Solutions to exercises

Exercise 1.1 When we calculated that the sky is as bright as the Sun, we assumed that the line of sight stopped on the star, i.e. stars are opaque. When we calculated the brightness of the sky for a $S^{-5/2}$ power law, we integrated down to zero flux, which (for any particular type of star) means integrating to $r = \infty$. So the lines of sight don't stop on stars in the latter case; stars are treated as transparent.

Exercise 1.2 We use $E = \gamma m_0 c^2$, where E is the energy, m_0 is the rest mass, $\gamma = (1 - v^2/c^2)^{-1/2}$, and c is the speed of light. We have that $10^{20}\,\text{eV} = \gamma c^2 \times 938.28\,\text{MeV}/c^2 \simeq \gamma \times 10^9\,\text{eV}$. The quoted accuracy of the energy does not justify carrying more than just the first significant figure on the proton's rest mass. The γ factor is then just $\gamma = 10^{20}/10^9 = 10^{11}$. The cosmic ray is moving at very close to the speed of light, so it would take about $100\,000$ years for the proton to cross the Galaxy in the Galaxy's rest frame. But moving clocks run slow, so it would take $100\,000/\gamma$ years in the proton's rest frame, i.e. $10^5/10^{11}$ years, or 10^{-6} years, or about 30 seconds!

Exercise 1.3 First we differentiate Equation 1.7 with respect to time t to get

$$2\dot{R}\ddot{R} = \frac{8\pi G}{3}\left(\dot{\rho}R^2 + 2\rho R\dot{R}\right) + \frac{2\Lambda c^2 R\dot{R}}{3}, \tag{S1.1}$$

where we write $\rho = \rho_\text{m} + \rho_\text{r}$ for brevity and the 'dot' notation is used to indicate differentiation with respect to time, i.e. $\dot{R} = \mathrm{d}R/\mathrm{d}t$ and $\ddot{R} = \mathrm{d}^2R/\mathrm{d}t^2$. The conservation of matter energy gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c^2 R^3\right) &= \dot{\rho}c^2R^3 + 3\rho c^2 R^2\dot{R} \\ &= -p\,\frac{\mathrm{d}(R^3)}{\mathrm{d}t} \\ &= -3pR^2\dot{R},\end{aligned}$$

so

$$\dot{\rho}c^2R^3 = -3pR^2\dot{R} - 3\rho c^2R^2\dot{R}.$$

Equation S1.1 has a term $\dot{\rho}R^2$, so we rearrange the above to find

$$\begin{aligned}\dot{\rho}R^2 &= \frac{-3pR\dot{R}}{c^2} - 3\rho R\dot{R} \\ &= -R\dot{R}\left(\frac{3p}{c^2} + 3\rho\right).\end{aligned}$$

Substituting this into Equation S1.1 gives

$$\begin{aligned}2\dot{R}\ddot{R} &= \frac{8\pi G}{3}\left\{2\rho R\dot{R} - R\dot{R}\left(\frac{3p}{c^2} + 3\rho\right)\right\} + \frac{2\Lambda c^2 R\dot{R}}{3} \\ &= \frac{8\pi G R\dot{R}}{3}\left(2\rho - \frac{3p}{c^2} - 3\rho\right) + \frac{2\Lambda c^2}{3}R\dot{R} \\ &= \frac{-8\pi G R\dot{R}}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{2\Lambda c^2}{3}R\dot{R} \\ &= -8\pi G\left(\rho + \frac{3p}{c^2}\right)\frac{R\dot{R}}{3} + \frac{2\Lambda c^2}{3}R\dot{R}.\end{aligned}$$

Dividing this by $2\dot{R}$ gives

$$\begin{aligned}\ddot{R} &= -4\pi G\left(\rho + \frac{3p}{c^2}\right)\frac{R}{3} + \frac{\Lambda c^2 R}{3} \\ &= -4\pi G\left(\rho_{\mathrm{m}} + \rho_{\mathrm{r}} + \frac{3p}{c^2}\right)\frac{R}{3} + \frac{\Lambda c^2 R}{3},\end{aligned}$$

as required.

Exercise 1.4 If $\Lambda = 0$, then Ω_Λ is always zero (Equation 1.17). From Equation 1.33, we therefore have that $(H/H_0)^2 = (1+z)^3$ when $\Omega_{\mathrm{m}} = 1$ and $\Lambda = 0$. Now, Equation 1.28 tells us that

$$H = \frac{-1}{1+z}\frac{\mathrm{d}z}{\mathrm{d}t},$$

so

$$\frac{1}{H_0^2}\frac{1}{(1+z)^2}\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = (1+z)^3,$$

which we may write more simply as $\mathrm{d}z/\mathrm{d}t \propto (1+z)^{5/2}$, or $\mathrm{d}t/\mathrm{d}z \propto (1+z)^{-5/2}$. Integrating this with respect to z, we get $t \propto (1+z)^{-3/2}$. But $1 + z = R_0/R$, so $t \propto R^{3/2}$, or

$$R = \alpha t^{2/3}, \tag{S1.2}$$

where α is some constant. In particular, at the current time $t = t_0$ we have

$$R_0 = \alpha t_0^{2/3}, \tag{S1.3}$$

and dividing Equation S1.2 by Equation S1.3 gives $R/R_0 = (t/t_0)^{2/3}$.

Exercise 1.5 We can rearrange Equation 1.35 to read

$$R = R_0\left(\frac{t}{t_0}\right)^{2/3}. \tag{S1.4}$$

From Equation 1.12, we have that $H = (1/R)\,\mathrm{d}R/\mathrm{d}t$. Differentiating Equation S1.4, we get

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{2}{3}\frac{R_0}{t_0^{2/3}}t^{-1/3}.$$

At a time $t = t_0$, this is just

$$\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t=t_0} = \frac{2}{3}\frac{R_0}{t_0}.$$

Therefore the Hubble parameter at a time $t = t_0$ in this model Universe is

$$H_0 = \frac{1}{R_0}\left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{t=t_0} = \frac{1}{R_0}\frac{2}{3}\frac{R_0}{t_0} = \frac{2}{3t_0},$$

or $t_0 = 2/(3H_0)$ as required. Putting in $H_0 = 72 \pm 3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, we find $t_0 = 9.1 \pm 0.4\,\mathrm{Gyr}$.

Exercise 1.6 The angular diameter in degrees will be inversely proportional to d_{A} (Equation 1.47), so the angular area (e.g. in square degrees) will vary as

$\theta^2 \propto d_A^{-2}$. The flux will be inversely proportional to d_L^2 (Equation 1.49), i.e. $S \propto d_L^{-2}$. The surface brightness will therefore vary as $S/\theta^2 \propto d_A^2/d_L^2$. But $d_L = (1+z)^2 d_A$ (Equation 1.50), so surface brightness must vary as $(1+z)^{-4}$.

Exercise 1.7 In Section 1.5 we are given that $H_0 = 72 \pm 3\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\Lambda,0} = 0.742 \pm 0.030$. One parsec is 3.09×10^{16} m, so in SI units, $H_0 = (2.3 \pm 0.1) \times 10^{-18}\,\mathrm{s^{-1}}$. Equation 1.17 relates these two quantities to Λ: $\Omega_{\Lambda,0} = \Lambda c^2/(3H_0^2)$, so $\Lambda = 3\Omega_{\Lambda,0}\, H_0^2/c^2$. Putting in the numbers, we get $\Lambda = (1.3 \pm 0.2) \times 10^{-52}\,\mathrm{m^{-2}}$. The horizon size will be $\sqrt{3/\Lambda} = (1.5 \pm 0.1) \times 10^{26}$ m, or 4900 ± 300 Mpc. This cosmological event horizon will be exceedingly distant; for comparison, the current radius of the observable Universe in Section 1.9 is about $3.53c/H_0 = 14\,900$ Mpc.

Exercise 2.1 The 13.6 eV photon *does* ionize another atom. However, the process of recombination needn't result in the emission of just *one* photon. Sometimes the electron will bind first in a high energy state (releasing one photon with an energy < 13.6 eV), then release the remaining energy in stages as the electron drops down the energy levels of the hydrogen atom. Each of these stages will involve the release of a photon, but none of these photons will have enough energy on its own to ionize hydrogen atoms.

Exercise 2.2 We are given that $T = 2.725 \pm 0.001$ K, so the energy density must be $\rho_{r,0}\, c^2 = 4\sigma T^4/c = 4 \times 5.67 \times 10^{-8} \times 2.725^4/(3.00 \times 10^8)$ joules per cubic metre, i.e. $\rho_{r,0}\, c^2 = 4.17 \times 10^{-14}\,\mathrm{J\,m^{-3}}$, or mass-equivalent density of $\rho_{r,0} = 4.64 \times 10^{-31}\,\mathrm{kg\,m^{-3}}$. Applying Equation 1.16, and remembering that $H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} = 3.24 \times 10^{-18} h\,\mathrm{s^{-1}}$, we find that

$$\Omega_{r,0} = \frac{8\pi G\, \rho_{r,0}}{3H_0^2} = 2.47h^{-2} \times 10^{-5}.$$

So

$$\Omega_{r,0}\, h^2 \simeq 2.5 \times 10^{-5},$$

as required.

Exercise 2.3 The matter energy density scales as R^{-3}, while the photon/neutrino energy density scales as R^{-4}. Therefore from Equations 1.15 and 1.16, $\Omega_r/\Omega_m = (1+z)\,\Omega_{r,0}/\Omega_{m,0}$. From Exercise 2.2 and the text following it, we have that $\Omega_{r,0}\, h^2 \simeq 4.2 \times 10^{-5}(T_{\mathrm{CMB},0}/2.725\,\mathrm{K})^4$. The epoch of matter–radiation equality must by definition satisfy $\Omega_r/\Omega_m = 1$, so

$$\begin{aligned} 1 + z_{\mathrm{eq}} &= \frac{\Omega_{m,0}}{\Omega_{r,0}} \\ &= \frac{h^2}{4.2 \times 10^{-5}}\, \Omega_{m,0}\, (T_{\mathrm{CMB},0}/2.725\,\mathrm{K})^{-4} \\ &\simeq 23\,800\, \Omega_{m,0}\, h^2 (T_{\mathrm{CMB},0}/2.725\,\mathrm{K})^{-4}, \end{aligned}$$

as required.

Exercise 2.4 The analysis is the same up to Equation 1.30, where ρ this time is ρ_r. However, instead of $\rho = \rho_0 \times R_0^3/R^3$, we must also take into account the fact that photons lose energy from redshifting, so $\rho_r = \rho_0 \times R_0^4/R^4$. With Λ set to zero, the equivalent of Equation 1.32 comes out as

$$\left(\frac{H}{H_0}\right)^2 = (1+z)^2 \left(1 - \Omega_{r,0} + \Omega_{r,0}\, (1+z)^2\right),$$

and inserting $\Omega_{\mathrm{r},0} = 1$ and using $H^2 = (1+z)^{-2}\,(\mathrm{d}z/\mathrm{d}t)^2$, we find that

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = H_0^2(1+z)^6$$

so

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\mathrm{d}(1+z)}{\mathrm{d}t} = H_0(1+z)^3.$$

Now the dimensionless scale factor a is related to redshift via $a = 1/(1+z)$, so we could write this as

$$\frac{\mathrm{d}(a^{-1})}{\mathrm{d}t} = H_0 a^{-3}$$

thus

$$-a^{-2}\frac{\mathrm{d}a}{\mathrm{d}t} = H_0 a^{-3}$$

hence

$$a\,\mathrm{d}a \propto \mathrm{d}t.$$

Integrating this gives $a^2 \propto t$, or $a \propto t^{1/2}$ as required.

Exercise 2.5 $\hbar$ is measured in J s. A Joule has dimensions of energy (like $\frac{1}{2}mv^2$) so it has dimensions $\mathrm{ML^2T^{-2}}$, where we write M for the dimension of mass, L for length, and T for time. (Note that numerical constants are ignored in dimensional analysis.) Therefore we can write the dimensions of $\hbar$ as $[\hbar] = \mathrm{ML^2T^{-1}}$. Similarly, the dimensions of c are $[c] = \mathrm{LT^{-1}}$. To find the dimensions of G, we can start with the familiar equation $F = GMm/r^2$, and note that force is mass times acceleration, so $ma = GMm/r^2$ or $G = ar^2/M$, so the dimensions of G are $[G] = \mathrm{LT^{-2}L^2/M} = \mathrm{M^{-1}L^3T^{-2}}$. Now let's suppose that the Planck time is given by a formula of the form $\hbar^x c^y G^z$, where the constants x, y and z are to be determined. The result must have the dimensions of time, so

$$\mathrm{T} = \left(\mathrm{ML^2T^{-1}}\right)^x \left(\mathrm{LT^{-1}}\right)^y \left(\mathrm{M^{-1}L^3T^{-2}}\right)^z.$$

Multiplying this out and rearranging gives

$$\mathrm{T} = \mathrm{M}^{x-z}\mathrm{L}^{2x+y+3z}\mathrm{T}^{-x-y-2z}.$$

The left-hand side has no mass M, so $x - z$ must equal zero, i.e. $x = z$. The left-hand side also has no length L, so $2x + y + 3z = 0$. The left-hand side has exactly one power of T, so $-x - y - 2z = 1$. We have three simultaneous equations for three unknowns. Substituting in $x = z$ into the other two equations gives $5x + y = 0$ and $-3x - y = 1$. Therefore $y = -1 - 3x = -5x$, or $x = 1/2$. Since $x = z$, we have $z = 1/2$. Finally, any of the equations involving y imply that $y = -5/2$. Therefore the characteristic time must be of the form $\hbar^x c^y G^z = \hbar^{1/2}c^{-5/2}G^{1/2} = \sqrt{\hbar G/c^5}$, as required.

Exercise 2.6 We have already that $(1/R)\,\mathrm{d}^2R/\mathrm{d}t^2 = \alpha(\alpha-1)t^{-2}$. Since t is positive and $\alpha > 1$, the right-hand side must be positive. Therefore the left-hand side must also be positive. Since R is also positive, $\mathrm{d}^2R/\mathrm{d}t^2 > 0$.

Exercise 2.7 We start with

$$3H\dot{\phi} = -V' \qquad \text{(Eqn 2.23)}$$

and then use

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2} V. \qquad \text{(Eqn 2.24)}$$

Now the $H\,\mathrm{d}t$ term in the integral in the question can also be expressed as

$$H\,\mathrm{d}t = H\,\frac{\mathrm{d}t}{\mathrm{d}\phi}\,\mathrm{d}\phi = H\,\frac{\mathrm{d}\phi}{\dot{\phi}}.$$

Next we use Equation 2.23 to get

$$H\,\mathrm{d}t = H\,\frac{\mathrm{d}\phi}{(-V'/3H)} = -3H^2\,\frac{\mathrm{d}\phi}{V'}.$$

Finally, using Equation 2.24 this comes out as

$$H\,\mathrm{d}t = \frac{-8\pi}{m_{\rm Pl}^2}\left(\frac{V}{V'}\right)\mathrm{d}\phi,$$

so we reach the required integral:

$$N = \frac{-8\pi}{m_{\rm Pl}^2}\int_{\phi_2}^{\phi_1}\frac{V}{V'}\,\mathrm{d}\phi.$$

For the next part, we set $V' \simeq V/\phi$ and $\phi_1 = 0$ (as advised in the question) to write this as

$$N = \frac{-8\pi}{m_{\rm Pl}^2}\int_{\phi_2}^{0}\frac{V}{V}\phi\,\mathrm{d}\phi.$$

Evaluating this integral gives

$$N = \frac{4\pi}{m_{\rm Pl}^2}\phi_2^2 = \left(\frac{2\sqrt{\pi}\phi_2}{m_{\rm Pl}}\right)^2.$$

Thus to have $N > 60$ we need $\phi_2 > m_{\rm Pl}\sqrt{60}/(2\sqrt{\pi})$, or in other words, $\phi_2 > 2.2m_{\rm Pl}$.

Exercise 2.8 No, not immediately. At first the CMB will appear very uniform, as you receive light from only your immediate neighbourhood. As time progresses you will receive light from larger and more distant parts of the Universe. You'll only be able to see the structures with wavelength λ once light has had time to travel the distance λ, i.e. after a time $\delta t = \lambda/c$, where c is the speed of light. The size of the largest acoustic peak is set by the sound horizon after inflation. Once light has had time to travel this distance, all the acoustics will start to become visible. Also, the acoustic peaks will have a different *angular* size on the sky, because the surface of last scattering was closer. Finally, the CMB wouldn't have peaked at microwave wavelengths then, so perhaps we shouldn't call it the CMB then!

Exercise 2.9 We found in Section 2.7 that the particle horizon radius at recombination was $2c/H = 0.46\,\mathrm{Mpc}$. The sound speed is $c_{\rm s} = c/\sqrt{3}$, so the sound horizon will be $2c_{\rm s}/H = (2c/H) \times (c_{\rm s}/c) = 0.46/\sqrt{3}\,\mathrm{Mpc} = 0.27\,\mathrm{Mpc}$.

Exercise 2.10 Dark matter clumps through gravitation, while dark energy appears to be smoothly distributed through space. Dark matter is also essentially pressureless, with $\Omega_{\rm m}$ dominated by the rest mass of the dark matter particles, while dark energy has a strong negative pressure. Dark matter makes up about 20% of the total energy density of the Universe, and at recombination made up about 70%. Dark energy, meanwhile, was negligible at recombination and yet dominates the present-day energy density of the Universe. (One hopes that it will soon be possible to add that the dark matter particle has been directly detected, though that is not yet true at the time of writing; certainly, the proposed particle physics mechanisms for generating dark matter and dark energy are very different.)

Exercise 2.11 One parsec is about 3.09×10^{16} m, so $H_0 = 72 \times 10^3/(10^6 \times 3.09 \times 10^{16}) \simeq 2.33 \times 10^{-18}\,\mathrm{s}^{-1}$. In Chapter 1 we saw that $\Omega_{\Lambda,0} = \Lambda c^2/(3H_0^2)$, so $\Lambda = 3\Omega_{\Lambda,0}\,H_0^2/c^2$. Putting in the numbers gives $\Lambda = 1.3 \times 10^{-52}\,\mathrm{m}^{-2}$.

Exercise 3.1 The luminosity contributed by a shell of radius $r \to r + \mathrm{d}r$ will be $I(r)$ times the area of the shell, $2\pi r\,\mathrm{d}r$. Summing these shells, the total luminosity will be $L = \int_0^\infty I(r)\,2\pi r\,\mathrm{d}r$. Let's define L_0 to be the luminosity with $I_0 = r_0 = 1$, i.e.

$$L_0 = \int_0^\infty f(r)\,2\pi r\,\mathrm{d}r.$$

Now let's calculate the luminosity in the more general case:

$$\begin{aligned} L &= \int_0^\infty I_0\,f\left(\frac{r}{r_0}\right)\,2\pi r\,\mathrm{d}r \\ &= I_0 r_0^2 \int_0^\infty f\left(\frac{r}{r_0}\right)\,2\pi\frac{r}{r_0}\,\mathrm{d}\left(\frac{r}{r_0}\right). \end{aligned}$$

But this integral has the same form as the integral defining L_0, which also integrates from 0 to ∞, so $L = I_0 r_0^2 L_0$.

Exercise 3.2 A shell of thickness $\mathrm{d}r$ and radius r will have mass $\mathrm{d}M = 4\pi r^2\rho\,\mathrm{d}r$. The gravitational potential energy of this shell will be

$$\mathrm{d}E_{\mathrm{GR}} = \frac{-G\,M(<r)\,\mathrm{d}M}{r}, \qquad \text{(S3.1)}$$

where $M(<r)$ is the mass enclosed within a radius r, i.e.

$$M(<r) = \tfrac{4}{3}\pi r^3\rho,$$

and the mass of the shell is

$$\mathrm{d}M = 4\pi r^2\rho\,\mathrm{d}r.$$

Substituting this into Equation S3.1 gives

$$\mathrm{d}E_{\mathrm{GR}} = \frac{-G\frac{4}{3}\pi r^3\rho}{r}\,\mathrm{d}M = -G\tfrac{4}{3}\pi r^2\rho \times 4\pi r^2\rho\,\mathrm{d}r$$

so

$$\mathrm{d}E_{\mathrm{GR}} = -3G \times \left(\tfrac{4}{3}\pi r^2\rho\right)^2\,\mathrm{d}r.$$

Integrating this from radius 0 to radius R gives

$$\begin{aligned}
E_{\mathrm{GR}} &= -3G\int_0^R \left(\tfrac{4}{3}\pi r^2\rho\right)^2\,\mathrm{d}r = -3G\left(\tfrac{4}{3}\pi\rho\right)^2\frac{R^5}{5}\\
&= \frac{-3G}{5R}\left(\tfrac{4}{3}\pi R^3\rho\right)^2\\
&= -\frac{3GM^2}{5R},
\end{aligned}$$

where $M = \frac{4}{3}\pi R^3\rho$ is the total mass of the sphere.

Exercise 3.3 The kinetic energy will be $E_{\mathrm{K}} = \frac{3}{2}NkT$, where N is the number of gas particles. Virial equilibrium is $2E_{\mathrm{K}} = -E_{\mathrm{GR}}$, i.e.

$$3NkT = \frac{3}{5}\frac{GM^2}{R}.$$

The requirement for gravitational collapse is therefore

$$3NkT < \frac{3}{5}\frac{GM^2}{R}.$$

To reach Equation 3.7, we need to eliminate N and R. To a good approximation, at recombination we can assume that the gas particle masses are the proton mass m_{p}, so the number of particles must be $N = M/m_{\mathrm{p}}$. We can also use $M = \frac{4}{3}\pi R^3$ to eliminate R, since $R = (3M/4\pi\rho)^{1/3}$. Inserting these substitutions gives

$$3\frac{M}{m_{\mathrm{p}}}kT < \frac{3}{5}GM^2\left(\frac{4\pi\rho}{3M}\right)^{1/3},$$

which when rearranged in terms of M gives the required equation.

The current temperature of the CMB is about 2.7 K, and the redshift of recombination is about $z = 1000$, so the photon temperature at recombination must be $T = 2.7(1+z) \simeq 3000\,\mathrm{K}$. Matter and radiation will just have been in thermal equilibrium, so this will have been the matter temperature too. The baryonic density will be proportional to $(1+z)^3$, and using Equation 1.26 and $\rho_{\mathrm{b}} = \Omega_{\mathrm{b}}\,\rho_{\mathrm{crit}}$ (Equation 1.22), we have that the baryonic density at $z = 1000$ will be

$$\begin{aligned}
\rho_{\mathrm{b}} &= \rho_{\mathrm{b},0}(1+z)^3\\
&= \rho_{\mathrm{crit}} \times \Omega_{\mathrm{b},0}\,(1+z)^3\\
&= 1.8789\times10^{-26}\times\Omega_{\mathrm{b},0}\,h^2(1+z)^3\,\mathrm{kg\,m^{-3}}\\
&= 1.8789\times10^{-26}\times2.273\times10^{-2}\times(1+1000)^3\,\mathrm{kg\,m^{-3}}\\
&\simeq 4.3\times10^{-19}\,\mathrm{kg\,m^{-3}}.
\end{aligned}$$

Putting in the numbers gives

$$\begin{aligned}
M &> \left(\frac{5\times(1.381\times10^{-23}\,\mathrm{J\,K^{-1}})\times3000\,\mathrm{K}}{(6.673\times10^{-11}\,\mathrm{N\,m^2\,kg^{-2}})\times(1.673\times10^{-27}\,\mathrm{kg})}\right)^{3/2}\times\left(\frac{3}{4\pi\times4.3\times10^{-19}\,\mathrm{kg\,m^{-3}}}\right)^{1/2}\\
&\simeq 2\times10^{36}\,\mathrm{kg},
\end{aligned}$$

or $M > 10^6\,\mathrm{M}_{\odot}$, as required.

Exercise 3.4 For a *flat* universe, the comoving distance is the same as the proper motion distance (Equation 1.56). This isn't true in general (watch out!) but it's true in a flat universe. The proper motion distance is related to the angular diameter distance d_A by Equation 1.50, which gives $d_A = d_{\text{comoving}}/(1+z)$. The definition of angular diameter distance in Equation 1.47 gives us a relationship between the size of an object *as it was at the time of redshift* z and the angular size as it appears today. The proper size of the BAO wiggles is just the comoving size divided by $(1+z)$, i.e. $L_{\text{BAO}}/(1+z)$. The angular diameter distance to redshift z is therefore $d_A = (L_{\text{BAO}}/(1+z))/\theta_{\text{BAO}}$. The comoving distance to redshift z must therefore be $d_{\text{comoving}} = d_A \times (1+z) = L_{\text{BAO}}/\theta_{\text{BAO}}$, as required.

Exercise 3.5 Here the trick is to use Equation 1.43. It follows from that relation that a small comoving interval along the redshift axis must equal $\delta d_{\text{comoving}} = c\,\delta z/H(z)$. Setting this comoving interval to L_{BAO} gives us $L_{\text{BAO}} = c\,\delta z/H(z)$, so $H(z) = c\,\delta z/L_{\text{BAO}}$, as required.

Exercise 3.6 No. The amplitude of the fluctuations could depend on the bias, but the scale length itself is bias-independent.

Exercise 4.1 First, we need to get Equation 1.7 into a form where the only time-dependent parameter is R. The density ρ is time-dependent and varies as $\rho = \rho_0(R_0/R)^3$ (where subscript 0 indicates present-day values), so we have

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{R_0}{R}\right)^3 R^2 - c^2 = \frac{8\pi G\rho_0 R_0^3}{3}R^{-1} - c^2$$

(where we've used $k = +1$). If we set $\mathrm{d}R/\mathrm{d}t = 0$ and solve, we find that $R_{\text{max}} = R = 8\pi G\rho_0 R_0^3/(3c^2)$. Therefore

$$\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2 = \frac{R_{\text{max}}}{R}c^2 - c^2.$$

Using the chain rule we have that

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\theta}\right)^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\left(\frac{\mathrm{d}t}{\mathrm{d}\theta}\right)^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2\left(\frac{R}{c}\right)^2$$

and so

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\theta}\right)^2 = \left(\frac{R}{c}\right)^2\left(\frac{R_{\text{max}}}{R}c^2 - c^2\right) = R_{\text{max}}\,R - R^2,$$

as required.

We're asked to verify that Equation 4.2 works rather than proving it, so all we have to do is substitute it in. Differentiating Equation 4.2 with respect to θ gives

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{R_{\text{max}}}{2}\sin\theta$$

so

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\theta}\right)^2 = \frac{R_{\text{max}}^2}{4}\sin^2\theta = \frac{R_{\text{max}}^2}{4}\left(1 - \cos^2\theta\right).$$

Meanwhile,

$$\begin{aligned}
R_{\max} R - R^2 &= \frac{R_{\max}^2}{2}(1-\cos\theta) - \frac{R_{\max}^2}{4}(1-\cos\theta)^2 \\
&= \frac{R_{\max}^2}{4}(2-2\cos\theta) - \frac{R_{\max}^2}{4}\left(1+\cos^2\theta - 2\cos\theta\right) \\
&= \frac{R_{\max}^2}{4}\left(2-2\cos\theta - 1 - \cos^2\theta + 2\cos\theta\right) \\
&= \frac{R_{\max}^2}{4}\left(1-\cos^2\theta\right),
\end{aligned}$$

which equals $(\mathrm{d}R/\mathrm{d}\theta)^2$ as above.

Finally, we just need to differentiate Equation 4.3, which gives

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{R_{\max}}{2c}(1-\cos\theta) = \frac{R}{c},$$

as required.

Therefore Equations 4.2 and 4.3 are a solution.

Exercise 4.2 To show this, we'll first get things in terms of H. It's a flat matter-dominated universe, so $\Omega_\mathrm{m} = 1 = 8\pi G\rho_\mathrm{m}/(3H^2)$, thus $4\pi G\rho_\mathrm{m} = 3H^2/2$. We also know that $H(t) = \dot{a}/a$. Substituting this into Equation 4.9, we have

$$\ddot{\delta} + 2H(t)\,\dot{\delta} = 3H^2(t)\,\delta/2.$$

Next we use $H(t) = 2/(3t)$ to reformulate this in terms of a differential equation involving just δ and time:

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} = \frac{3}{2}\left(\frac{2}{3t}\right)^2 \delta = \frac{2}{3t^2}\delta.$$

Next, let's try power law solutions $\delta = bt^c$ where b and c are constants. Then $\dot{\delta} = bct^{c-1}$ and $\ddot{\delta} = bc(c-1)t^{c-2}$. Substituting in, we find

$$bc(c-1)t^{c-2} + \frac{4}{3t}bct^{c-1} = \frac{2}{3t^2}bt^c.$$

Collecting the terms together, we find that

$$bc(c-1)t^{c-2} + \tfrac{4}{3}bct^{c-2} = \tfrac{2}{3}bt^{c-2},$$

and dividing through by bt^{c-2} gives

$$c(c-1) + \tfrac{4}{3}c = \tfrac{2}{3}.$$

The solution to this quadratic equation is $c = 2/3$ or $c = -1$. The -1 solution is known as the decaying mode, and is not physically relevant in this universe (it decays more rapidly than the growing mode grows and is quickly negligible). The $2/3$ power law time-dependence (which we found ultimately from linearized fluid dynamic equations) is identical to Equation 4.8, which is why the latter is known as the linear theory.

Exercise 4.3 The redder colour will be the one with the larger V-band to B-band flux ratio $S_\mathrm{V}/S_\mathrm{B}$. The fluxes are related to the magnitudes by

$V = -2.5 \log_{10} S_V + c_V$ and $B = -2.5 \log_{10} S_B + c_B$, where c_V and c_B are constants (not necessarily identical). Therefore

$$\begin{aligned}(\text{B–V}) &= -2.5 \log_{10} S_B + c_B + 2.5 \log_{10} S_V - c_V \\ &= -2.5(\log_{10} S_B - \log_{10} S_V) + (c_B - c_V) \\ &= -2.5 \log_{10}(S_B/S_V) + (c_B - c_V) \\ &= 2.5 \log_{10}(S_V/S_B) + (c_B - c_V),\end{aligned}$$

which gives

$$2.5 \log_{10}(S_V/S_B) = (\text{B–V}) - (c_B - c_V)$$

so

$$\log_{10}(S_V/S_B) = (\text{B–V})/2.5 - (c_B - c_V)/2.5$$

thus

$$\begin{aligned}(S_V/S_B) &= 10^{(\text{B–V})/2.5-(c_B-c_V)/2.5} \\ &= 10^{(\text{B–V})/2.5} \times 10^{-(c_B-c_V)/2.5} \\ &= 10^{(\text{B–V})/2.5} \times \text{constant}.\end{aligned}$$

Therefore the larger the value of (B–V), the larger the value of S_V/S_B. Therefore $(\text{B–V}) = 1$ is redder than $(\text{B–V}) = 0$.

Exercise 4.4 We haven't specified the geometry yet, so let's keep things simple. Let's take the dust and stars to be in a cylinder facing us, with cross-sectional area A. Let's set the length of the cylinder to be h, and measure distances along this length with the variable x. An infinitesimal layer would have thickness $\mathrm{d}x$ and volume $A\,\mathrm{d}x$. The bigger the volume, the more stars it will contain, so let's set the luminosity of the shell to be $\mathrm{d}L = \rho A\,\mathrm{d}x$, where ρ is a constant (the luminosity density). By the time the light emerges from the end of cylinder, it will have been extinguished by a factor of $\mathrm{e}^{\tau(x)}$, where $\tau(x)$ is the optical depth at a distance x into the cylinder. This optical depth must be proportional to x, because each increment δx will suppress the light by the same factor, which we could write as $\mathrm{e}^{\delta\tau}$, so let's write that as $\tau = kx$. We could, for example, write the optical depth from one end of the cloud to the other as $\tau_{\text{total}} = kh$. The light that emerges from the shell at $x \to x + \mathrm{d}x$ will therefore be $\mathrm{d}L_{\text{out}} = L \times \mathrm{e}^{-\tau(x)} = \rho A\,\mathrm{d}x \times \mathrm{e}^{-kx}$. If we integrate that from $x = 0$ to $x = h$, we get

$$L_{\text{out}} = \int_{x=0}^{h} \rho A\,\mathrm{e}^{-kx}\,\mathrm{d}x = \frac{\rho A}{k}\left(1 - \mathrm{e}^{-kh}\right).$$

Some quick checks: note that k has dimensions of one over length (because $\tau = kx$ and τ is dimensionless), so A/k has dimensions of volume, and so $\rho A/k$ is luminosity density times volume, which is a luminosity. Note also that kh is dimensionless.

Now, what would happen if there were no dust? The luminosity would just be $L_{\text{no dust}} = \rho A h$. The dust has therefore reduced the output luminosity by a factor

$$\frac{L_{\text{out}}}{L_{\text{no dust}}} = \frac{\rho A/k}{\rho A h}\left(1 - \mathrm{e}^{-kh}\right) = \frac{1}{kh}\left(1 - \mathrm{e}^{-kh}\right).$$

This ratio is independent of the geometrical cross section A and of the luminosity density ρ. If the cloud is deep enough, then the term in brackets is $\simeq 1$, so we just have $L_{\text{out}}/L_{\text{no dust}} = 1/(kh)$. We can now write this for Hα light:

$$\frac{L_{\text{out}}(\text{H}\alpha)}{L_{\text{no dust}}(\text{H}\alpha)} = \frac{1}{k_{\text{H}\alpha}\, h}.$$

For Hβ, we have that $\tau_{\text{H}\beta} \simeq 1.45\,\tau_{\text{H}\alpha}$, so $k_{\text{H}\beta} = 1.45\,k_{\text{H}\alpha}$, thus

$$\frac{L_{\text{out}}(\text{H}\beta)}{L_{\text{no dust}}(\text{H}\beta)} = \frac{1}{k_{\text{H}\beta}\, h} = \frac{1}{1.45\,k_{\text{H}\alpha}\, h} = \frac{1}{1.45}\,\frac{L_{\text{out}}(\text{H}\alpha)}{L_{\text{no dust}}(\text{H}\alpha)}.$$

Therefore

$$\frac{L_{\text{out}}(\text{H}\alpha)}{L_{\text{out}}(\text{H}\beta)} = 1.45\,\frac{L_{\text{no dust}}(\text{H}\alpha)}{L_{\text{no dust}}(\text{H}\beta)}. \qquad \text{(S4.1)}$$

This is independent of h, so we've now removed all dependence on the geometry. So even if kh is enormous and $L_{\text{out}} \ll L_{\text{no dust}}$, the luminosity ratio of Hα and Hβ is only ever 1.45 times the ratio that you get with no dust, when enough dust is evenly mixed with the gas emitting the emission lines.

Now suppose that you wrongly assumed that it's a simple dust screen with an optical depth of $\tau_{\text{H}\alpha}$ for Hα and $\tau_{\text{H}\beta} = 1.45\,\tau_{\text{H}\alpha}$ for Hβ. Your luminosities would be

$$L_{\text{out}}(\text{H}\alpha) = L_{\text{no dust}}(\text{H}\alpha) \times \mathrm{e}^{-\tau_{\text{H}\alpha}},$$
$$L_{\text{out}}(\text{H}\beta) = L_{\text{no dust}}(\text{H}\beta) \times \mathrm{e}^{-1.45\,\tau_{\text{H}\alpha}},$$

so the luminosity ratio would be

$$\frac{L_{\text{out}}(\text{H}\alpha)}{L_{\text{out}}(\text{H}\beta)} = \frac{L_{\text{no dust}}(\text{H}\alpha)}{L_{\text{no dust}}(\text{H}\beta)}\,\mathrm{e}^{0.45\,\tau_{\text{H}\alpha}}. \qquad \text{(S4.2)}$$

Comparing this to Equation S4.1, we have $1.45 = \mathrm{e}^{0.45\,\tau_{\text{H}\alpha}}$, or $\tau_{\text{H}\alpha} = \ln(1.45)/0.45 \simeq 0.83$. Since $\tau_{\text{H}\alpha} \simeq 0.7A_{\text{V}}$, we have $A_{\text{V}} \simeq 1.2$. So, if you have an optically-thick cloud in which the dust is well-mixed with the gas, but you wrongly assumed a foreground dust screen, you'd infer a V-band extinction of just 1.2 magnitudes, regardless of what the real extinction τ_{total} is from one end of the cloud to the other.

Exercise 4.5 Astronomical absolute magnitudes are defined as $m = -2.5\log_{10} L + \text{constant}$, so

$$\mathrm{d}m = -2.5\,\mathrm{d}(\log_{10} L) = -2.5\,\frac{\mathrm{d}(\ln L)}{\ln 10} = \frac{-2.5}{\ln 10}\,\frac{1}{L}\,\mathrm{d}L. \qquad \text{(S4.3)}$$

Therefore

$$\frac{\mathrm{d}N}{\mathrm{d}m} = \frac{-\ln 10}{2.5}\,L\,\frac{\mathrm{d}N}{\mathrm{d}L}. \qquad \text{(S4.4)}$$

The $-$ sign just indicates that the magnitude increment $\mathrm{d}m$ is in the opposite sense to the luminosity increment $\mathrm{d}L$, and is usually neglected.

Exercise 4.6 The variance of a probability distribution $p(x)$ is the mean of the squares minus the square of the mean, i.e.

$$\text{Var}(x) = \int_0^1 x^2\,p(x)\,\mathrm{d}x - \left(\int_0^1 x\,p(x)\,\mathrm{d}x\right)^2.$$

Now, our probability distribution is uniform, so $p(x) = 1$ for all x from 0 to 1, hence this is just

$$\begin{aligned}
\mathrm{Var}(x) &= \int_0^1 x^2\,\mathrm{d}x - \left(\int_0^1 x\,\mathrm{d}x\right)^2 \\
&= \left[\frac{x^3}{3}\right]_{x=0}^{x=1} - \left[\left(\frac{x^2}{2}\right)^2\right]_{x=0}^{x=1} \\
&= \tfrac{1}{3} - \tfrac{1}{4} = \tfrac{1}{12},
\end{aligned}$$

as required. The standard deviation is the square root of the variance, so the standard deviation of the uniform distribution is $1/\sqrt{12}$. The central limit theorem states that if you have N measurements, each with an uncertainty σ (i.e. taken from the same distribution with standard deviation σ), then the standard deviation of the mean average of these measurements is $\sigma/\sqrt{N}$. Now, if our null hypothesis holds, then V/V_{max} is uniformly distributed, so each measurement of V/V_{max} is taken from a distribution with standard deviation $1/\sqrt{12}$. Therefore the standard deviation of the average N measurements of V/V_{max} must be $1/\sqrt{12N}$, as required.

Exercise 4.7 Yes, provided that the selection function has been correctly stated.

Exercise 4.8 No, not necessarily. Suppose that you had a volume-limited sample with $V_{\mathrm{max}} = V(z_{\mathrm{max}})$ for all galaxies. Now suppose that half the galaxies exist at exactly $z = 0$, half are at $z = z_{\mathrm{max}}$, and there are none in between. Clearly, the numbers of galaxies are evolving very strongly and discontinuously, but $\langle V/V_{\mathrm{max}}\rangle = 1/2$.

Exercise 4.9 The amount of light emitted per unit volume will be given by the number density of galaxies multiplied by their luminosity, i.e. $L \times \phi(L)$. At luminosities far below the break, $\phi(L) \propto L^{-\alpha}$, so $L\,\phi(L) \propto L^{1-\alpha}$. Since we're given that the faint-end slope α satisfies $\alpha < 1$, this must be increasing with luminosity. At the bright end we have that $\phi \propto \exp(-L/L_*)$, which tends to zero faster than $1/L$, so $L\,\phi(L)$ (which is proportional to $L\exp(-L/L_*)$) must also tend to zero. We'd expect one turning point — but where? We can differentiate $L\,\phi(L)$, set the result equal to zero and rearrange. This gives

$$\frac{\mathrm{d}(L\phi)}{\mathrm{d}L} = \phi_*(-\mathrm{e}^{-L/L_*}(L/L_*)^{-\alpha+1} + (1-\alpha)\mathrm{e}^{-L/L_*}(L/L_*)^{-\alpha}) = 0.$$

Dividing by $\phi_*\mathrm{e}^{-L/L_*}$ gives $(L/L_*)^{1-\alpha} = (1-\alpha)(L/L_*)^{-\alpha}$. Further dividing by $(L/L_*)^{-\alpha}$ gives $L/L_* = 1-\alpha$, or $L = (1-\alpha)L_*$. The galaxies that dominate the cosmic luminosity density are therefore those with luminosities of $(1-\alpha)L_*$.

Exercise 4.10 PDE is vertical translations, while PLE is horizontal translations.

Exercise 4.11 Active galaxies can be seen to much higher redshifts than the elliptical galaxies used in the Tolman test in Chapter 3, and as the predicted redshift-dependence of surface brightness is strong, i.e. $(1+z)^4$, it might appear that the radio lobes of radiogalaxies have a strong advantage. The attraction of the Tolman test is that the $(1+z)^4$ surface brightness prediction is independent of the cosmological parameters. In order to apply it, we need a population of objects whose luminosity per unit area (in, for example, square parsecs) is constant. In this case, rearranging the relation in the question gives us $L/r^2 \propto Q^{7/6} r^{-4/3} \rho^{7/12}$. We might hope to find active galaxies with the same Q on average if we match other properties of the central engine (e.g. optical emission lines and continuum) on average. We might also be able to calibrate out any variations in density through other observations as indicated in the question, but we're still left with a surface brightness that depends on the linear size of the system. Without additionally having a standard rod as a comparison, we can't apply the Tolman test as it stands.

Exercise 4.12 There are $60 \times 60 = 3600$ arcseconds in a degree, so there are $3600^2 \simeq 1.30 \times 10^7$ square arcseconds in a square degree. Therefore the number of random 5σ noise spikes in one square degree would be $(1.30 \times 10^7)/(3.5 \times 10^6) \simeq 3.7$. So we'd expect one 5σ noise spike in $1/3.7$ square degrees, or about 0.27 square degrees. In practice, noise spikes can occur more frequently than this for a variety of reasons (including instrumental effects).

Exercise 4.13 Suppose that your camera or detector covers an area A on the sky. Let's say that you invest all your time in a pencil-beam survey, and it reaches a flux S. The number counts are Euclidean, so $N(>S) = kS^{-1.5}$, where k is some constant. Therefore the number of galaxies seen in the pencil-beam survey is

$$n_{\text{pencil}} = A \times N(>S) = AkS^{-1.5}.$$

Now suppose that instead of doing a pencil-beam survey, you spread your integration time over m fields of view, each of which has area A. The total area that you cover is $m \times A$, but the images would be shallower by a factor of $\sqrt{m}$, so the total number of galaxies in the wide-field survey would be

$$n_{\text{wide}} = mA(\sqrt{m}S)^{-1.5} = mAm^{-0.75}S^{-1.5} = m^{0.25}AS^{-1.5}.$$

Comparing this to n_{pencil}, we see that $n_{\text{wide}} = m^{0.25}\, n_{\text{pencil}}$, so the wide-field survey finds more galaxies by a factor of $m^{0.25}$.

A similar calculation shows that if the source counts are steeper than $N(>S) \propto S^{-2}$, then the pencil-beam survey would see more. However, only rarely are source counts that steep (we'll see an example in Chapter 5). In the vast majority of cases, wide-field surveys find more objects in a given observing time than pencil-beam surveys. In practice, though, there's often a limit to how wide you can make a survey, because the time spent simply moving the telescope or reading out the detector becomes significant (we've neglected both effects here).

Exercise 5.1 $I_\nu \, \mathrm{d}\nu$ is the background intensity in an interval $\nu \to \nu + \mathrm{d}\nu$. The background per decade is the background in a logarithmic interval, $\nu \to \nu + \mathrm{d}\log_{10}\nu$. Let's write this as $B \, \mathrm{d}\log_{10}\nu$. If we can set this equal to something times $\mathrm{d}\nu$, then that something must be I_ν. Now, $\mathrm{d}\log_{10}\nu = (\mathrm{d}\ln\nu)/\ln(10)$, so $B\,\mathrm{d}\log_{10}\nu = (B/\ln(10))\,\mathrm{d}\ln\nu$. But $\mathrm{d}\ln\nu = (1/\nu)\,\mathrm{d}\nu$, so

$$B\,\mathrm{d}\log_{10}\nu = B\frac{1}{\nu\ln(10)}\,\mathrm{d}\nu.$$

Therefore

$$I_\nu = B\frac{1}{\nu\ln(10)},$$

so $B = \ln(10)\,\nu I_\nu$. Therefore the background intensity per decade of frequency is proportional to νI_ν. Looking at Figure 5.1, we see that the far-infrared bump has a similar height and area to the optical/near-infrared bump, each over roughly the same logarithmic frequency interval of about $\Delta \log_{10} \nu = 1.5$. Therefore there's about the same energy output in the far-infrared bump as in the optical/near-infrared bump.

Exercise 5.2 This will be the one in which $S_\nu \,\mathrm{d}N/\mathrm{d}\ln S_\nu$ is a maximum, and since $\mathrm{d}\ln S_\nu = S_\nu^{-1}\,\mathrm{d}S_\nu$, we can also express this as $S_\nu^2\,\mathrm{d}N/\mathrm{d}S_\nu$. This is similar (though not quite identical) to Figure 5.2.

Exercise 5.3 The angular resolution in radians is $1.22\lambda/D = 1.22 \times 500 \times 10^{-6}/3.5 = 1.7429 \times 10^{-4}$ (we'll carry some extra significant figures until the end of the calculation). In degrees this is $1.7429 \times 10^{-4} \times 360°/(2\pi) = 0.009\,985\,8°$. In arcseconds this is $0.009\,985\,8 \times 3600 = 35.95''$, or $36.0''$ to the accuracy of the initial numbers.

Exercise 5.4 (a) The fractional range would be $0.09/0.15 = 0.6$ or 60%, which we could also quote as a possible variation of a factor of $1/0.6 = 1.7$.

(b) The variation in β changes the extrapolation from the 800 μm quoted to the rest frame, which is $850/(1+z)\,\mu\mathrm{m} = 850/4\,\mu\mathrm{m} = 212.5\,\mu\mathrm{m}$. The wavelength dependence is $\lambda^{-\beta}$, so

$$\frac{k_\mathrm{d}(800\,\mu\mathrm{m})}{k_\mathrm{d}(212.5\,\mu\mathrm{m})} = \left(\frac{800}{212.5}\right)^{-\beta} = 3.765^{-\beta},$$

i.e. 0.0705–0.2656 when $\beta = 1$–2, or a further variation of a factor of 3.8. The total variation so far is $1.7 \times 3.8 \simeq 6.5$.

(c) Using the black body spectrum given in Equation 2.2 and putting in the numbers for a wavelength of 212.5 μm (i.e. $\nu = c/212.5\,\mu\mathrm{m} = 2.998 \times 10^8\,\mathrm{m\,s^{-1}}/(212.5 \times 10^{-6}\,\mathrm{m}) = 1.411 \times 10^{12}\,\mathrm{Hz}$) and temperatures of $T = 20$ K and 40 K, we find that

$$\begin{aligned}\frac{B(1.411\,\mathrm{THz}, 40\,\mathrm{K})}{B(1.411\,\mathrm{THz}, 20\,\mathrm{K})} &= \frac{\exp(h\nu/kT_1) - 1}{\exp(h\nu/kT_2) - 1}\\ &= \frac{\exp(6.626 \times 10^{-34}\,\mathrm{J\,s} \times 1.411 \times 10^{12}\,\mathrm{Hz}/1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \times 20\,\mathrm{K}) - 1}{\exp(6.626 \times 10^{-34}\,\mathrm{J\,s} \times 1.411 \times 10^{12}\,\mathrm{Hz}/1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}} \times 40\,\mathrm{K}) - 1}\\ &= 6.435.\end{aligned}$$

The range of allowed temperatures therefore gives an additional fractional range of 6.4, so the total fractional range is $1.7 \times 3.8 \times 6.4 \simeq 41$, i.e. we cannot even quote a dust mass to within an order of magnitude!

However, if we measure fluxes at more wavelengths, we might be able to reduce these uncertainties by constraining the value of β on the Rayleigh–Jeans tail, and determining the temperature from the wavelength λ_max of the location of the peak of the spectral energy distribution. This is quantified with the Wien displacement law, which can be expressed in astrophysically-useful quantities as

$$\frac{\lambda_\mathrm{max}}{100\,\mu\mathrm{m}} = 1.45\frac{20\,\mathrm{K}}{T}.$$

There is, however, still the issue that galaxies do not have single temperatures.

Exercise 5.5 Suppose that there were no background. In some fixed observing time, suppose that we collect N photons from a distant object. Using Poisson statistics, the variance on this number will also be N, so the standard deviation (i.e. the noise) will be $\sqrt{N}$. The signal-to-noise ratio will therefore be $N/\sqrt{N} = \sqrt{N}$. Now suppose that there's a strong background, so we observe $N + N_{\text{back}}$ photons, with $N_{\text{back}} \gg N$. The noise on this will be $\sqrt{N + N_{\text{back}}} \simeq \sqrt{N_{\text{back}}} \gg \sqrt{N}$. What we want is N and not $N + N_{\text{back}}$, so we have to observe an additional blank bit of sky to estimate N_{back}. This can be done if we have a small object in our camera, so we can use blank bits of the image, but if our detector has only one or a small number of pixels, we have to spend extra time observing blank sky. However, even neglecting the uncertainty on our N_{back} estimate, we still have a signal-to-noise ratio of $N/\sqrt{N_{\text{back}}}$, which is much less than the $N/\sqrt{N}$ that we'd have in the case of no background. So once $N_{\text{back}} \geq N$ we enter the *background-limited* regime where good signal-to-noise is harder to get. In the case of the SCUBA camera, the faintest objects are $\simeq 10^5$–10^6 times fainter than the sky background. Worse, the background varies on timescales of less than a second, so observing techniques at submm wavelengths are often geared towards making the best background subtraction.

Exercise 5.6 See Figure S5.1.

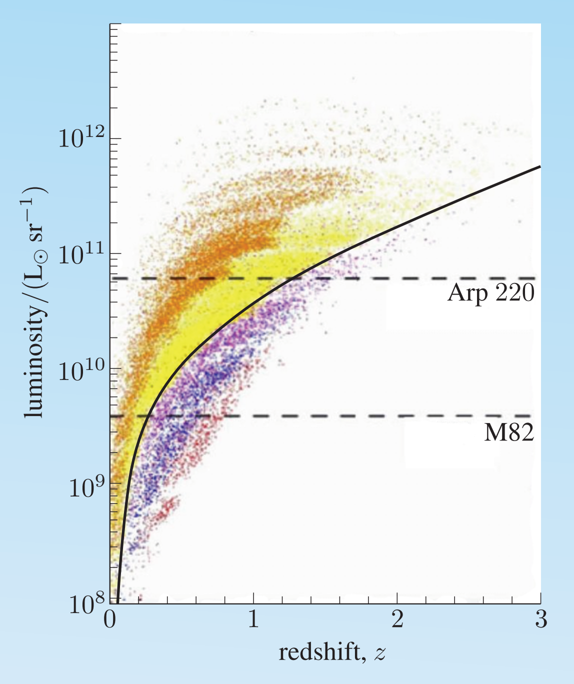

Figure S5.1 This is the same as Figure 5.15, but with the approximate location of one possible flux limit marked as a thick black line.

Exercise 6.1 Suppose that we wanted to separate a human being into protons and electrons, then hold them one metre apart. For a 60 kg mass, the force required would be $F = (ne)^2/(4\pi\varepsilon_0 r^2)$, where $r = 1$ m, $n = 60\,\text{kg}/m_{\text{p}}$ and ε_0 is the vacuum permittivity of free space. This comes out as a gigantic $F \simeq 3 \times 10^{29}\,\text{kg m s}^{-2}$. The luminosity of the Sun is $L_\odot = 3.83 \times 10^{26}$ W, so the momentum flux from the Sun is $L_\odot/c = 1.28 \times 10^{18}\,\text{kg m s}^{-2}$. If we could employ all the momentum flux from all the $\simeq 10^{11}$ stars in the Galaxy in keeping the positive and negative parts separate, it would be just sufficient to maintain a 1 m separation for just 60 kg. The potential barrier for separating the charged components of a plasma accreting around a black hole is clearly insuperable for radiation pressure.

Exercise 6.2 Putting the numbers into Equation 6.6 gives

$$\begin{aligned} L_\mathrm{E} &= \frac{4\pi \times (6.67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}) \times (3.00\times 10^{8}\,\mathrm{m\,s^{-1}}) \times (1.99\times 10^{30}\,\mathrm{kg}) \times (1.67\times 10^{-27}\,\mathrm{kg})}{6.65\times 10^{-29}\,\mathrm{m^2}} \\ &= 1.26\times 10^{31}\,\mathrm{W}. \end{aligned}$$

The luminosity of the Sun is 3.83×10^{26} W, which is far below the Eddington limit.

Exercise 6.3 Assuming that the mass of a 100 W light bulb is (say) about 50 g, we get an Eddington limit of just 0.2 W . Clearly, a light bulb radiates at much more than the Eddington limit. Light bulbs don't blow themselves apart because they are not gravitationally bound.

Exercise 6.4 To obtain Equation 6.26 we start with Equation 1.53, then use Equation 1.41. It immediately follows that

$$\mathrm{d}V = d_\mathrm{A}^2 (1+z)^3 \frac{4\pi c\,\mathrm{d}z}{(1+z)\,H(z)} = 4\pi d_\mathrm{A}^2 (1+z)^2 \frac{c\,\mathrm{d}z}{H(z)}.$$

(We ignore the $-$ sign, which just refers to the directions in which the infinitesimal increments are measured.) Next, putting in the relationship between angular diameter and luminosity distance, $d_\mathrm{L} = (1+z)^2 d_\mathrm{A}$ (Equation 1.50), gives

$$\mathrm{d}V = \frac{4\pi d_\mathrm{L}^2}{(1+z)^4}(1+z)^2 \frac{c\,\mathrm{d}z}{H(z)} = \frac{4\pi d_\mathrm{L}^2}{(1+z)^2}\,\frac{c\,\mathrm{d}z}{H(z)}.$$

Dividing by $\mathrm{d}z$ and multiplying by H_0/H_0 gives

$$\frac{\mathrm{d}V}{\mathrm{d}z} = \frac{4\pi c d_\mathrm{L}^2}{(1+z)^2 H(z)} = \frac{c}{H_0}\,\frac{4\pi d_\mathrm{L}^2}{(1+z)^2 H(z)/H_0},$$

as required.

We can rearrange this as

$$\frac{4\pi d_\mathrm{L}^2}{\mathrm{d}V/\mathrm{d}z} = (1+z)^2 \frac{H(z)}{c}.$$

Finally, we use Equation 1.28: $|\mathrm{d}z/\mathrm{d}t| = (1+z)\,H(z)$ (again we'll not worry about the sign). Therefore

$$\frac{4\pi d_\mathrm{L}^2}{\mathrm{d}V/\mathrm{d}z}\,\mathrm{d}t = \frac{1}{c}(1+z)\,\mathrm{d}z,$$

which is Equation 6.24, as required.

Exercise 6.5 The angular size θ will satisfy $\theta \simeq \tan\theta = r_h/D$, where $D = 10\,\mathrm{Mpc}$ and r_h is given by Equation 6.29: $r_h = (10^8/10^8) \times (220/200)^{-2}\,\mathrm{pc} = 0.83\,\mathrm{pc}$. Plugging in the numbers, we have $\theta \simeq r/D = 0.83\,\mathrm{pc}/10\,\mathrm{Mpc} = 8.3\times 10^{-8}$ radians. In arcseconds this is

$\theta = 8.3 \times 10^{-8} \times (360°/2\pi) \times 60 \times 60 = 0.017''$. This is clearly a lot smaller than the seeing limit of ground-based telescopes.

Exercise 6.6 The e-folding timescale for Eddington-limited black hole growth is the Salpeter timescale t_E divided by the efficiency η, i.e. $t_{\text{e-fold}} = 4 \times 10^8/\eta$ yr. There have been $3 \times 10^9/t_{\text{e-fold}}$ e-foldings since the start of the Universe, or $0.75(\eta/0.1)$ e-foldings. To reach $10^6\,M_\odot$, one needs $\log_e(10^6/10^1) = 11.5$ e-foldings. Even if $\eta = 1$, you have only 7.5 e-foldings, so 3 Gyr is not long enough.

Exercise 7.1 Comoving distances add, so $r_S = r_L + r_{LS}$. Therefore $r_{LS} = r_S - r_L$. In flat space, angular diameter distance is simply comoving distance divided by $(1+z)$ (Chapter 1), but in this case we need the redshift of the background source *as seen from the lens*. We could write this factor as $(1 + z_{LS})$. This is the factor by which the Universe expanded between the source redshift and the lens redshift, i.e. R_L/R_S, where R is the scale factor. But

$$\frac{R_L}{R_S} = \frac{R_L/R_0}{R_S/R_0} = \frac{R_0/R_S}{R_0/R_L}$$

(where the subscript 0 refers to the present day), so $(1 + z_{LS}) = (1 + z_S)/(1 + z_L)$. Therefore our final expression for the angular diameter distance D_{LS} is

$$D_{LS} = (r_S - r_L) \times \frac{(1 + z_L)}{(1 + z_S)}.$$

Exercise 7.2 First, matching distances along the top of Figure 7.7 shows that $\boldsymbol{\theta} D_S = \boldsymbol{\beta} D_S + \widehat{\boldsymbol{\alpha}} D_{LS}$. But $\boldsymbol{\alpha} = \widehat{\boldsymbol{\alpha}} D_{LS}/D_S$, so $\boldsymbol{\theta} D_S = \boldsymbol{\beta} D_S + \boldsymbol{\alpha} D_S$. Dividing out the scalar D_S gives $\boldsymbol{\theta} = \boldsymbol{\beta} + \boldsymbol{\alpha}$, which we can rearrange to $\boldsymbol{\beta} = \boldsymbol{\theta} - \boldsymbol{\alpha}$, as required.

Exercise 7.3 We set $\beta = 0$ in Equation 7.8. We can rearrange this to show that

$$\theta = \sqrt{\frac{4GM}{c^2}\,\frac{D_{LS}}{D_L D_S}}\,.$$

But what would this look like? The background object is exactly behind the lens and it's deflected by an angle θ. Is it deflected to the left or right or up or down? In fact, there is nothing to give the deflection any particular direction, so the background source is lensed into a *ring*. These are very rare, but an example is shown in Figure S7.1.

Exercise 7.4 $\beta^2 + 4\theta_E^2$ is always positive, but the square root of it can be positive or negative. $\sqrt{\beta^2 + 4\theta_E^2} > \beta$ unless $\theta_E = 0$, so the negative root must always give a negative θ. This is indeed a physical solution and represents an angle measured in the opposite direction: as shown in Figure 7.7, the image is on the other side of the lens. Note that one image is at $\theta > \theta_E$ and the other is at $\theta < \theta_E$, unless $\theta = \theta_E$ and the system is an Einstein ring.

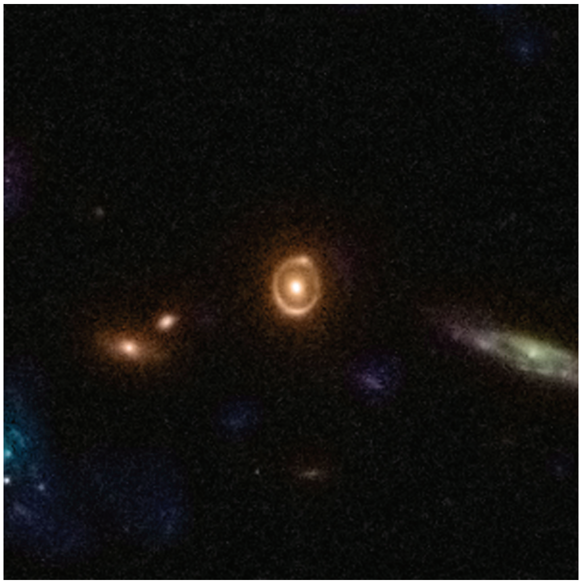

Figure S7.1 The gravitational lens 0038+4133 (an Einstein ring) from the COSMOS survey, taken by the HST . The image is $15''$ by $15''$.

Exercise 7.5 From the previous exercise, a source can have multiple images, so there is *not* necessarily a unique image position $\boldsymbol{\theta}$ for a given source position $\boldsymbol{\beta}$. In mathematical terms, we would speak of the mapping $\boldsymbol{\beta} \to \boldsymbol{\theta}$ as being one-to-many. However, each image position $\boldsymbol{\theta}$ *does* map in a one-to-one way onto a source position $\boldsymbol{\beta}$, i.e. each image position can correspond to only one position in the background source. To see why, consider Equation 7.4. The function $\boldsymbol{\alpha}(\boldsymbol{\theta})$ must be a single-valued function, i.e. any particular input $\boldsymbol{\theta}$ can give only one possible output $\boldsymbol{\alpha}$. Therefore there can be only one value of $\boldsymbol{\beta}$ for a given input $\boldsymbol{\theta}$.

Exercise 7.6 We're asked to differentiate Equation 7.12, which gives $\mathrm{d}\beta/\mathrm{d}\theta = 1 + (\theta_\mathrm{E}^2/\theta^2)$. This gives us one of the fractions in Equation 7.16. The magnification is therefore

$$\begin{aligned}\frac{\theta}{\beta}\,\frac{\mathrm{d}\theta}{\mathrm{d}\beta} &= \frac{\theta}{\beta}\left(1+\frac{\theta_\mathrm{E}^2}{\theta^2}\right)^{-1} = \theta\left(\theta-\frac{\theta_\mathrm{E}^2}{\theta}\right)^{-1}\left(1+\frac{\theta_\mathrm{E}^2}{\theta^2}\right)^{-1}\\ &= \left(1-\frac{\theta_\mathrm{E}^2}{\theta^2}\right)^{-1}\left(1+\frac{\theta_\mathrm{E}^2}{\theta^2}\right)^{-1} = \left(1+\frac{\theta_\mathrm{E}^2}{\theta^2}-\frac{\theta_\mathrm{E}^2}{\theta^2}-\frac{\theta_\mathrm{E}^4}{\theta^4}\right)^{-1}\\ &= \left(1-\frac{\theta_\mathrm{E}^4}{\theta^4}\right)^{-1},\end{aligned}$$

as required.

Exercise 7.7 A negative magnification means that the image is mirror-reversed. For example, a positive change $\mathrm{d}\beta$ would have a corresponding $\mathrm{d}\theta$ in the opposite direction, so $\mathrm{d}\theta$ is negative. Therefore $\mathrm{d}\theta/\mathrm{d}\beta$ is negative in Equation 7.16.

Exercise 7.8 We start from Equation 7.24. The mass enclosed is $\Sigma\pi\xi^2$ and we set $\xi = D_\mathrm{L}\theta$:

$$\widehat{\alpha} = \frac{4GM(\xi)}{c^2\xi} = \frac{4G}{c^2\xi} \times \Sigma\pi\xi^2 = \frac{4G}{c^2} \times \Sigma\pi \times D_\mathrm{L}\theta.$$

Now,

$$\alpha = \frac{D_{\mathrm{LS}}}{D_{\mathrm{S}}}\widehat{\alpha}, \qquad \text{(Eqn 7.6)}$$

so

$$\alpha = \frac{4\pi G\Sigma}{c^2}\,\frac{D_{\mathrm{L}}D_{\mathrm{LS}}}{D_{\mathrm{S}}}\,\theta,$$

as required.

If we then set $\Sigma = \Sigma_{\mathrm{cr}}$, we find that $\alpha(\theta) = \theta$ for any θ, so $\beta = 0$. This means that the gravitational lens is acting as a perfect focusing lens! However, this is a very special case — gravitational lenses in general do *not* focus light. As 'lenses' in the optical sense, they have all forms of aberration, except of course chromatic aberration since gravitational lensing is strictly achromatic.

Exercise 7.9 From left to right, they are a saddle point, a maximum and a minimum.

Exercise 7.10 The time delay of the image at the centre increases. In a diagram like Figure 7.15, the central panel showing the gravitational time delay would be acquiring a sharper and higher point in the centre. When the lens potential becomes a singular isothermal sphere, the time delay becomes infinite, so the image disappears. Photons would take an infinite amount of time to climb out of the infinitely-deep potential well, and (by symmetry) spend another infinite amount of time falling in beforehand. But a more thoughtful answer is that this deep potential well would form a black hole. Right from Equation 7.1, we've been assuming a weak-field limit, so a better answer is that these simple assumptions break down as the potential becomes more extreme.

Exercise 7.11 The background objects have the same redshift, so we could think of the luminosity function as differential source counts, thus $\mathrm{d}N/\mathrm{d}S \propto S^{-\alpha}$. Therefore the number of objects per unit area brighter than a flux S_0 will be $N(> S_0) \propto S_0^{1-\alpha}$, which we could write as

$$N(> S_0) = kS_0^{1-\alpha}.$$

If the background galaxies are gravitationally magnified by a factor of μ, the intrinsic fluxes will be $S_{\mathrm{intrinsic}} = S/\mu$, while the comoving volume sampled will be smaller by a factor of $1/\mu$. Therefore the number of galaxies brighter than an *observed* flux S_0 will be

$$N_{\mathrm{lensed}}(> S_0) = \frac{k}{\mu}\left(\frac{S_0}{\mu}\right)^{1-\alpha} = k\mu^{-1}S_0^{1-\alpha}\mu^{\alpha-1} = kS_0^{1-\alpha}\mu^{\alpha-2} = N(> S_0)\,\mu^{\alpha-2}.$$

Therefore for a magnification of μ (where $\mu > 1$), the lensing changes the number of background galaxies per unit area by a factor of $\mu^{\alpha-2}$. For this factor to be bigger than 1 we need

$$\mu^{\alpha-2} > 1,$$

so $\quad \log(\mu^{\alpha-2}) > \log(1) = 0,$
thus $\quad (\alpha - 2)\log(\mu) > 0.$
We already know that $\log(\mu) > 0$ (because $\mu > 1$), so this can happen only if $\alpha > 2$. For example, if the source counts have a Euclidean slope ($\alpha = 2.5$), then

lensing would increase the number of objects. The effect of sampling less volumes due to lensing, and so finding fewer objects than the flux magnification on its own would suggest, is known as the *Broadhurst effect*. (See Broadhurst, T.J., Taylor, A.N. and Peacock, J.A., 1995, *Astrophysical Journal*, **438**, 49.)

Exercise 8.1 There's no guarantee that the re-emitted photon comes out in the same direction — in fact, it probably won't. A corollary is that any Lyman α cloud should glow faintly in Lyman α light in all directions from these re-emitted photons, even if the cloud is not intercepting our line of sight to a quasar (because there will always be *some* line of sight that does). This re-emission is in general too faint to detect. However, Lyman α emission can sometimes be seen if there are internal ionizing sources (e.g. star formation) within damped Lyman α systems, which you will meet later in the chapter.

Exercise 8.2 The column density through the centre will be the same as that seen through a cubical cloud with a side 2 Mpc, facing the observer (because the absorption doesn't depend on the distribution of material that the light *doesn't* pass through). One Mpc is about 3×10^{24} cm, so we can write the density as $(3 \times 10^{24})^3\,\mathrm{cm}^{-3} = 2.7 \times 10^{73}\,\mathrm{Mpc}^{-3}$. The total number of neutral hydrogen atoms in the cube must be $2.7 \times 10^{73}\,\mathrm{Mpc}^{-3} \times 8\,\mathrm{Mpc}^3 = 21.6 \times 10^{73}$, which is spread over a projected area of $2 \times 2\,\mathrm{Mpc}^2 = 36 \times 10^{48}\,\mathrm{cm}^2$. Therefore the column density must be $21.6 \times 10^{73}/(36 \times 10^{48})\,\mathrm{cm}^{-2} \simeq 6 \times 10^{24}\,\mathrm{cm}^{-2}$.

Exercise 8.3 In order for a hydrogen atom to absorb an Hα photon, the photon must have the right energy, and there must be an atom with an electron in the $n = 2$ energy level ready to absorb the photon. This energy level is at $E = -13.6/n^2\,\mathrm{eV} = -13.6/4\,\mathrm{eV} = -3.4\,\mathrm{eV}$. In order to be in such a state, the atom must have absorbed a photon of energy $(-3.4\,\mathrm{eV}) - (-13.6\,\mathrm{eV}) = 10.2\,\mathrm{eV}$. Photons of this energy require a black body temperature of order

$$T \simeq E/k = \frac{10.2\,\mathrm{eV} \times 1.602 \times 10^{-19}\,\mathrm{J\,eV}^{-1}}{1.381 \times 10^{-23}\,\mathrm{J\,K}^{-1}} = 120\,000\,\mathrm{K}.$$

This is hotter than the surface of an O star, and is much hotter than the typical temperatures in the intergalactic medium. Lyman α clouds are too cold to have many atoms with electrons already excited to the $n = 2$ level, so the clouds have almost no Hα absorption.

Exercise 8.4 We can write $\sigma(\nu) = \sigma_0(\nu/\nu_{\mathrm{limit}})^{-3}$, where $\sigma_0 = 7.88 \times 10^{-22}\,\mathrm{m}^{-2}$, and ν_{limit} is the frequency of the Lyman limit. Writing $J_\nu = k\nu^{-\alpha}$ and plugging the terms in, we find

$$\begin{aligned}
\tau = N_{\mathrm{H\,I}} \frac{\int_{\nu_{\mathrm{limit}}}^{\infty} (\sigma J_\nu/(h\nu))\,\mathrm{d}\nu}{\int_{\nu_{\mathrm{limit}}}^{\infty} (J_\nu/(h\nu))\,\mathrm{d}\nu} &= N_{\mathrm{H\,I}}\,\sigma_0 \frac{\int_{\nu_{\mathrm{limit}}}^{\infty} (\nu/\nu_{\mathrm{limit}})^{-3}\, k\nu^{-\alpha-1}\,\mathrm{d}\nu}{\int_{\nu_{\mathrm{limit}}}^{\infty} k\nu^{-\alpha-1}\,\mathrm{d}\nu} \\
= \frac{N_{\mathrm{H\,I}}\,\sigma_0}{\nu_{\mathrm{limit}}^{-3}} \frac{\int_{\nu_{\mathrm{limit}}}^{\infty} \nu^{-\alpha-4}\,\mathrm{d}\nu}{\int_{\nu_{\mathrm{limit}}}^{\infty} \nu^{-\alpha-1}\,\mathrm{d}\nu} &= \frac{N_{\mathrm{H\,I}}\,\sigma_0}{\nu_{\mathrm{limit}}^{-3}} \frac{\nu_{\mathrm{limit}}^{-\alpha-3}}{\alpha+3} \frac{\alpha}{\nu_{\mathrm{limit}}^{-\alpha}} \\
= \frac{N_{\mathrm{H\,I}}\,\sigma_0 \alpha}{\alpha+3},
\end{aligned}$$

where we first cancelled the h terms, then cancelled the k terms. Setting $\tau > 1$, we find $N_{\mathrm{H\,I}} > 1.3\,((\alpha+3)/\alpha) \times 10^{21}\,\mathrm{m}^{-2}$, as required.

Exercise 8.5 Equation 1.28 relates $\mathrm{d}z/\mathrm{d}t$ to $H(z)$. Taking the modulus and reciprocal of that equation gives $(1+z)\,|\mathrm{d}t/\mathrm{d}z| = 1/H(z)$. A population with constant proper sizes has constant A in Equation 8.2, and a constant comoving density is constant n_{co} in the same equation. Therefore $\mathrm{d}^2\mathcal{N} \propto (1+z)^3\,|\mathrm{d}t/\mathrm{d}z| \propto (1+z)^2/H(z)$. If we write $\mathrm{d}X/\mathrm{d}z = (1+z)^2 H_0/H(z)$, then

$$\mathrm{d}^2\mathcal{N} = n_{\mathrm{co}}\,A \times (1+z)^2 c \left|\frac{1}{H(z)}\right| \mathrm{d}N_{\mathrm{H\,I}}\,\mathrm{d}z$$

gives

$$\mathrm{d}^2\mathcal{N} = n_{\mathrm{co}}\,A\frac{c}{H_0}\,\mathrm{d}X\,\mathrm{d}N_{\mathrm{H\,I}},$$

which is constant.

Exercise 8.6 Gravitational lensing of the background quasar by the damped Lyman α system could cause such an effect. The strength of this effect, and the biases that it creates on the measured cosmic evolution of neutral gas, are still the subject of debate. However, it turns out that this is probably only a 10–20% effect on $\Omega_{\mathrm{H\,I}}$ at $z > 2$.

Exercise 8.7 Dust in the damped Lyman α systems should redden the quasar spectra, so one might compare the optical spectral indices or B–V colours of quasars with and without damped Lyman α absorbers. However, if damped systems are very dusty, they may induce so much reddening that the quasars drop out of the parent sample, so bright quasar catalogues would be biased to detecting low-reddening systems. Statistical analyses suggest that this latter effect does not dominate, but direct results on quasar reddening are currently conflicting.

Exercise 8.8 The energy of the hydrogen Lyman limit is $E = 13.6\,\mathrm{eV}$, i.e. $E = 13.6 \times 1.602 \times 10^{-19}\,\mathrm{J} = 2.179 \times 10^{-18}\,\mathrm{J}$. This corresponds to a frequency of $\nu = E/h$, where h is Planck's constant, which comes out as $\nu = 3.289 \times 10^{15}\,\mathrm{Hz}$. The wavelength of this light is $\lambda = c/\nu$, where c is the speed of light, which comes out as $\lambda = 9.116 \times 10^{-8}\,\mathrm{m}$, or 91.2 nm (i.e. 912 Å) to three significant figures. For the helium Lyman limit, $\lambda_{\mathrm{He}} = \lambda \times 13.6/54.4 = 22.8\,\mathrm{nm}$.

The redshifted hydrogen Lyman limit in Figure 8.20 is at a wavelength of $912 \times (1+z)\,\text{Å} = 912 \times (1+3.2)\,\text{Å} = 3830\,\text{Å}$.