we study analytically and through numerical simulations the system in Exercise 3.4(d) which has been modified by the introduction of a parameter as follows:

$$\begin{aligned} \dot{x} &= y - x^2 + 2 \\ \dot{y} &= 2a(x^2 - y^2) \end{aligned} \qquad (i)$$

The Jacobian matrix of (i) is:

$$J = \begin{pmatrix} -2x & 1 \\ 4ax & -4ay \end{pmatrix} \qquad (ii)$$

and fixed points are easily calculated from (i) as $A = (-1, -1)$, $B = (1, -1)$, $C = (-2, 2)$, $D = (2, 2)$. Then the Jacobian matrices calculated in the four fixed points are

$$J(A) = \begin{pmatrix} 2 & 1 \\ -4a & 4a \end{pmatrix} \quad \text{eigenvalues} \quad \lambda_{12}^{A} = 1 + 2a \pm \sqrt{1 - 8a + 4a^2}$$

$$J(B) = \begin{pmatrix} -2 & 1 \\ 4a & 4a \end{pmatrix} \quad \text{eigenvalues} \quad \lambda_{12}^{B} = -1 + 2a \pm \sqrt{1 + 8a + 4a^2}$$

$$J(C) = \begin{pmatrix} 4 & 1 \\ -8a & -8a \end{pmatrix} \quad \text{eigenvalues} \quad \lambda_{12}^{C} = 2\left(1 - 2a \pm \sqrt{1 + 2a + 4a^2}\right)$$

$$J(D) = \begin{pmatrix} -4 & 1 \\ 8a & -8a \end{pmatrix} \quad \text{eigenvalues} \quad \lambda_{12}^{D} = -2\left(1 + 2a \pm \sqrt{1 - 2a + 4a^2}\right).$$

We now have all the information necessary to study the local properties of the fixed points of system (i). Select model `cona` and simulate orbits in the state space, varying the value of parameter a. Compare the behaviour of the system with information on the linearised system. For example:

con $a = -1$

eq	λ_1	λ_2	*linearized*	*nonlinear system*
A	-4.61	2.6	saddle point	locally unstable
B	-3-1.73i	-3+1.73i	stable focus	locally stable
C	2.54	9.46	unstable node	locally unstable
D	-3.29	7.29	saddle point	locally unstable

con $a = 0.14$

eq	λ_1	λ_2	*linearized*	*nonlinear system*
A	1.28-0.2i	1.28+i	unstable focus	locally unstable
B	-2.2	0.76	saddle point	locally stable
C	-0.89	3.77	saddle point	locally unstable
D	-4.35	-0.77	stable node	locally stable

con $a = 0.5$

eq	λ_1	λ_2	*linearized*	*nonlinear system*
A	2-1.41i	2+1.41i	unstable focus	locally unstable
B	-2.45	2.45	saddle point	locally unstable
C	-3.46	3.46	saddle point	locally unstable
D	-6	-2	stable node	locally stable

Choose a different parameter value for a and determine the local stability of the fixed points. Use the `Variation` routine for initial conditions to conferm the local stability.

SET 3

In an analogous manner study the following system

$$\begin{aligned}\dot{x} &= x^2 + ax + xy^2\\ \dot{y} &= -y + by^{3/2}\end{aligned}$$

(model name `conb`), varying the values of parameters a and b and determining the associated stability of the fixed points of $1A = (0,0)$, $B = \left(0, \frac{1}{b^2}\right)$, $C = (-a, 0)$, $D = \left(-\frac{ab^4+1}{b^4}, \frac{1}{b^2}\right)$.

SET 4

This is the last of the continuous systems in the plane to be considered. We take the famous model for predator-prey relations known as the Lotka-Volterra system, given as Exercise 3.4(g):

$$\begin{aligned}\dot{x} &= \alpha x - \beta xy\\ \dot{y} &= -\gamma y + \delta xy \qquad \alpha, \beta, \gamma, \delta \geq 0\end{aligned} .$$

There also exists a famous economic growth model leading to the same equations (due to Goodwin, see the appendix to chapter 4 for more details).

There are two equilibria, the origin and $(\gamma/\delta, \alpha/\beta)$. The latter, setting all parameters to one is at $(1,1)$. From the linear approximation (see response to exercise 3.4(g)) it is known that the origin behaves locally as a saddle point, while the fixed point in the positive quadrant has a two–dimensional centre manifold and the Hartman–Grobman Theorem is not applicable. However, from the exact solution (see Exercise 3.7(a)) it is known that there are closed curve orbits around the equilibria, that is, the fixed point is a centre.

Select `lv` and use `Variation` on initial conditions to obtain a plot of several of these curves. Suppose the system can take on negative values. Set x and vary the initial conditions of y to see how the saddle at the origin attracts and then repells orbits.

SET 5

In this set we consider the set of difference equations:

$$\begin{aligned}x_{n+1} &= 2bx_n + 10\\ y_{n+1} &= 2ay_n^2\end{aligned}$$

with fixed points $A = (-\frac{10}{2b-1}, 0)$ and $B = (-\frac{10}{2b-1}, \frac{1}{2a})$. Check that only fixed point A can be locally stable for certain parameter values. Select the model `disa` and use the `Variation` routine to simulate two orbits, one starting in the basin of attraction of A and another starting outside the basin but close to the basin boundary (recall that with the technique of the linear approximation we have established only the local stability of A).

SET 6

In this set we consider Lyapunov's direct method for demonstrating the global stability of equilibria. This method requires that a Lyapunov function be determined whose time derivative is negative along the orbits of the system. For further details refer to Section 3.3 and, in particular, Theorem 3.2. For system Exercise 3.11(b)

$$\begin{aligned} \dot{x} &= -x - y^2 \\ \dot{y} &= kxy \end{aligned} \qquad 3.11(b)$$

a Lyapunov function is

$$V(x,y) = \frac{k}{2}x^2 + \frac{1}{2}y^2.$$

Select `conlyapa` and use the `Variation` routine to show that the stability is independent of initial conditions for selected ranges of k and that the fixed point (0,0) is stable over the specified plane. Try even large subsets of the plane.

Repeat the same procedure with Exercise 3.11(d)

$$\begin{aligned} \dot{x} &= y + kx(x^2 + y^2) \\ \dot{y} &= -x \end{aligned} \qquad 3.11(d)$$

for which the following is a Lyapunov function

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$$

and the model name is `conlyapb`.

computer exercises chapter 4

SET 1

In this set we study of the period-2 and period-4 cycles of the simple logistic map

$$x_{n+1} = \mu x_n(1 - x_n).$$

As determined in exercise 4.9(b), there exists a period-2 cycle for the map with $\mu = 3.2$, which is stable and attracting over $x \in (0, 1)$.

a. Use `Variation` to produce a plot of the time evolution that illustrates that the period-2 cycle is attracting over a selected range of initial conditions. It will probably be necessary to select `Big dots` and possibly `Connect dots` from the `Plot` options. Include in the plot at least one trajectory that is not attracted. Use manual bounds over (0,1) and only a few (say 20) iterations. Check that the values of the periodic points approximate those calculated in exercise 4.9(b).

b. Use `Variation` to get trajectories for $\mu = 3.2, 3.3, 3.4$ on the same plot, using 0 transients and 50 iterates. Note that the trajectory with $\mu = 3.4$ takes a number of iterations which suggest a period-4 cycle, before settling down on the period-2 cycle. Simulate a trajectory of the period-4 cycle for $\mu = 3.5$.

c. From the `New plot` menu select `Shifted and cobweb` and from the `Plot` select `Cobweb animation`. Set the order to 1, which plots values in the (x_n, x_{n+1}) plane (an order of 2 would plot values in the (x_n, x_{n+2}) plane). Use 0 transients and define both axes over (0,1). To begin, use an initial value which is in the basin of attraction of the period-2 cycle. To slow down the convergence drag the arrow above the plot to the right. Once the cycle has been reached click `Stop` to stop the algorithm. In order to see the values of the periodic points it will be helpful to repeat the simulation using some transients (say 100). Repeat the procedure for the period-4 cycle. The period-8 cycle can be simulated for $\mu = 3.55$.

SET 2

In Section 4.4.2 we defined periodic points x^* of a system of difference equations, or map,

$$x_{n+1} = G(x_n)$$

x^* is a periodic point of period k if $x^* = G^k(x^*)$ and $G^n(x) \neq x$ for $1 \leq n < k$, where $G^n(x)$ denotes the k-th iterate of the map G. We study the periodic points of period k of the map G by studying the fixed points of its K-th iterate $F \equiv G^k$. It is then possible to use the linear approximation (see Chapter 3) of the map F to determine the local properties of the nonlinear map if the periodic points are hyperbolic.

Consider the famous model of Hénon:

$$x_{n+1} = a - x_n^2 + by_n$$
$$y_{n+1} = x_n.$$

The four periodic points of period 2 are determined as solutions to the system

$$\bar{x} = a - (a - \bar{x}^2 + b\bar{y})^2 + b\bar{x} + b\bar{y}$$
$$\bar{y} = a - \bar{x}^2 + b\bar{y}$$

and are

$$A = \left(\frac{b - 1 + \sqrt{b^2 - 2b + 1 + 4a}}{2}, \frac{b - 1 + \sqrt{b^2 - 2b + 1 + 4a}}{2} \right)$$
$$B = \left(\frac{b - 1 - \sqrt{b^2 - 2b + 1 + 4a}}{2}, \frac{b - 1 - \sqrt{b^2 - 2b + 1 + 4a}}{2} \right)$$
$$C = \left(\frac{1 - b + \sqrt{4a - 3 + 6b - 3b^2}}{2}, \frac{b - 1 - \sqrt{4a - 3 + 6b - 3b^2}}{2} \right)$$
$$D = \left(\frac{1 - b - \sqrt{4a - 3 + 6b - 3b^2}}{2}, \frac{b - 1 + \sqrt{4a - 3 + 6b - 3b^2}}{2} \right).$$

The points A and B are also fixed points of the system and are real for $a \geq \frac{1}{4}\left(-b^2 + 2b - 1\right)$. The periodic points C and D are real for $a \geq \frac{3}{4}\left(b^2 - 2b + 1\right)$. Notice that when C and D are real so are A and B, since $\frac{3}{4}\left(b^2 - 2b + 1\right) \geq \frac{1}{4}\left(-b^2 + 2b - 1\right) \; \forall b$.

In the following set one parameter fixed, let $b = 0.1$, then for approximately $0 < a < 0.6$ trajectories converge to a fixed point, for approximately $0.6 < a < 1.1$ trajectories converge to a period-2 cycle.

a. Select the model `hénon`. Again it will be useful to switch to bigger dots and connected dots, as well as slow down the plotting, in order to follow the orbits. Insert values for convergence to the fixed point $a = b = 0.1$ from initial conditions $x = 0, y = 0.5$ using 0 tranasients and only a few iterations, say 100. Notice that in converging to the equilibrium the orbit jumps back and forth as the case of improper oscillations. The fixed point attractor in this case is $A = \left(\frac{b-1+\sqrt{b^2-2b+1+4a}}{2}, \frac{b-1+\sqrt{b^2-2b+1+4a}}{2} \right) = (0.1, 0.1)$. Use the time evolution plot to see how a single variable converges.

b. Next we want to set a for a period-2 cycle. The minimum value that guarantees existence of the cycle is $a_r = 0.6075$, because for $a \geq a_r$ the determinant $4a - 3 + 6b - 3b^2$ is nonnegative and the points C e D are real. In that interval also points A and B are real, but both are unstable and do not attract orbits. Set $a = 0.9$ to see convergence to the period-2 cycle. Start with 0 transients and then to better see the values of the cycle rerun the simulation with some transients. The Hénon system for these parameter values asymptotically visits only

$$C = \left(\frac{1}{2}1 - b + \sqrt{4a - 3 + 6b - 3b^2}, \frac{1}{2}b - 1 - \sqrt{4a - 3 + 6b - 3b^2}\right)$$
$$= (0.99, -0.09)$$
$$D = \left(\frac{1}{2}1 - b - \sqrt{4a - 3 + 6b - 3b^2}, \frac{1}{2}b - 1 + \sqrt{4a - 3 + 6b - 3b^2}\right)$$
$$= (-0.09, 0.99).$$

c. The `Cycles` routine, chosen from the `New plot` menu, represents the periodic points of both stable and unstable cycles of the specified period. Use the routine to confirm that the period 2 cycle periodic points are those given in the above exercise. For the algorithm fields choose a small value for the epsilon (which defines the point as within the epsilon radius of the value so that lower values give ever more precision), and maximum tries around 10. The horizontal and vertical axes can be set the same at (-2,2).

d. Use the `Basin of attraction` routine from the `Files` menu to see the initial conditions for which the orbits converge to this period-2 cycle. Because there is only one attractracting set we can plot the basin of infinity, that is, all the initial conditions that eventually diverge to infinity. Then what is not the basin of infinity is the basin of attraction for the period-2 cycle. Select `Basin of infinity` from the `Plot` menu and `Big dots` from the `Options` menu. Set $a = 0.9, b = 0.1$ and set the value at which the algorithm assigns the state infinity at a low number, say 10. In order to get a quick picture of the basin set transients low, say 50, and iterations also low, say 100. The number of trials (different initial conditions) also can be a small number, say 4. Set the horizontal axis (-4,4) and the vertical axis (-10, 10). The two periodic points are visible as pink dots in the black basin. If you chose the `All basins` routine you will arrive, after a little more time, at the same figure but with black representing initial conditions that converge to infinity, the colored basin of attraction for the period-2 cycle and the pink periodic points.

e. The stability of the period-2 cycle is lost around $a \approx 1.15$ and a period-4 cycle appears that is initially stable (we will return to this period-doubling scenario in sections 5.4 and 8.1). Use the routines suggested above to study this period-4 cycle.

SET 3

In this exercise a quasiperiodic trajectory on a torus is simulated using the system

$$\begin{aligned}\dot{x} &= (a-b)x - cy + xz + dx(1-z^2)\\ \dot{y} &= cx + (a-b)y + yz + dy(1-z^2)\\ \dot{z} &= az - x^2 - y^2 - z^2.\end{aligned}$$

Select the `quasi` model. Suggested values are: `Initial values` near the unstable fixed point at the origin, say $(0.1, 0.1, 0.1)$; `Parameters` 2.005, 3, 0.25, 0.2; `step size` 0.05, starting with 0 transients and 5000 iterations. Set the number of iterations for the automatic calculation of the axes at 1000. The orbit moves toward the torus, and then, once there, it continues to wind around the surface. Drag the arrow above the plot to slow down the orbit plotting. Press continue to calculate another 5000 iterations and see how the surface fills in (and again if you like). To see only the torus remove transient behaviour by setting transients at a fairly high value, say 2000.

computer exercises chapter 5

SET 1

In this set we use the `Bifurcation` routine to re-visit models from the computer exercise in Chapter 4 and study the flip bifurcation.

a. Choose the logistic model. From the `Files`, `New plot` menu select the `Bifurcation` routine to produce the numeric bifurcation diagram. While analytically unstable branches are determined and can be plotted (customarily, as a dashed line), iDMC does not determine these unstable fixed points so that the bifurcation diagram represents only stable asymptotic behaviour. All transients must be eliminated in bifurcation diagram calculations. If the number of iterations assigned as transients is insufficient, the variable values represented in the diagram are not truly part of the limit set and may therefore be misleading (an equilibrium point appears as an equilibrium set, etc.). Obviously, the diagram will be finer the smaller the interval of the paramter. However, for a close-up of any region left-click and select the area to be zoomed. This area will be re-calculated.

Specify an initial value in (0,1), give the range for the parameter of (2,4), a vertical axis of (0,1) and for the algorithm 500 transients, 2000 iterations. These values of μ give the bifurcation diagram before the first flip bifurcation and cover the entire range of attracting sets. We will return to the period-doubling scenario in Chapter 8, Section 8.1. Notice that if you choose `Transparency` from the `Options` menu it is clearer which points are being visited more often.

b. Choose the Hénon 2 model and for the `Bifurcation` plot use initial values (0, 0.5), set b at 0.1 and set a to vary over (0, 1.8), a vertical range of (-2, 2), 500 transients and 2000 iterations. Again, using the `Trasparency` option gives a clearer idea of what values are most visited. You should be able to produce orbits for a variety of periodic cycles using the `Trajectory` routine and estimate the values of the periodic points with the `Cycles` algorithm.

SET 2

In this set various views of the fold and transcritical bifurcations are simulated. The equation is that of Exercise 5.6 at the end of chapter 5, the model is called `conbif`:

$$\dot{x} = x^3 + x^2 - (2 + \mu)x + \mu.$$

It is suggested that analytical results are obtained before turning to the simulations.

First study typical transient behaviour of trajectories for each of three intervals of μ: $\mu < -1$; $-1 < \mu < 3; \mu > 3$. It should be pointed out that if there is no stable fixed point the trajectory races off to infinity. It is therefore useful to use a value of μ close to the bifurcation value $\mu = -1$ and to define a limited range for the variable x in the `Manual bounds` option, say (-2, 2). The algorithm data given below allows for a plot in which the trajectory values are still small if μ is set at -1.01. It is then possible to use `Variation` to increase μ until all local dynamic behaviours are evident in the plot. Set the `Initial value` at $x = 0.5$, the algorithm step size at 0.01, use 0 transients and 1500 iterations. Check that the asymptotic behaviour (and approximate value of fixed points) is as found analytically in Exercise 5.6.

SET 3

In this set the fold and flip bifurcations in a discrete-time equation are studied. The equation is that of Exercise 5.8(c) and the model is called `disbif`:

$$x_{n+1} = \mu + x_n - x_n^2.$$

It is suggested that analytical results are obtained before turning to the simulations.

a. First produce the numeric bifurcation diagram. To avoid transient dynamics set the number of `transients` to 500 and set `iterations` 500. Select `Bifurcation` and produce a diagram for $\mu \in (0, 1.2)$. Note where the fold and flip bifurcations occur. Simulate typical trajectories representing behaviour in each of three intervals of μ, setting the `Manual bounds` to limit the range of x, say from -5 to 2. Use the `Variation` procedure to get trajectories in each interval (it will be helpful to select the `connect-dots` option).

SET 4

In this set of simulations the Neimark–Sacker bifurcation is studied using an economic model of the class known as *overlapping generations model* which is presented in Exercise 5.12 at the end of chapter 5 (other references given there). The model, called `olgns` in iDMC, reduces to the system

$$\begin{aligned} c_{n+1} &= l_n^{\mu} \\ l_{n+1} &= b(l_n - c_n). \end{aligned}$$

a. Using the sketch of the relevant parameter subspace of the OLG model given in your answer to Exercise 5.11 of the text, choose a value of μ that, by varying b, results in three types of transient dynamic behaviour. Use the `Variation` routine to get examples of orbits of each type plotted in the state space. `Initial conditions` should be fractional, e.g. at $(c, l) = (0.1, 0.3)$. In order to follow the orbits use the option for `Connect dots` under `Plot` and a small number of iterates, say 200 (no transients). Plot the time evolutions of the variables.

b. Focus the `Variation` routine so as to get a number of invariant circles in the state space over the limited parametric subspace of their existence. For example, for $\mu = 6$ the curves exist over $b \in (1.20, 1.26)$ using the starting point suggested in exercise a. Change the `Plot type` back to `dots` and set `transients` to 1000, `iterates` at 2000 so as to avoid long transients and slow filling in of invariant curves. The invariant circles are most likely to appear quasiperiodic, but there may also be periodic curves (with few points on the invariant circle visited), depending on the value chosen.

c. Use the `Bifurcation` routine to compute the bifurcation diagram, using the same fixed value of μ as used in exercise b., varying the parameter b so as to capture the Neimark–Sacker bifurcation and a few periodic invariant curves. Use the variable l which has greater variation. Having chosen the same parameter values the bifurcation diagram plots the values of the labour variable l taken on by orbits on the invariant circles in the state space as simulated in the previous exercise.

d. Finally, use the `Basin of attraction` routine from the `Files` menu to get a sample of the size and shape of the basin of attraction for one of the invariant circles plotted in b. As there is only one attractracting set we can plot the basin of infinity, that is, all the initial conditions that eventually diverge to infinity. Then what is not the basin of infinity is the basin of attraction for the invariant circle. Select `Basin of infinity` from the `Plot` menu and `Big dots` from the `Options` menu. Set the paramter values so as to reproduce a limit set of one of the invariant circles simulated in b. Set the value at which the algorithm assigns the state infinity at a low number, say 10. In order to get a quick picture of the basin set transients low, say 20, and iterations at 500. The number of trials can be set to 1. Set the horizontal axis (-2, 2) and the vertical axis (-2, 2).

SET 5

The Hopf bifurcation occurs in continuous–time systems when the real part of a pair of complex, conjugate eigenvalues passes through zero. The following numerical simulations are to study the Hopf bifurcation in the system given at the end of Chapter 5, in Exercise 5.7:

$$\dot{x} = y + kx(x^2 + y^2)$$
$$\dot{y} = -x + \mu y$$

for which the bifurcation leads to locally stable limit cycles. The model is called `hopf` in DMC.

a. In the first simulation the transient behaviour is plotted for a few values of μ for a value of k for which the Hopf bifurcation is supercritical and the fixed points and limit cycles are stable. For the simulation set $k = -2$, and choose a small negative value for μ (e.g. $\mu = -0.4$). It is convenient to start close to the fixed point and cycles, say at $(1, 1)$. Setting the step size at 0.05 for 5000 iterations gives an evolution of 250 time periods. Increase the value of μ using `Variation` to show transient behaviours in the state space: convergence to the fixed point, the limit cycle of zero amplitude, a limit cycle of positive amplitude, a divergent trajectory. Adequate `Manual bounds` are (-1, 1.5), (-1, 1.5). (Also try a start point near the origin with μ at say 0.4 and increase the value of the starting point with the procedure `Variation`. In this case trajectories sometimes spiral out to the cycle, and sometimes spiral in to it.)

b. To see the limit cycle orbits set the `transients` value at 4000 and use the `Variation` routine to get a number of cycles.

computer exercises chapter 6

SET 1

In these simulations the map $G(x_n) = \mu - x_n^2$ (model `flip`), found in Exercises 5.8(a) and 6.1, is used to study of various aspects of flip bifurcations and chaotic trajectories.

a. Select values to simulate the time evolution for G over 100 iterations (no transients), $x \in (-0.5, 2)$ and $x_0 = 0.5$. Use the `Variation` procedure to produce a plot of trajectories converging to a fixed point and at least one trajectory converging to a period-2 cycle. The trajectory will be clearer if the starting point is simultaneously varied by 1 at each run. Simulate a period-4 cycle.

b. Recall that a characteristic of chaotic trajectories of strange attractors is their sensitive dependence on initial conditions. To ensure that the system is on its attractor and not merely experiencing chaotic transients set transients to 500 and iterations to 550. Select a value near the strange attractor ($\mu \approx 1.9$ will do) and use `Variation` to simulate a second trajectory beginning within 0.005 of the first. Note the maximum and minimum distances between the two trajectories. How would the initial difference have evolved on a non-chaotic attractor?

c. Changes in the dynamical behaviour arising from varying μ, studied as time evolving trajectories in exercise a. above, are here viewed from the point of view of the *asymptotic* stable dynamics using the `Bifurcation` routine. Set transients to 500, iterations to 1000. For the first run use $\mu \in (-0.25, 1.3)$, beginning with the first stable fixed point, at the fold bifurcation value, followed by the first flip at $\mu = 0.75$ and the second flip at $\mu \approx 1.28$. Take a closer look at the period-doubling scenario using the left-click to select the interval of $\mu \in (0.7, 1.5)$. Finally, take a closer look at the neighbourhood of the period-3 cycle $\mu \in (1.65, 1.8)$.

d. In this exercise we use the `Cobweb animation` routine to display the maps of G, G^2 and G^4, using the map G with $\mu = 1.9$, by setting the order to 1, 2, 4 respectively. It may be necessary to set the delay in order to follow the trajectory and use 0 transients.

SET 2

In this set we study the dynamical behaviour of the Lorenz model:

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= -xz + rx - y \\ \dot{z} &= xy - bz \end{aligned} \tag{6.7}$$

where $x, y, z \in \mathbb{R}$; $\sigma, r, b > 0$. Recall that system (6.7) is symmetrical under the transformation $(x, y, z) \to (-x, -y, z)$ and has potentially 3 equilibria, depending on the value of r. If $0 < r < 1$ the only equilibrium is $E_1 : (0, 0, 0)$. For $r > 1$ there exist two other equilibria, namely $E_2 : \left(+\sqrt{b(r-1)}, +\sqrt{b(r-1)}, r-1\right)$ and $E_3 : \left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right)$. Local stability also depends on the value of r. Let $r > 0$ be the bifurcation parameter, and set other parameter values as $\sigma = 10$, $b = 2.667$. Then, as discussed in section 6.8, the following behaviour occurs around $r = 1$

$r < 1$	1 stable fixed point at origin
$r = 1$	pitchfork bifurcation
$r > 1$	2 stable fixed points, unstable fixed point at origin.

a. A homoclinic connection occurs at $r \approx 13.927$, but it is very difficult to determine the exact value of r for which it takes place. The presence of the homoclinic orbits can be imagined by observing the transition from values slightly smaller to values slightly larger than the homoclinic connection value. For a very smooth curve and precise integration set `step size` to 0.002 and use 2000 iterates. Begin close to the origin (0.001, 0, 0) and use the delay if necessary to follow the obit as it slowly curves around E_2 moving along the z-axis and curving again around E_2 to which it converges for $r = 13.927$. For a value just past the connection value the orbit crosses over the z-axis and curves around (converging to) E_3. (Use, for example, `Variation` with 1 count, changes in r at 0.001, `Manual bounds` $x \in (-15, 15)$, $z \in (0, 25)$.)

b. The fixed points E_2 and E_3 are stable over $(1, r_H)$, $r_H \approx 24.74$, but in simulations a chaotic attractor is often detected for $r > r_c$, $r_c \approx 24.06$. The local neighbourhood for which the fixed point is attracting gets smaller the closer the system is to the Hopf bifurcation (see Figure 6.9 in the book). At $r = 23$, for instance, an orbit beginning at $(1, 1, 1)$ clearly converges to E_3 (use `step size` 0.2 and 5000 iterates). At $r = 24.2$ the orbit from that initial point does not converge, but try a closer initial value $(-8, -8, 25)$ (and set `step size` to 0.05).

c. Simulate the so–called "butterfly attractor", that is, the chaotic attractor of the Lorenz model, by selecting a value of r slightly greater than r_H. The choice of variables x and z ensures a good view.

SET 3

In this section we study a two-dimensional system describing the "backward dynamics" of an overlapping generations model. The basic framework is the same as that of the model studied in the exercises of Chapter 5, Set 4, except that an exponential utility function has been assumed which makes it impossible to invert the system. The time indices have been exchanged in order to study the dynamics of the system moving backwards in time (for further details see Medio, **Choatic Dynamics**, Cambridge University Press, 1992, Section 12.3). The variables are again consumption c and labour l in per-capita terms, the system is

$$l_{n+1} = \left(rce^{-c}\right)^{\frac{1}{\beta}}$$
$$c_{n+1} = \left(rce^{-c}\right)^{\frac{1}{\beta}} - \frac{l_n}{b}$$

with $r > 0$, $b, \beta > 1$ and for iDMC the model name is `olg`.

a. In our analysis so far we have studied how dynamics change through a bifurcation diagram with a single parameter. In this exercise we explore the dynamics over a subspace of the 2-parameter space. Open the `Bifurcation` procedure and click on the `Double parameter` option from the `Plot` menu. Use `Initial values` $l = 5$, $c = 2$. In this plot we set `gamma`$= 0.5$ and use the subspace of the (b, r) parameter plane defined by a $b \in (1.5, 3)$, $r \in (10, 75)$.

The algorithm requests a number of input values. For the approximation process the user must define the precision and infinity. Epsilon is an indication at how fine-grained the user wants the plot to be. The smaller epsilon, the closer a value must be to the point to be defined as that point. First set `epsilon`$= 10e-4$, which is 0.183, rather coarse. Next try with more precision, say $10e - 10 \approx 0.0005$ and try to explain the differences in the two plots. The algorithm needs to know at what point the user considers that a trajectory is on its way to infinity. For the current plot let `infinity`$= 100$. The plot will represent periodic behaviour and the user must set the highest number of periods to be considered for a given simulation (up to 35). It will be tempting to take all higher-order cycles as quasiperiodic, but that is not the case. Here we consider cycles of up to 32 periods.

Recall that the bifurcation diagram is meant to represent asymptotic behaviour and for computational methods, the user must decide at what iteration the dynamical behaviour is no longer transient. The time it takes to reach a limit set varies greatly. A common problem in interpreting results is that the value for tranients is set too low and the plot contains more than the limit set. For the current exercise we can get by with ignoring the first 500 iterations.

The diagram presents a period-doubling scenario that can be followed as a sequence by holding either parameter constant and following changes in the other parameter. For example, follow changes in r with $b = 1.6$. As parameter values for r change, a fixed point becomes a period-2 cycle, and so on, as a cycle of period 2^k loses stability and a stable cycle of period 2^{k+1} replaces it as the limit set.

b. Use the plot produced above to explore the dynamics of the model over the given parameter ranges. For example, fix $b = 1.55$ and produce a bifurcation diagram for $r \in (42, 82)$ (chose from the plot menu `One paramter`) using `Vertical range` = (3.5,6) `Vertical axis` = l. The cascade of flip bifurcations resembles that of the one-dimensional logistic model produced in Chapter 5 exercises, Set 1.

c. Choose a value for r in the chaotic zone, such as $r = 80$ and simulate an orbit using `Trajectory` excluding transients. If the number of iterations is set high enough the attractor should be visible. Select `Time plot` and choose one of the variables to see that, while the variable does not escape from the attractor, neither does it settle down to a periodic set. Return to the state space representation and click on `Continue` a number of times until the attractor seems to be on a continuous curve. Select an area of the attractor to magnify. Notice that the piece of attractor selected is no longer appears a continuous curve, but has a fractal structure. No matter how many iterations used, there is always a magnification level that reveals the underlying fractal structure of the strange attractor (more on this in Section 7.2).

d. Consider again the bifurcation diagram in the single paramter r. Observe that within the chaotic zone there are intervals of r representing periodic limit sets. An example is the periodic window for $r \in (77.95, 78.12)$, in which there are stable period-6 cycles. Exclude transient behaviour and represent the periodic attractor in the state space.

SET 4

In this section we study Maynard Smith's two-dimensional map with very complicated dynamics

$$x_{n+1} = \epsilon x_n + \mu - y_n^2$$
$$y_{n+1} = x_n$$

a. Open the model `msmith`. Begin by producing a `Bifurcation` plot with two paramters, setting a `Initial values`: $x = 0.2$, $y = 0.1$; `Parameter` values $\epsilon \in$

$(-2, 2)$, $\mu \in (-0.5, 4)$ and use a maximum of 32 periods. In the parameter space many periodic " tonuges" can be observed. You should be able to distinguish areas where the equilibrium point loses stability through a flip bifurcation (e.g. ϵ is near 0) and where it loses stabilty through a Neimark bifuraction.

b. Now fix $\epsilon = 0.6$ and produce a bifurcation diagram for $\mu \in (0.3, 1.1)$, using `Vertical range` = (-1.7,1.8). The plot represents how as μ varies, the asymptotic dynamical behaviour of orbits changes from periodic to quasiperiodic to chaotic. At $\mu = 0.48$, for example, the orbits converge to a closed curve with quasiperiodic motion. Fix $\mu = 0.9$ and draw the trajectory using 5000 transients and 10000 iterations. The orbits converge to the chaotic attractor with is distributed in six regions of the (x, y) plane. Increase the value of μ to see the changing form of the attractor. Represent the basin of attraction for different, increasing values of μ over the chaotic range (0.9, 1.07). Observe that as μ increases the attractor pieces extend until at $\mu = \mu_c \approx 1.07$ they become tangent at various points of the basin. For $\mu > \mu_c$ the basin of attraction no longer exists and orbits tend to infinity.

computer exercises chapter 7

SET 1

In these exercises trajectories for each of five values of μ are simulated and the single Lyapunov characteristic exponent for each is calculated. The basic equation for simulations is the map used to study flip bifurcations and chaotic orbits in set 1 of the computer exercises for chapter 6 (the model name is `flip`)

$$G(x_n) = \mu - x_n^2.$$

a. Before calculating the LCEs at various parameter values it is good practice to take a look at the trajectories and get at least an expected sign for the exponent. Set the input values to simulate the time evolution for G over 100 iterations, with $x \in (-1.5, 2)$ and $x_0 = 1.5$. Plot trajectories for the following set of parameter values: $\mu = 0.9$ (period-2 cycle); $\mu = 1.38$ (period-8 cycle); $\mu = 1.6$ (chaotic trajectory); $\mu = 1.76$ (period-3 cycle); $\mu = 1.9$ (chaotic trajectory).

b. Select `Lyapunov exponents` from the `New plot` menu. There are three `Plot` options: `Time`, `Parameter`, `Parameter space`. Begin with a `Time` plot which represents the of the average estimate of the exponent converging in time. Set the `Algorithm` values for long trajectories (5000) as the Lyapunov characteristic exponents are time averages and are more precise the longer is the series used for their approximation. Choose the `Vertical range` to include positive and negative values, but the optimal range will depend on the dynamical behaviour at that parameter value. For each of the trajectories used in a. above plot the Lyapunov exponent and convince yourself that such a value makes sense. Note that this equation at $\mu = 1.9$ can be shown to have the same Lyapunov characteristic exponent that the logistic map has for $\mu = 4$, and that value is the natural logarithm of 2, approximately 0.7.

c. Use the `Bifurcation` routine to get the diagram over a range of μ including all of the dynamical behaviours studied in exercise a. and b. Start iDMC in another window and calculate the Lyapunov exponents over the same range of parameter values by selecting the option for `Parameter` from the Lyapunov exponent routine. Clicking on the `Crosshair` option and holding a line on zero makes it easier to see where exponents become positive. Place the bifurcation diagram above the Lyapunov exponent window to see how the exponents vary as the dynamics change.

SET 2

In this exercise we return to the OLG model (`oldns`) introduced in the computer exercises for Chapter 5, Set 4. Recall that for certain parameter values there are invariant circle solutions which may occasionally appear to be chaotic rather than quasiperiodic as all points on the circle seem to be visited and trajectories seem random. Use the value of μ chosen for exercise b. of that set to produce a quasiperiodic orbit of say 5000 iterates with no transients. It may be useful to slow down the plotting by dragging the speed arrow to the right. For those same parameter values plot the Lyapunov explonents. Is the evidence in favor of quasiperiodic or chaotic orbits?

SET 3

Consider again the 3-dimensional continuous-time system introduced in computer exercise Set 3, Chapter 4 to simulate quasiperiodic orbits (`quasi`). Use the same values for parameters and initial conditions as given there, but set the `Algorithm` for `step size` at 0.1, `transients` at 0, `iterates` at 10000 to get a long series. Look at the orbit in the state space, then plot the LCE's. COmpare the estimated values for the LCEs to those in Set 2 above, regarding the OLG model.

SET 4

In this set we study a simple oligopoly model (due to Tonu Puu) with interesting dynamics. Consider a market composed of only two firms that produce the same good. Let x and y be the supply of firm 1 and firm 2, respectively. We define the function of inverse demand as

$$p = \frac{1}{x+y},$$

where p indicates the price. Suppose that marginal costs of production are respectively a and b. Profits of the two firms are then:

$$\Pi_1 = \frac{x}{x+y} - ax \qquad\qquad \Pi_2 = \frac{y}{x+y} - bx.$$

The "best reply" (or "reaction function") for each firm to the supply of the other is found by solving:

$$\max_x \Pi_1 \qquad \text{such that } x, y \geq 0$$
$$\max_y \Pi_2 \qquad \text{such that } x, y \geq 0$$

from which we have

$$x = \sqrt{\frac{y}{a}} - y \qquad\qquad y = \sqrt{\frac{x}{b}} - x.$$

Suppose that the action of each firm coincides at every instant of time with its best reply. We can describe the supply adjustment process as follows

$$x_{n+1} = \begin{cases} \sqrt{\frac{y_n}{a}} - y_n & \text{if } ay_n \le 1; \\ 0 & \text{if } ay_n > 1, \end{cases} \qquad y_{n+1} = \begin{cases} \sqrt{\frac{x_n}{b}} - x_n & \text{if } bx_n \le 1; \\ 0 & \text{if } bx_n > 1, \end{cases}$$

It should be observed that in the intervals $ay_t > 1$ and $bx_t > 1$ the best reply is to offer a negative quantity. To avoid orbits taking on negative values the following condition must be satisfied $\frac{a}{b}, \frac{b}{a} \le \frac{25}{4}$.

This repeated game has 2 Nash equilibria (or Cournot points, see Puu **Attractors, Bifurcation and Chaos**, Springer, 2000 for further details), the trivial one $(0, 0)$ and a second in the positive quadrant $\left(\frac{b}{(a+b)^2}, \frac{a}{(a+b)^2}\right)$, which are also the fixed points of the dynamical model. It is easy to demonstrate that the trivial equilibrium is unstable while the second is stable over $3 - 2\sqrt{2} < a/b < 3 + 2\sqrt{2}$.

a. Open the model `cournot` and plot the bifurcation diagram using $x = y = 0.01$, $a = 1$, $b \in (5.75, 6.25)$, `Vertical range`= $(0, 0.2)$, 5000 transients and 200 iterations. Choose a value of b for which there exists a period-4 cycle, e.g. $b = 6.15$. Now use the `Basin of attraction` routine with n $b = 6.15$, a horizontal range of $(0, 0.17)$ and vertical range of $(0, 0.045)$. The basin has a checkerboard structure because of the coexistence of two period-8 cycles, each with its own basin of attraction, deriving from the combination of a period-4 cycle in the varible x and another in the variable y. Plot the basin for $b = 6.192$, with the coexistence of two period-24 cycles (a period-12 cycle in each variable).

In the above it was assumed that adjustment was istantaneous. Suppose, instead, that firms adjust their best reply, on the basis of previous decisions, in the direction of the optimal supply without necessarily reaching the optimal immediately. The resulting system is:

$$x_{n+1} = \begin{cases} (1-\gamma)x_n + \gamma\left(\sqrt{\frac{y_n}{a}} - y_n\right) & \text{if } ay_n \le 1 \\ (1-\gamma)x_n & \text{if } ay_n > 1 \end{cases}$$

$$y_{n+1} = \begin{cases} (1-\delta)y_n + \delta\left(\sqrt{\frac{x_n}{b}} - x_n\right) & \text{if } bx_n \le 1 \\ (1-\delta)y_n & \text{if } bx_n > 1 \end{cases}$$

with $0 < \gamma, \delta < 1$. This system has the same equilibria as the previous model and if $\delta = \gamma = 1$ it reduced to the instantaneous adjustment model.

b. Open the `cournotad` model. Notice that for simplicity we have set $\gamma = \delta = c$. Select the `Bifurcation` plot e the `Double parameter` option. Use $x = y =$

0.01, $a = 1$, `Horizontal axis:` $c \in (0.1, 1)$, `Vertical axis:` $b \in (0, 50)$, `Epsilon=` $10e-7 \approx 0.009$, `Infinity=` 10, `Transients=` 500, `Period=` 15. In the plot periodic "tongues" can be observed, that is, combinations of parameter values for which orbits converge to cycles of less than or equal to 15 periods. The curve bounding the red area represents the Neimark bifurcation curve (see Chapter 5.) on which the determinant of the Jacobian matrix calculated at the fixed point is equal to 1. Using the `Crosshair`, find values of the parameter for which the orbits converge to odd-period cycles and simulate the asymptotic state using `Trajectory` with a high number of transients.

computer exercises chapter 8

SET 1

In this set of computer simulations a closer look is taken at the period-doubling route to chaos using the logistic map `logistic`

$$x_{n+1} = \mu x_n(1 - x_n)\mu \in (1, 4).$$

Begin by printing out a bifurcation diagram for the map over $\mu \in (3, 4)$, starting from $x_0 = 0.2$, using 500 transients and 1000 iterates. The diagram will be a useful reference during the following exercises.

a. For the first simulations it may prove easier to follow the trajectories if the `Connect dots` option is chosen. Set the `Manual bounds` to give the evolution of $x \in (0, 1)$ and consider a limited time interval, say (2850, 2900). First look at a period-8 cycle, at $\mu = 3.54$ for example. Next look at a trajectory from the Feigenbaum attractor by setting $\mu = \mu_\infty \approx 3.569446$. The trajectory may appear to be a period-8 cycle similar to that found in the first run, but it is actually aperiodic. However, the Lyapunov characteristic exponent for the Feigenbaum attractor is zero, there is no divergence of nearby trajectories. Leaving μ set to simulate the Feigenbaum attractor, use `Variation` to start at a point only slightly distant from the initial value of the previous run. Do these trajectories appear to diverge or converge? Calculate the Lyapunov characteristic exponent for a trajectory of the Feigenbaum attractor using 5000 iterates. Does the value make sense for this attractor?

b. In this exercise a chaotic trajectory is viewed from several points of view. Simulate a long series using a slightly larger value of μ such as $\mu = 3.7$ for which trajectories are chaotic (using 300 transients and 3000 iterates, for example).

Choose `Algorithm` values such that a short time interval of long-run behaviour is clear, say (2850, 2900), and simulate a chaotic trajectory. Use `Variation` to show sensitive dependence on initial conditions, noting how many iterates are necessary before the distance between trajectories has reached 0.5 (half of the interval).

Select the `Shifted and cobweb` option and plot the series (no transients) in the plane (x_n, x_{n+1}). Much of the curve of the logistic equation for $\mu = 3.7$ appears, but the attractor does not cover the entire unit interval. Click on the `Cobweb animation` routine and note how the trajectory moves over the attractor.

Plot the time convergence of the Lyapunov characteristic exponent for the same parameter value. Do these plots provide evidence as to the chaoticity of the trajectory?

3. Repeat the steps in b. above for $\mu = 4$. At this value the map is characterized by a strange attractor with LCE$= \ln 2 \approx 0.7$. Again consider whether these plots provide evidence for chaoticity, and notice how the various plots change with respect to those for $\mu = 3.7$.

SET 2

a. Again simulate a long series from which transients have been removed using $\mu = 3.83$, a value for which the map has a period-3 cycle. Choose `Algorithm` values to get a time interval over (2850, 2900). What does the plot suggest regarding sensitive dependence on initial conditions and the attractiveness of the period-3 cycle? As further evidence, calculate the LCE, which should converge to a negative number (≈ -0.52).

b. Recall that intermittency refers to the aspect of a trajectory for which regular behaviour is occasionally interrupted by irregular behaviour. Consider values of μ on the right boundary of the period-3 cycle window (the case is described in Section 8.2 and pictorially represented in Figure 8.4). Set $\mu = 3.828427$ and the `Algorithm` values so as to obtain the time evolution of x over (2800, 2900). The three periodic points of the period-3 cycle should be evident. Next try $\mu = 3.828$, the trajectory still spends a lot of time around the three periodic points until it gets through the channel (see Figure 8.4(d)), after which it is erratic until it gets reinjected near the channel once again. For $\mu = 3.82$ the memory of the period-3 cycle is all but lost, although the trajectory does spend a great deal of time near the largest value of the period-3 cycle. The intermittency can be observed using the `Cobweb animation` routine for these values. If the order is set to 1, giving a plot in the $(x_n, x_{n+1}$ plane, from say 0.2 with no transients, the trajectory is seen to wander around and eventually converge to a period-3 cycle for which the 3 periodic points are obvious. To better see these points set transients at 1000. Choose order 3 and see the curve formed by the map G^3, that is $G(G(G(x_n)))$. In the $(x_n, x_{n+3}$ plane the period-3 cycle is represented by a fixed point, on which the trajectory eventually settles.