

Digital Logic Design: a rigorous approach ©

Chapter 7: Asymptotics

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Book Homepage:

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Order of Growth Rates

Consider the Fibonacci sequence $(0, 1, 1, 2, 3, 5, \dots)$ where $g(n)$ is defined by

(i) Base case: $g(0) = 0$ and $g(1) = 1$.

(ii) Reduction rule: $g(n+2) = g(n+1) + g(n)$.

The exact value of $g(n)$, or an analytic equation for $g(n)$ is interesting, but sometimes, all we need to know is how “fast” does $g(n)$ grow?

Does it grow faster than $f(n) = n$, $f(n) = n^2$, $f(n) = 2^n$? We wish to capture the notion of “ $g(n)$ does not grow faster than $f(n)$ ”.

big-O, big-Omega, big-Theta

Definition

Let $f, g : \mathbb{N} \rightarrow \mathbb{R}^{\geq}$ denote two functions.

- 1 We say that $g(n) = O(f(n))$, if there exist constants $c_1, c_2 \in \mathbb{R}^{\geq}$ such that, for every $n \in \mathbb{N}$,

$$g(n) \leq c_1 \cdot f(n) + c_2.$$

- 2 We say that $g(n) = \Omega(f(n))$, if there exist constants $c_3 \in \mathbb{R}^+$ and $c_4 \in \mathbb{R}^{\geq}$ such that, for every $n \in \mathbb{N}$,

$$g(n) \geq c_3 \cdot f(n) + c_4.$$

- 3 We say that $g(n) = \Theta(f(n))$, if $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$.

If $g(n) = O(f(n))$, then $g(n)$ does not grow faster than $f(n)$.

If $g(n) = \Omega(f(n))$, then $g(n)$ grows as least as fast as $f(n)$.

Finally, if $g(n) = \Theta(f(n))$, then $g(n)$ grows as fast as $f(n)$.

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Examples - 1

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- 2 Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.

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- 2 Let $g(n) \triangleq 2 \cdot n$. We claim that $g(n) = \Theta(n)$.
- 3 Let $g(n) \triangleq n^2 + n + 1$. We claim that $g(n) = \Theta(n^2)$.

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- 4 Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.

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- ❹ Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.
- ❺ $\sum_{i=1}^n i = \Theta(n^2)$.

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- ❹ Let $g(n) = c \cdot f(n)$, then $g(n) = \Theta(f(n))$.
- ❺ $\sum_{i=1}^n i = \Theta(n^2)$.
- ❻ $\log(n) = O(n)$.

① If $q > 1$, then $\sum_{i=1}^n q^i = \Theta(q^n)$.

Examples - 2

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- 2 [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Examples - 2

- 1 If $q > 1$, then $\sum_{i=1}^n q^i = \Theta(q^n)$.
- 2 [transitivity] If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.
- 3 [addition,max] If $f(n), g(n) = O(h(n))$, then $f(n) + g(n), \max\{f(n), g(n)\} = O(h(n))$.

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- ④ [addition,min] If $f(n), g(n) = \Omega(h(n))$, then $f(n) + g(n), \min\{f(n), g(n)\} = \Omega(h(n))$.

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- ❹ [addition,min] If $f(n), g(n) = \Omega(h(n))$, then $f(n) + g(n), \min\{f(n), g(n)\} = \Omega(h(n))$.
- ❺ [asymmetry] $f(n) = O(g(n))$ does **not** imply that $g(n) = O(f(n))$.

Big-O: equivalent definition

We study functions that describe the cost of a circuit, the delay of a circuit, the running time of an algorithm, etc. In all these cases it is natural to assume that

$$\forall n \in \mathbb{N} : f(n) \geq 1.$$

Claim

Assume $f(n) > 1$, for every n . Then, $g(n) = O(f(n))$ iff there exist constants $c \in \mathbb{R}^{\geq}$ and $N \in \mathbb{N}$, such that,

$$\forall n > N : g(n) \leq c \cdot f(n).$$

Big-Omega: equivalent definition

Claim

Assume $f(n) > 1$, for every n . Then, $g(n) = \Omega(f(n))$ iff there exist constants $c \in \mathbb{R}^{\geq}$ and $N \in \mathbb{N}$, such that,

$$\forall n > N : g(n) \geq c \cdot f(n).$$

Corollary

Assume $f(n), g(n) \geq 1$, for every n . Then,

$$f(n) = O(g(n)) \quad \Leftrightarrow \quad g(n) = \Omega(f(n)).$$

Big-Omega: equivalent definition

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Assume $f(n) > 1$, for every n . Then, $g(n) = \Omega(f(n))$ iff there exist constants $c \in \mathbb{R}^{\geq}$ and $N \in \mathbb{N}$, such that,

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Recurrence Equations

In this section we deal with the problem of solving or bounding the rate of growth of functions $f : \mathbb{N}^+ \rightarrow \mathbb{R}$ that are defined recursively. We consider the typical cases that we will encounter later.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \quad (1)$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n)$.

What about $f(n) = n + f(\lceil \frac{n}{2} \rceil)$?

Is it enough to solve for powers of 2?

In the following lemma we show that, under reasonable conditions, it suffices to consider powers of two when bounding the rate of growth.

Lemma

Assume that:

- ① *The functions $f(n)$ and $g(n)$ are both monotonically nondecreasing.*
- ② *The constant ρ satisfies, for every $k \in \mathbb{N}$,*

$$\rho \geq \frac{g(2^{k+1})}{g(2^k)}.$$

If $f(2^k) = O(g(2^k))$, then $f(n) = O(g(n))$.

What about big-Omega?

Lemma

Assume that:

- 1 The functions $f(n)$ and $g(n)$ are both monotonically nondecreasing.
- 2 The constant ρ satisfies, for every $k \in \mathbb{N}$,

$$\rho \leq \frac{g(2^{k+1})}{g(2^k)}.$$

If $f(2^k) = \Omega(g(2^k))$, then $f(n) = \Omega(g(n))$.

Recurrence 2.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \quad (2)$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n \log n)$.

Recurrence 3.

Consider the recurrence

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + 3 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1. \end{cases} \quad (3)$$

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n^{\log_2 3})$.

hint: $f(2^k) = 3^{k+1} - 2^{k+1}$.

Example - 1

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases} \quad (4)$$

where a, b, c are constants.

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n)$.

proof: $f(2^k) = 2a \cdot 2^k + b \cdot k + c - 2a...$

Example -2

Consider the recurrence

$$f(n) \triangleq \begin{cases} c & \text{if } n = 1 \\ a \cdot n + b + 2 \cdot f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases} \quad (5)$$

where $a, b, c = O(1)$.

Lemma

The rate of growth of the function $f(n)$ is $\Theta(n \log n)$.

proof: We claim that $f(2^k) = a \cdot k2^k + (b + c) \cdot 2^k - b \dots$

Example - 3

Consider the recurrence

$$F(k) \triangleq \begin{cases} 1 & \text{if } k = 0 \\ 2^k + 2 \cdot F(k-1) & \text{if } k > 0, \end{cases} \quad (6)$$

Lemma

$$F(k) = (k+1) \cdot 2^k.$$

Proof: Define $f(n) \triangleq F(\lceil \log_2 n \rceil)$. Observe that $f(2^x) \triangleq F(x) \dots$

Examples with floor and ceiling

1

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ 1 + f(\lfloor \frac{n}{2} \rfloor) & \text{if } n > 1, \end{cases}$$

2

$$f(n) \triangleq \begin{cases} 1 & \text{if } n = 1 \\ n + f(\lfloor \frac{n}{2} \rfloor) + f(\lceil \frac{n}{2} \rceil) & \text{if } n > 1, \end{cases}$$