A Student's Guide to Waves (Click-throughs)

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Wave Fundamentals click-throughs

Example 1 (page 8 of text)

[Statement] **Example:** *How much does the phase of a wave with period (T) of 20 seconds change in 5 seconds?*

Hint 1: Since the wave period T is 20 seconds, a time interval Δt of 5 seconds represents 1/4 period ($\Delta t/T = 5/20 = 1/4$).

Hint 2: Multiplying this fraction by 2π gives $\pi/2$ radians.

Hint 3: The phase of the wave advances by $\pi/2$ radians (90°) every 5 seconds.

Example 2 (page 15 of text)

[Statement] **Example:** *If vector $\vec{F} = \hat{\imath} + 4\hat{\jmath}$ and vector $\vec{G} = -7\hat{\imath} - 2\hat{\jmath}$, what are the magnitude and direction of vector $\vec{H}$ that results from adding $\vec{F}$ to $\vec{G}$?*

Hint 1: Using the component approach, the x- and y-components of vector $\vec{H}$ are

$$H_x = F_x + G_x = 1 - 7 = -6$$
$$H_y = F_y + G_y = 4 - 2 = 2.$$

Hint 2: Since $H_x = -6$ and $H_y = +2$, the vector $\vec{H} = -6\hat{\imath} + 2\hat{\jmath}$.

Hint 3: The magnitude of $\vec{H}$ is

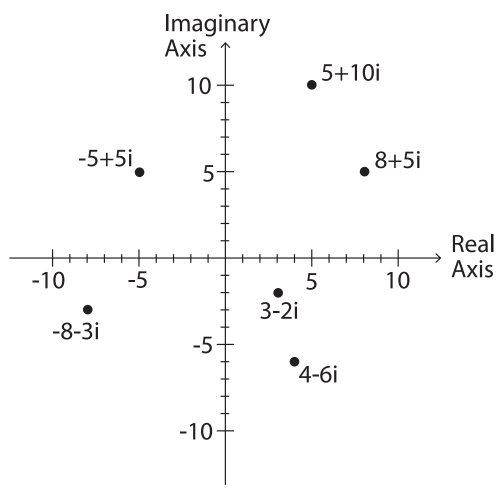
$$|\vec{H}| = \sqrt{H_x^2 + H_y^2} = \sqrt{(-6)^2 + (2)^2} = 6.32.$$

Hint 4: The direction of $\vec{H}$ is

$$\theta = \arctan\left(\frac{H_y}{H_x}\right) = \arctan\left(\frac{2}{-6}\right) = -18.4^\circ$$

but since the denominator of the arctan argument is negative, the angle of vector $\vec{H}$ measured anti-clockwise from the positive x-axis is $-18.4^\circ + 180^\circ = 161.6^\circ$.

Example 3 (page 20 of text)



[Statement] **Example:** *Find the magnitude and angle of each of the complex numbers in Figure 1.10.*

Hint 1: The rectangular-to-polar conversion equations (Eqs. 1.12 and 1.13) can be applied to the complex numbers in Figure 1.10 to determine the magnitude and angle of each.

Hint 2: For the complex number $z = 5 + 10i$, Re(z)=5 and Im(z)=10.

Hint 3: For this number

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(5)^2 + (10)^2} = 11.18$$

and the angle measured anti-clockwise from the positive real axis is

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\left(\frac{10}{5}\right) = 63.4°.$$

Hint 4: For the complex number $-5+5i$, Re(z)=-5 and Im(z)=5, so

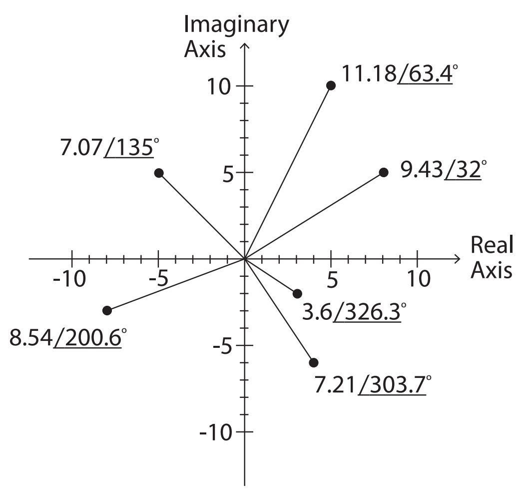
$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(-5)^2+(5)^2} = 7.07$$

and the angle from the positive real axis is

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\left(\frac{5}{-5}\right) = -45°.$$

Once again, since the denominator of the arctan argument is negative, the angle measured anti-clockwise from the positive real axis is $-45° + 180° = 135°$.

Hint 5: The magnitude and angle values for all six of the complex numbers in Fig. 1.10 are shown in Figure 1.12.



Example 4 (page 30 of text)

[Statement] **Example:** *Consider a wave with wavefunction given by*

$$y(x,t) = A\sin(kx+\omega t) \tag{1.1}$$

where the wave amplitude (A) is 3 meters, the wavelength (λ) is 1 meter, and the wave period (T) is 5 seconds. Find the value of the displacement

$y(x,t)$ at the position $x = 0.6$ m and time $t = 3$ seconds.

Hint 1: One approach would be make a flipbook of this wave. The wave amplitude tells you how big to make the peaks of your wave, the wavelength tells you how far apart to space the peaks on each page, and the wave period tells you how much to shift the wave between the pages of your book (since it has to move a distance of one wavelength in the direction of propagation during each period). You could then turn to the page in your flipbook corresponding to a time of 3 seconds and measure the y-value (the displacement) of the wave at a distance of 0.6 meters along the x-axis.

Hint 2: Alternatively, you can just plug each of the variables into Eq. 1.1.

Hint 3: The wavelength of 1 meter means that the wavenumber is $k = 2\pi/1 = 2\pi$ rad/m

Hint 4: The wave period of 5 seconds tells you that the frequency is $f = 1/5$ s $= 0.2$ Hz (and the angular frequency is $\omega = 2\pi f = 0.4\pi$ rad/s).

Hint 5: Plugging in these values gives

$$\begin{aligned} y(x,t) &= A\sin(kx + \omega t) \\ &= (3 \text{ m})\sin[(2\pi \text{ rad/m})(0.6 \text{ m}) + (0.4\pi \text{ rad/s})(3 \text{ s})] \\ &= (3 \text{ m})\sin(2.4\pi \text{ rad}) = 2.85 \text{ m}. \end{aligned}$$

Problem 1 (page 44 of text)

[Statement] Find the frequency and angular frequency of the following waves:

[Statement] a. A string wave with period of 0.02 s.

Hint 1: To find the frequency (f) and angular frequency (ω) if you know the period (T), use

$$f = \frac{1}{T}.$$

and

$$\omega = 2\pi f = \frac{2\pi}{T}$$

Hint 2: Since the period $T = 0.02$ s,

$$f = \frac{1}{T} = \frac{1}{0.02 \text{ s}} = 50 \text{ Hz}$$

and

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{0.02 \text{ s}} = 314.16 \text{ rad/s}.$$

[Statement] b. An electromagnetic wave with period of 1.5 ns.

Hint: In this case, the period $T = 1.5$ ns $= 1.5 \times 10^{-9}$ s, so

$$f = \frac{1}{T} = \frac{1}{1.5 \times 10^{-9} \text{ s}} = 6.67 \times 10^{8} \text{ Hz}$$

and

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{1.5 \times 10^{-9}} \text{ s} = 4.19 \times 10^{9} \text{ rad/s}.$$

[Statement] c. A sound wave with period of 1/3 ms.

Hint: In this case, the period $T = 1/3$ ms $= 3.33 \times 10^{-4}$ s, so

$$f = \frac{1}{T} = \frac{1}{3.33 \times 10^{-4}} \text{ s} = 3,000.00 \text{ Hz}$$

and

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{3.33 \times 10^{-4}} \text{ s} = 18,849.56 \text{ rad/s}.$$

Problem 2 (page 44 of text)

[Statement] Find the period of the following waves:

[Statement] a. A mechanical wave with frequency of 500 Hz.

Hint 1: To find the period (T) if you know the frequency (f), use $T = 1/f$.

Hint 2: For frequency $f = 500$ Hz,

$$T = \frac{1}{f} = \frac{1}{500 \text{ Hz}} = 2 \times 10^{-3} \text{ s}.$$

[Statement] b. A light wave with frequency of 5.09×10^{14} Hz.

Hint: In this case, frequency $f = 5.09 \times 10^{14}$ Hz, so

$$T = \frac{1}{f} = \frac{1}{5.09 \times 10^{14} \text{ Hz}} = 1.96 \times 10^{-15} \text{ s}.$$

[Statement] c. An ocean wave with angular frequency of 0.1 rad/s.

Hint 1: To find the period (T) if you know the angular frequency (ω), use $T = 2\pi/f$.

Hint 2: For angular frequency $\omega = 0.1$ rad/s,

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.1 \text{ rad/s}} = 62.83 \text{ s}.$$

Problem 3 (page 44 of text)

[Statement] What is the speed of an electromagnetic wave with wavelength of 2 meters and frequency of 150 MHz?

Hint 1: To find the wave speed (v) when you know the wavelength (λ) and frequency (f), use $v = \lambda f$.

Hint 2: For a wave with wavelength $\lambda = 2$ m and frequency $f = 150$ MHz $= 1.5 \times 10^8$ Hz,

$$v = \lambda f = (2 \text{ m})(1.5 \times 10^8 \text{ Hz}) = 3 \times 10^8 \text{ m/s}$$

which is the speed of light in a vacuum.

[Statement] What is the wavelength of a sound wave with frequency of 9.5 kHz, if the speed of sound is 340 m/s?

Hint 1: To find the wavelength (λ) if you know the frequency (f) and wave speed (v), use $\lambda = v/f$.

Hint 2: For a wave with frequency $f = 9.5$ kHz $= 9.5 \times 10^3$ Hz and speed ($v = 340$ m/s),

$$\lambda = \frac{v}{f} = \frac{340 \text{ m/s}}{9.5 \times 10^3 \text{ Hz}} = 0.036 \text{ m}.$$

Problem 4 (page 44 of text)

[Statement] How much does the phase of an electromagnetic wave with frequency of 100 kHz change in 1.5 μs at a fixed location?

Hint 1: To find the phase change at a fixed location $(\Delta\phi)_{\text{constant x}}$ if you know the time interval (Δt) and period ($T = 1/f$), use Eq. 1.4:

$$(\Delta\phi)_{\text{constant x}} = \omega\Delta t = \left(\frac{2\pi}{T}\right)\Delta t = 2\pi\left(\frac{\Delta t}{T}\right).$$

Hint 2: In this case, the time interval $\Delta t = 1.5\mu\text{s} = 1.5 \times 10^{-6}$ s, and the period $T = 1/f = 1/100$ kHz $= 1/1 \times 10^5$ Hz$= 1 \times 10^{-5}$ s.

Hint 3: Plugging the values into Eq. 1.4 gives

$$(\Delta\phi)_{\text{constant x}} = \left(\frac{2\pi}{T}\right)\Delta t = \left(\frac{2\pi}{1 \times 10^{-5} \text{ Hz}}\right)(1.5 \times 10^{-6} \text{ s}) = 0.94 \text{ rad}.$$

[Statement] What is the difference in phase of a mechanical wave with period of 2 seconds and speed of 15 m/s at two locations separated by 4 meters at some instant?

Hint 1: To find the phase change at a fixed time $(\Delta\phi)_{\text{constant t}}$ if you know the distance interval (Δx) and wavelength (λ), use Eq. 1.6:

$$(\Delta\phi)_{\text{constant t}} = k\Delta x = \left(\frac{2\pi}{\lambda}\right)\Delta x.$$

Hint 2: In this case, the distance interval $\Delta x = 4$ m, and the wavelength (λ) can be found from the wave period (T) and wave speed (v).

Hint 3: Since the wave period $T = 2$ s and the wave speed $v = 15$ m/s, the wavelength (λ) is

$$\lambda = \frac{v}{f} = vT = (15 \text{ m/s})(2 \text{ s}) = 30 \text{ m}.$$

Hint 4: Plugging the values from Hints 2 and 3 into Eq. 1.6 gives

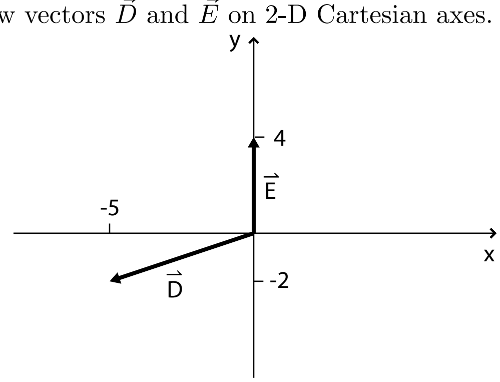
$$(\Delta\phi)_{\text{constant t}} = \left(\frac{2\pi}{\lambda}\right)\Delta x = \left(\frac{2\pi}{30 \text{ m}}\right)(4 \text{ m}) = 0.84 \text{ rad}.$$

Problem 5 (page 44 of text)

[Statement] If vector $\vec{D} = -5\hat{\imath} - 2\hat{\jmath}$ and vector $\vec{E} = 4\hat{\jmath}$, find the magnitude and direction of the vector $\vec{F} = \vec{D} + \vec{E}$ both graphically and algebraically.

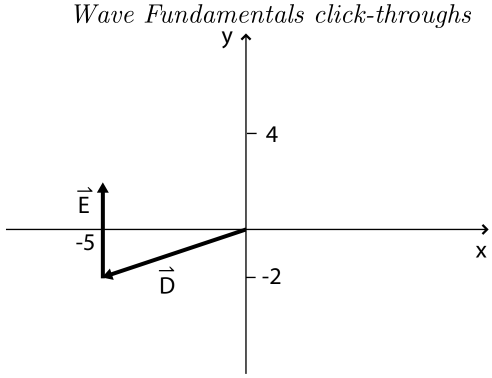
[Statement] Graphical Approach

Hint 1: Draw vectors $\vec{D}$ and $\vec{E}$ on 2-D Cartesian axes.

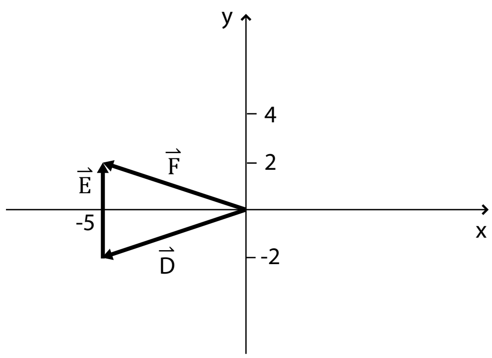


Hint 2: To graphically add the two vectors $\vec{D}$ and $\vec{E}$, imagine moving vector $\vec{E}$ without changing its length or direction so that its tail is at the position of the head of vector $\vec{D}$, as shown in the figure below.

Hint 3: Now draw a new vector (called vector $\vec{F}$) from the beginning (tail) of vector $\vec{D}$ to the end (head) of vector $\vec{E}$. Vector $\vec{F}$ is the sum of



vectors $\vec{D}$ and $\vec{E}$ and is called the "resultant" vector (so $\vec{F} = \vec{A} + \vec{B}$. The result would have been the same had you chosen to displace the tail of vector $\vec{D}$ to the head of vector $\vec{E}$ without changing the length or direction of $\vec{D}$.



Hint 4: To determine the magnitude of the resultant vector $\vec{F}$, use a ruler to measure its length. You should get about 5.4 units.

Hint 5: To determine the direction of the resultant vector $\vec{F}$, use a protractor to measure the angle of $\vec{F}$ measured anti-clockwise from the positive x-axis. You should get about 158 degrees.

[Statement] Algebraic Approach

Hint 1: To find the sum of vectors $\vec{D}$ and $\vec{E}$ algebraically, use

$$F_x = D_x + E_x$$
$$F_y = D_y + E_y.$$

Hint 2: Since $D_x = -5$ and $E_x = 0$,

$$F_x = D_x + E_x = -5 + 0 = -5.$$

Hint 3: Since $D_y = -2$ and $E_y = 4$,

$$F_y = D_y + E_y = -2 + 4 = +2.$$

Hint 4: Using unit vectors the vector $\vec{F}$ may be written as

$$\vec{F} = -5\hat{\imath} + 2\hat{\jmath}.$$

Hint 5: To find the magnitude of vector $\vec{F}$, use Eq. 1.5:

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(-5)^2 + (+2)^2} = 5.39 \text{ units}.$$

Hint 6: To find the direction of vector $\vec{F}$, use Eq. 1.8:

$$\theta = \arctan\left(\frac{F_y}{F_x}\right) = \arctan\left(\frac{+2}{-5}\right) = -21.8^\circ$$

but since the denominator of the arctan argument is negative, the angle of vector $\vec{F}$ measured anti-clockwise from the positive x-axis is $-21.8^\circ + 180^\circ = 158.2^\circ$.

Problem 6 (page 44 of text)

[Statement] Verify that each of the complex numbers in Fig. 1.10 have the polar form shown in Fig. 1.12.

Hint 1: To convert from rectangular to polar coordinates, use

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2}$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right).$$

Hint 2: For the complex number $z = 5 + 10i$, Re(z)=5 and Im(z)=10, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(5)^2 + (10)^2} = 11.18$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{10}{5} = 63.4^\circ.$$

Hint 3: For the complex number $z = -5 + 5i$, Re(z)=-5 and Im(z)=5, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(-5)^2 + (5)^2} = 7.07$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{5}{-5} = -45^\circ.$$

but since the denominator of the arctangent function is negative, to get the angle from the positive real axis you must add 180° to the result (assuming your calculator has a two-quadrant arctan function). So in this case the angle measured anti-clockwise from the positive real axis is $-45^\circ + 180^\circ = 135^\circ$.

Hint 4: For the complex number $z = -8 - 3i$, Re(z)=-8 and Im(z)=-3, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(-8)^2 + (-3)^2} = 8.54$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{-3}{-8} = 20.55^\circ$$

but once again you must add 180° to the result since the denominator of the arctangent function is negative, so the angle measured anti-clockwise from the positive real axis is $20.55^\circ + 180^\circ = 200.55^\circ$.

Hint 5: For the complex number $z = 3 - 2i$, Re(z)=3 and Im(z)=-2, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(3)^2 + (-2)^2} = 3.6$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{-2}{3} = -33.69^\circ$$

Note that this result is negative because the numerator of the arctangent function is negative, and not because the denominator is negative. So this is not a case in which you must add 180° to the result; instead, the negative result means that the direction is 33.69° below the real axis, which is the same as 326.31° measured anti-clockwise from the real axis.

Hint 6: For the complex number $z = 4 - 6i$, Re(z)=4 and Im(z)=-6, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(4)^2 + (-6)^2} = 7.21$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{-6}{4} = -56.31^\circ$$

which is the same as 303.69° measured anti-clockwise from the positive real axis.

Hint 7: For the complex number $z = 8 + 5i$, Re(z)=8 and Im(z)=5, so the magnitude and angle of z are

$$|z| = \sqrt{[\mathrm{Re(z)}]^2 + [\mathrm{Im(z)}]^2} = \sqrt{(8)^2 + (5)^2} = 9.43$$

and

$$\theta = \arctan\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right) = \arctan\frac{5}{8} = 32.0°$$

Problem 7 (page 44 of text)

[Statement] Solve the differential equation $dz/d\theta = iz$ for z.

Hint 1: To solve this equation for z, collect the terms involving z on the left side of the equation

$$\frac{dz}{z} = id\theta.$$

Hint 2: Integrate both sides of the rearranged equation

$$\int\frac{dz}{z} = \int id\theta.$$

Hint 3: Integrating the left side gives

$$\int\frac{dz}{z} = ln(z)$$

Hint 4: Integrating the right side gives

$$\int id\theta = i\theta.$$

Hint 5: Plugging in these results gives

$$\ln z = i\theta.$$

Hint 6: Raising "e" to the powers of $\ln z$ and $i\theta$ gives

$$e^{\ln z} = e^{i\theta}$$

or

$$z = e^{i\theta}.$$

Problem 8 (page 44 of text)

[Statement] Use the power-series representation of $\sin\theta$, $\cos\theta$, and $e^{i\theta}$ to prove the Euler relation $e^{i\theta} = \cos\theta + i\sin\theta$.

Hint 1: The power series for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

Hint 2: The power series for $\cos\theta$ is

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

Hint 3: The power series for e^z is

$$e^z = \sum_{n\geq 0} \frac{z^n}{n!}$$

Hint 4: This expression with $z = i\theta$ is

$$e^{i\theta} = \sum_{n\geq 0} \frac{(i\theta)^n}{n!} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

or

$$e^{i\theta} = 1 + i\theta - \frac{(\theta)^2}{2!} - \frac{i\theta^3}{3!} + \frac{(\theta)^4}{4!} + \frac{i\theta^5}{5!} + \cdots$$

Hint 5: Separating the real and imaginary parts of this expression gives

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{(\theta)^4}{4!} + \cdots\right) + i\left(\theta - \frac{(\theta)^3}{3!} + \frac{(\theta)^5}{5!} + \cdots\right).$$

Hint 6: Inserting the power series of $\sin\theta$ and $\cos\theta$ into this expression gives

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Problem 9 (page 44 of text)

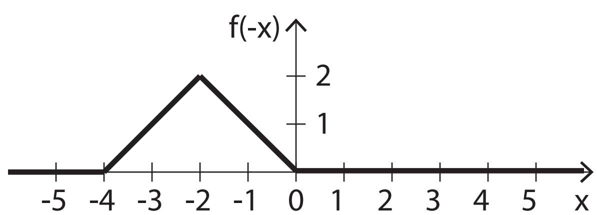
[Statement] Show that the wavefunction $f(-x-1)$ is shifted in the negative-x direction relative to the wavefunction $f(-x)$.

Hint 1: For the function $f(x)$ shown in Fig. 1.19, make a table for $f(-x)$ and $f(-x-1)$ such as that shown in Fig. 1.20 (start with $x = -5$).

Hint 2: Your table should look like this:

x	f(-x)	f(-x-1)
-5	f(+5)=0	f(+5-1)=f(+4)=0
-4	f(+4)=0	f(+4-1)=f(+3)=1
-3	f(+3)=1	f(+3-1)=f(+2)=2
-2	f(+2)=2	f(+2-1)=f(+1)=1
-1	f(+1)=1	f(+1-1)=f(0)=0
0	f(0)=0	f(0-1)=f(-1)=0
1	f(-1)=0	f(-1-1)=f(-2)=0
2	f(-2)=0	f(-2-1)=f(-3)=0

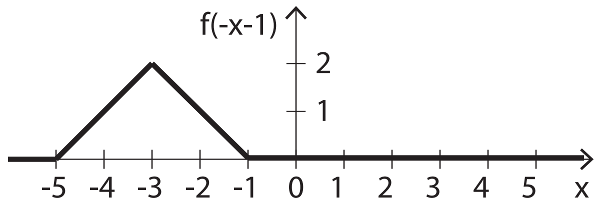
Hint 3: Plotting $f(-x)$ gives



Hint 4: Plotting $f(-x-1)$ gives

which is shifted in the negative-x direction relative to $f(-x)$.

Problem 10 (page 44 of text)



[Statement] Find the phase speed and the direction of propagation of each of the following waves (all units are SI):

a) $f(x,t) = 5\sin(3x - \frac{t}{2})$

Hint 1: To find the phase speed of a wave with wavefunction in the form of $f(kx \pm \omega t)$, use $v = \omega/k$.

Hint 2: For $f(x) = 5\sin(3x - t/2)$, $k = 3$ and $\omega = 1/2$, so

$$v = \frac{\omega}{k} = \frac{(\frac{1}{2})}{3} = \frac{1}{6} \text{ m/s}.$$

Hint 3: To determine the direction of this wave, note that the x-term and the t-term have opposite signs. This means that the direction of wave propagation is toward positive x in this case.

b) $\psi(x,t) = g{-}4x - 20t$

Hint 1: For $\psi(x,t) = g{-}4x - 20t$, $k = 4$ and $\omega = 20$, so

$$v = \frac{\omega}{k} = \frac{20}{4} = 5 \text{ m/s}.$$

Hint 2: In this case, the x-term and the t-term have the same sign, so the direction of wave propagation is toward negative x.

c) $h(y,t) = \frac{1}{2(2t+x)} + 10$.

Hint 1: For $h(y,t) = \frac{1}{2(2t+x)} + 10$, the ratio of the x-term coefficient to the t-term coefficient is 1/2, so

$$v = \frac{\omega}{k} = \frac{1}{2} = 0.5 \text{ m/s}.$$

Hint 2: In this case, the x-term and the t-term have the same sign, so the direction of wave propagation is toward negative x.

2

The Wave Equation click-throughs

Example 1 (page 48 of text)

[Statement] **Example:** *For the function $y(x,t) = 3x^2 - 5t$, find the partial derivative of y with respect to x and with respect to t.*

Hint 1: To take the partial derivative of y with respect to x, treat t as a constant:

$$\begin{aligned}\frac{\partial y}{\partial x} &= \frac{\partial(3x^2 - 5t)}{\partial x} = \frac{\partial(3x^2)}{\partial x} - \frac{\partial(5t)}{\partial x} \\ &= 3\frac{\partial(x^2)}{\partial x} - 0 = 6x.\end{aligned}$$

Hint 2: For the partial derivative of y with respect to t, treat x as a constant:

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial(3x^2 - 5t)}{\partial t} = \frac{\partial(3x^2)}{\partial t} - \frac{\partial(5t)}{\partial t} \\ &= 0 - 5\frac{\partial t}{\partial t} = -5.\end{aligned}$$

Example 2 (page 65 of text)

[Statement] **Example:** *Consider two sine waves with the following wavefunctions:*

$$\begin{aligned}y_1(x,t) &= A_1 \sin(k_1 x + \omega_1 t + \epsilon_1) \\ y_2(x,t) &= A_2 \sin(k_2 x + \omega_2 t + \epsilon_2).\end{aligned}$$

If these two waves have the same amplitude, $A_1 = A_2 = A = 1$, the same wave number, $k_1 = k_2 = k = 1$ rad/m, and the same angular frequency, $\omega_1 = \omega_2 = \omega = 2$ rad/s, but the first wave $y_1(x,t)$ has a phase constant

$\epsilon_1 = 0$ and the second wave $y_2(x,t)$ has a phase constant of $\epsilon_2 = +\pi/3$, determine the charateristics of the wave that results from the addition of these waves.

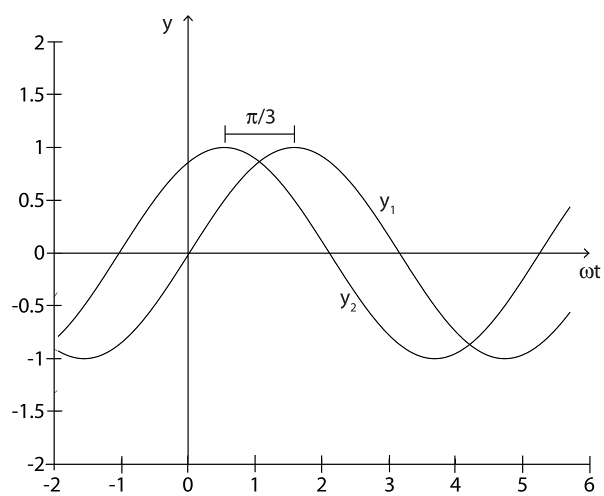
Hint 1: Since the distance term and the time term have the same sign for both of these waves, you know that both waves are traveling in the negative-x direction, and since the wave phase speed $v = \omega/k$ (see Eq. 1.36), they also have the same speed.

Hint 2: By comparing the phase constants for the two waves, you also know that $y_2(x,t)$, leads $y_1(x,t)$ by a phase difference of $\pi/3$ (if you don't recall why more-positive phase constant results in a leading wave in this case, look back to Section 1.6 of Chapter 1).

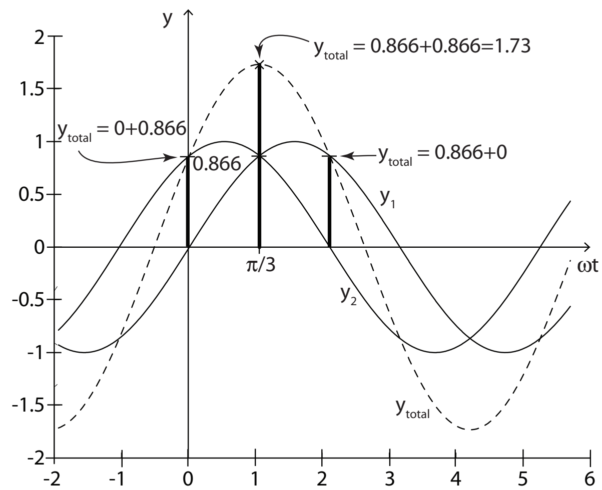
Hint 3: Inserting the values given above, the two wavefunctions may be written as

$$\begin{aligned} y_1(x,t) &= A_1 \sin(k_1 x + \omega_1 t + \epsilon_1) = \sin(x + 2t + 0) \\ y_2(x,t) &= A_2 \sin(k_2 x + \omega_2 t + \epsilon_2) = \sin(x + 2t + \pi/3) \end{aligned} \tag{2.21}$$

and, at $x = 0$, they look like this:



Hint 4: To understand how these two waves add to produce a new wave, take a look at the figure below. In this figure, the graphical addition of the two waves is shown to result in another sinusoidal wave, drawn with



a dashed line. This resultant wave has the same frequency as the two original waves, but it has a different phase constant and larger amplitude.

Hint 5: Some algebra can show the same result, starting with the expression for the resultant wave:

$$y_{total}(x,t) = \sin(x+2t) + \sin(x+2t+\pi/3). \tag{2.21}$$

Hint 6: A useful trigonometric identity here is

$$\sin\theta_1 + \sin\theta_2 = 2\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right). \tag{2.23}$$

Hint 7: Plugging in $\theta_1 = x+2t$ and $\theta_2 = x+2t+\pi/3$ gives

$$y_{total}(x,t) = 2\sin\left(\frac{2(x+2t)+\pi/3}{2}\right)\cos\left(\frac{-\pi/3}{2}\right). \tag{2.24}$$

Hint 8: The sine term in this expression simplifies to $\sin(x+2t+\pi/6)$, which is a wave with wave number $k = 1$ rad/m and angular frequency $\omega = 2$ rad/s (hence the same wavelength and frequency as the original waves), but with a phase constant $\epsilon = \pi/6$ (in this case, the average of the original phases of zero and $\pi/3$).

Hint 9: What about the amplitude? The rest of Eq. 2.24 gives $A = 2\cos(-\pi/6) \approx 1.73$. So the amplitude is larger than the original $A = 1$, but not twice as large (since in this case the two original waves don't reach their peak values at the same time).

Problem 1 (page 74 of text)

[Statement] Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial t}$ for the function $f(x,t) = 3x^2t^2 + \frac{1}{2}x + 3t^3 + 5$.

Hint 1: To find the partial derivative with respect to x, treat t as a constant.

Hint 2: The $3t^3$ and 5 terms are both constants and have derivatives of zero. The derivatives of the other terms are

$$\frac{\partial}{\partial x}(3x^2t^2) = 3(2x)t^2 = 6xt^2$$

and

$$\frac{\partial}{\partial x}\left(\frac{1}{2}x\right) = \frac{1}{2}.$$

All together, the solution is $\frac{\partial f}{\partial x} = 6xt^2 + \frac{1}{2}$.

Hint 3: To find the partial derivative with respect to t, treat x as a constant.

Hint 4: The $\frac{1}{2}x$ and 5 terms are both constants and have derivatives of zero. The derivatives of the other terms are

$$\frac{\partial}{\partial t}(3x^2t^2) = 3x^2(2t) = 6x^2t$$

and

$$\frac{\partial}{\partial t}\left(3t^3\right) = 3(3t^2) = 9t^2.$$

All together, the solution is $\frac{\partial f}{\partial t} = 6x^2t + 9t^2$.

Problem 2 (page 74 of text)

[Statement] For the function $f(x,t)$ of Problem 1, find $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial t^2}$.

Hint 1: To find the second partial derivative with respect to x, take the partial derivative with respect to x of the first derivative, found in

Problem 1:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

Don't forget to treat t as a constant.

Hint 2: The first derivative is $\frac{\partial f}{\partial x} = 6xt^2 + \frac{1}{2}$. The partial derivative of this function with respect to x is

$$\frac{\partial}{\partial x}\left(6xt^2 + \frac{1}{2}\right) = 6t^2(1) + 0 = 6t^2.$$

So, $\frac{\partial^2 f}{\partial x^2} = 6t^2$.

Hint 3: To find the second partial derivative with respect to t, take the partial derivative with respect to t of the first derivative, found in Problem 1:

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right).$$

Don't forget to treat x as a constant.

Hint 4: The first derivative is $\frac{\partial f}{\partial t} = 6x^2t + 9t^2$. The partial derivative of this function with respect to t is

$$\frac{\partial}{\partial t}\left(6x^2t + 9t^2\right) = 6x^2(1) + 9(2t) = 6x^2 + 18t.$$

So, $\frac{\partial^2 f}{\partial t^2} = 6x^2 + 18t$.

Problem 3 (page 74 of text)

[Statement] For the function $f(x,t)$ of Problem 1, show that $\frac{\partial^2 f}{\partial x \partial t}$ gives the same result as $\frac{\partial^2 f}{\partial t \partial x}$.

Hint 1: To find $\frac{\partial^2 f}{\partial x \partial t}$, take the partial derivative with respect to x of the first derivative with respect to t, found in Problem 1:

$$\frac{\partial^2 f}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial t}\right).$$

Don't forget to treat t as a constant.

Hint 2: The first derivative is $\frac{\partial f}{\partial t} = 6x^2t + 9t^2$. The partial derivative of this function with respect to x is

$$\frac{\partial}{\partial x}\left(6x^2t + 9t^2\right) = 6(2x)t + 0 = 12xt.$$

So, $\frac{\partial^2 f}{\partial x \partial t} = 12xt$.

Hint 3: To find $\frac{\partial^2 f}{\partial t \partial x}$, take the partial derivative with respect to t of the first derivative with respect to x, found in Problem 1:

$$\frac{\partial^2 f}{\partial t \partial x} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\right).$$

Don't forget to treat x as a constant.

Hint 4: The first derivative is $\frac{\partial f}{\partial x} = 6xt^2 + \frac{1}{2}$. The partial derivative of this function with respect to t is

$$\frac{\partial}{\partial t}\left(6xt^2 + \frac{1}{2}\right) = 6x(2t) + 0 = 6x^2 + 12xt.$$

So, $\frac{\partial^2 f}{\partial t \partial x} = 12xt$, the same as $\frac{\partial^2 f}{\partial x \partial t}$.

Problem 4 (page 74 of text)

[Statement] Does the function $Ae^{i(kx-\omega t)}$ satisfy the classical wave equation? If so, prove it. If not, say why not.

Hint 1: In order for a function to satisfy the classical wave equation, the left side of $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ must equal the right side.

Hint 2: The left side of the classical wave equation can be found by taking the second partial derivative with respect to x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(Ae^{i(kx-\omega t)}\right)\right).$$

Hint 3: The derivative is easier if you rewrite the function as $Ae^{ikx}e^{-i\omega t}$. Because you're taking a partial derivative with respect to x, the constant terms are $Ae^{-i\omega t}$, which come out of the derivative:

$$\begin{aligned} Ae^{-i\omega t}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(e^{ikx}\right)\right) &= Ae^{-i\omega t}\frac{\partial}{\partial x}ike^{ikx} \\ &= Ae^{-i\omega t}(ik)^2 e^{ikx} \\ &= -k^2 Ae^{-i\omega t}e^{ikx} \end{aligned}$$

Hint 4: Because $y = Ae^{i(kx-\omega t)}$, you can rewrite the left side as $-k^2 y$.

Hint 5: The right side of the classical wave equation can be found by taking the second partial derivative with respect to t and multiplying by $\frac{1}{v^2}$:

$$\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(Ae^{i(kx-\omega t)}\right)\right).$$

Hint 6: Again, the derivative is easier if you rewrite the function as $Ae^{ikx}e^{-i\omega t}$. Because you're taking a partial derivative with respect to t, the constant terms are Ae^{ikx}, which come out of the derivative:

$$\begin{aligned} \frac{1}{v^2}Ae^{ikx}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(e^{-i\omega t}\right)\right) &= \frac{1}{v^2}Ae^{ikx}\frac{\partial}{\partial t} - i\omega e^{-i\omega t} \\ &= \frac{1}{v^2}Ae^{ikx}(-i\omega)^2 e^{-i\omega t} \\ &= -\frac{1}{v^2}\omega^2 Ae^{ikx}e^{-i\omega t} \end{aligned}$$

Hint 7: Because $y = Ae^{i(kx-\omega t)}$, you can rewrite the left side as $-\frac{\omega^2}{v^2}y$.

Hint 8: Set the left side equal to the right side and see if the result is true:

$$-k^2 y = -\frac{\omega^2}{v^2}y.$$

All terms cancel except for $k^2 = \frac{\omega^2}{v^2}$.

Hint 9: By rearranging this equation into $v^2 = \frac{\omega^2}{k^2}$ or $v = \frac{\omega}{k}$, you can see that it matches Eq. 1.36 and the wave equation is satisfied.

Hint 10: An easier way to do all of this is to recognize that $Ae^{i(kx-\omega t)}$ is a function of the form $f(kx \pm \omega t)$ and hence will be a solution.

Problem 5 (page 74 of text)

[Statement] Does the function $A_1 e^{i(kx-\omega t)} + A_2 e^{i(kx+\omega t)}$ satisfy the classical wave equation? If so, prove it. If not, say why not.

Hint 1: You know from Problem 4 that the first term satisfies the wave equation. Because the wave equation is linear, if two functions are solutions, then their sum is also a solution.

Hint 2: The spatial part of y is the same in both terms, so the left side of the classical wave equation is the same for $A_2 e^{i(kx+\omega t)}$.

Hint 3: The time derivatives are taken the same way as in Problem 4, but with a changed sign ($+\omega$ instead of $-\omega$):

$$\begin{aligned}
\frac{1}{v^2} A e^{ikx} \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\left(e^{i\omega t}\right)\right) &= \frac{1}{v^2} A e^{ikx} \frac{\partial}{\partial t} i\omega e^{i\omega t} \\
&= \frac{1}{v^2} A e^{ikx} (i\omega)^2 e^{i\omega t} \\
&= -\frac{1}{v^2} \omega^2 A e^{ikx} e^{i\omega t}
\end{aligned}$$

Hint 4: The right side still condenses down to $-\frac{\omega^2}{v^2}$. Because the right and left sides are the same for $A_2 e^{i(kx+\omega t)}$ as it was for $A_1 e^{i(kx-\omega t)}$, it is also a solution, as is their sum. This is the form of two traveling waves, one to the right (the A_1 wave) and one to the left (the A_2 wave).

Problem 6 (page 74 of text)

[Statement] Does the function $Ae^{(ax+bt)^2}$ satisfy the classical wave equation? If so, what is the speed of the wave described by this function?

Hint 1: The method to solve this problem is the same as for Problem 4: find the left and right sides independently, then see if they equal each other with a reasonable velocity (that is, a constant value, not a function of x or t).

Hint 2: The left side of the classical wave equation can be found by taking the second partial derivative with respect to x:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(Ae^{(ax+bt)^2}\right)\right).$$

Hint 3: Take the first partial derivative, $\frac{\partial f}{\partial x}$:

$$\begin{aligned}\frac{\partial}{\partial x}\left(Ae^{(ax+bt)^2}\right) &= A(2)(ax+bt)(a)e^{(ax+bt)^2}\\ &= 2Aa(ax+bt)e^{(ax+bt)^2}\end{aligned}$$

Hint 4: Take the second derivative by

$$\begin{aligned}\frac{\partial}{\partial x}\left(2Aa(ax+bt)e^{(ax+bt)^2}\right) &= 2Aa(a)(ax+bt)e^{(ax+bt)^2}\\ &\quad + 2Aa(ax+bt)(2)(ax+bt)(a)e^{(ax+bt)^2}\\ &= 2a^2\,(ax+bt)\,(1+2(ax+bt))\left(Ae^{(ax+bt)^2}\right)\\ &= 2a^2\,(ax+bt)\,(1+2(ax+bt))y\end{aligned}$$

Hint 5: You can find the second derivative with respect to t in the same way, but especially in an algebraically intense situation like this it's worth looking for an easier method. Consider that x and t are not inherently different from each other; they're just names for variables, just as a and b are arbitrary names for constants. If you take the partial derivative of a function $f = ax$ with respect to x, it will have the same form as the partial derivative of a function $g = bt$ taken with respect to t. So, $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$, with all $x \to t$ and $a \to b$:

$$\frac{\partial^2 f}{\partial t^2} = 2b^2\,(ax+bt)\,(1+2(ax+bt))y$$

Hint 6: Set the left and right sides equal, remembering to add in the $1/v^2$ term to the right side:

$$2a^2\,(ax+bt)\,(1+2(ax+bt))y = 2\frac{b^2}{v^2}\,(ax+bt)\,(1+2(ax+bt))y.$$

Everything cancels now except for

$$a^2 = \frac{b^2}{v^2}.$$

Hint 7: Solving for v^2 gives

$$v^2 = \frac{b^2}{a^2}$$

or

$$v = \pm\frac{b}{a}.$$

These are two possible velocities, one to the left and one to the right.

Hint 8: You can even check the units of the velocity. The argument of e has to be unitless (or in radians, just like the argument of sine or cosine). So, you know that if x is in meters, a is in 1/m; similarly if t is in seconds, b is in 1/s. The ratio of b/a therefore has units of (1/s)/(1/m), or m/s, as expected.

Problem 7 (page 74 of text)

[Statement] Sketch the solutions to the hyperbolic equation $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ for various values of a and b.

Hint 1: First, solve the equation for y, which will let you plot it in mathematical software:

$$\begin{aligned}
\frac{y^2}{a^2} - \frac{x^2}{b^2} &= 1 \\
\frac{y^2}{a^2} &= 1 + \frac{x^2}{b^2} \\
y^2 &= a^2(1 + \frac{x^2}{b^2}) \\
y &= \pm a\sqrt{1 + \frac{x^2}{b^2}}.
\end{aligned}$$

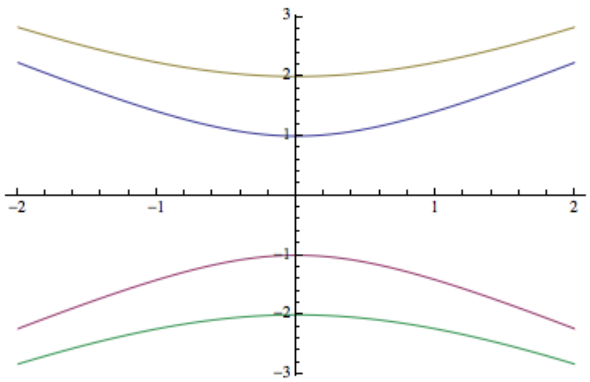
Hint 2: Decide on which values of a and b to use in the plot. For example, you could plot the following combinations:

- $a = b = 1$ (same and small values)
- $a = b = 2$ (same and large values)
- $a = 2, b = 1$ (different, with $a > b$)
- $a = 1, b = 2$ (different, with $a < b$)

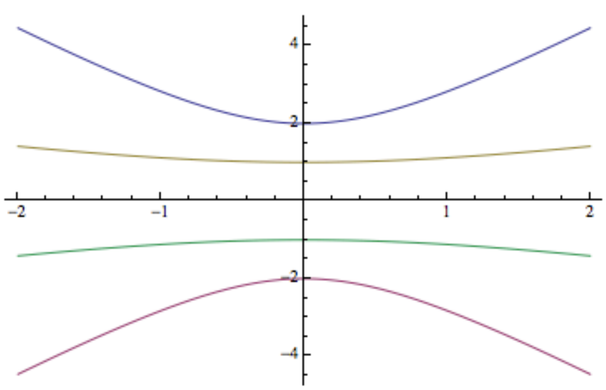
Hint 3: Find a reasonable range of x to plot over. For the given values of a and b, plotting from $x = -2$ to 2 works well.

Hint 4: The first figure below shows the first two cases, when $a = b$. The second figure below shows the last two cases, when $a \neq b$.

```
a1 = 1;
b1 = 1;
a2 = 2;
b2 = 2;
Plot[{a1 Sqrt[1 + x^2 / b1^2], -a1 Sqrt[1 + x^2 / b1^2], a2 Sqrt[1 + x^2 / b2^2],
  -a2 Sqrt[1 + x^2 / b2^2]}, {x, -2, 2}, PlotRange → All]
```




```
a3 = 2;
b3 = 1;
a4 = 1;
b4 = 2;
Plot[{a3 Sqrt[1 + x^2 / b3^2], -a3 Sqrt[1 + x^2 / b3^2], a4 Sqrt[1 + x^2 / b4^2],
  -a4 Sqrt[1 + x^2 / b4^2]}, {x, -2, 2}, PlotRange → All]
```

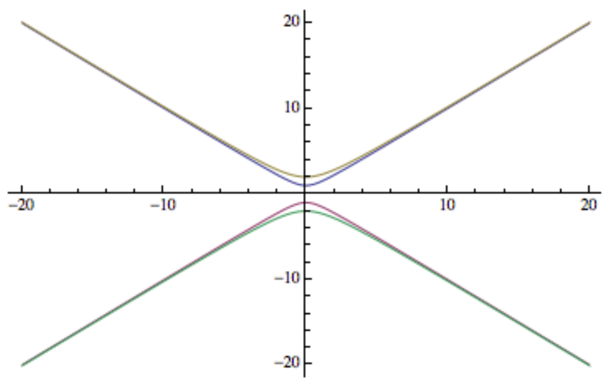


Hint 5: The two main differences between any of these plots is 1. where they intersect the y-axis and 2. how steeply they leave that intersection. The question is, how do the values of a and b control those properties?

Hint 6: The y-intercept is the same as the value of $\pm a$.

Hint 7: The plots with the same value of a and b eventually leave the y-intercept with the same slope (this is more obvious in the third and fourth figures below, which plot the functions between -20 and 20). If $a > b$, then the slope is larger than if $a < b$.

```
a1 = 1;
b1 = 1;
a2 = 2;
b2 = 2;
Plot[{a1 Sqrt[1 + x^2 / b1^2], -a1 Sqrt[1 + x^2 / b1^2], a2 Sqrt[1 + x^2 / b2^2],
  -a2 Sqrt[1 + x^2 / b2^2]}, {x, -20, 20}, PlotRange → All]
```



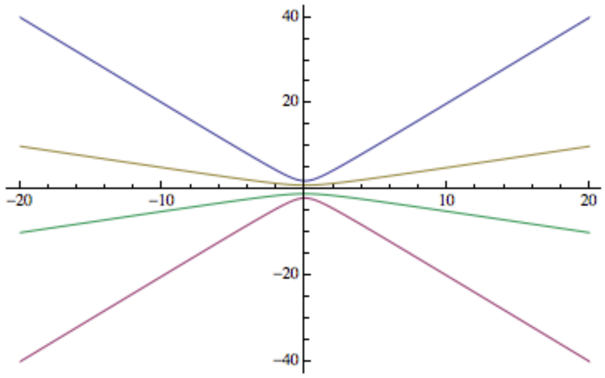
Hint 8: To make sense of the previous observations, look at the form of the function:

$$y = \pm a\sqrt{1 + \frac{x^2}{b^2}} = \pm\sqrt{a^2 + \frac{a^2}{b^2}x^2}.$$

It appears that the first term, $\sqrt{a^2} = a$, is the y-intercept. The a and b part of the second term $\sqrt{\frac{a^2}{b^2}} = \frac{a}{b}$ gives the slope at large x.

Hint 9: Now compare to the homogeneous wave equation, which maps

```
a3 = 2;
b3 = 1;
a4 = 1;
b4 = 2;
Plot[{a3 Sqrt[1 + x^2 / b3^2], -a3 Sqrt[1 + x^2 / b3^2], a4 Sqrt[1 + x^2 / b4^2],
  -a4 Sqrt[1 + x^2 / b4^2]}, {x, -20, 20}, PlotRange → All]
```



to $\frac{x^2}{a^2} - \frac{t^2}{b^2} = 0$. Solutions would look like

$$\begin{aligned}\frac{x^2}{a^2} - \frac{t^2}{b^2} &= 0\\ \frac{x^2}{a^2} &= \frac{t^2}{b^2}\\ x^2 &= a^2(\frac{t^2}{b^2})\\ x &= \pm\frac{a}{b}t.\end{aligned}$$

There is no first term, so the y-intercept is zero (you could think of it as $a \cdot 1$ in the previous case; now it's $a \cdot 0$). The fraction $\frac{a}{b}$ is now literally the slope at all values of t. If you multiply the original equation by a^2, you'll see that the $1/v^2$ term identifies with a^2/b^2 (it's not $v^2 = a^2/b^2$ because the x and t in the wave equation are in the denominators as ∂x and ∂t). This is a nice connection and support for the analogy between the hyperbolic functions and the PDEs.

Problem 8 (page 74 of text)

[Statement] Make time-domain plots of $y_1(x,t) = A_1 \sin(k_1 x + \omega_1 t + \epsilon_1)$

and $y_2(x,t) = A_2 \sin(k_2 x + \omega_2 t + \epsilon_2)$ and their sum for $A = 1$ m, $k = 1$ rad/m, $\omega = 2$ rad/s, $\epsilon_1 = 1.5$ rad, and $\epsilon_2 = 0$ rad, at positions $x = 0.5$ m and $x = 1.0$ m over at least one full period of oscillation.

Hint 1: Before plotting, you need to know what the period is. Because the angular frequency ω is given, you can get the period from Eq. 1.3:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi.$$

Hint 2: At $x = 0.5$ m, the wave functions simplify to

$$y_1(x,t) = \sin((1)(0.5) + 2t + 1.5) = \sin 2.0 + 2t$$

and

$$y_2(x,t) = \sin((1)(0.5) + 2t + 0) = \sin 0.5 + 2t.$$

Hint 3: Plotting these functions at $x = 0.5$ m from $t = 0$ to π s results in the graph in the top portion of the figure shown below.

Hint 4: Similarly, to plot at $x = 1$ m, plug into the functions:

$$y_1(x,t) = \sin((1)(1) + 2t + 1.5) = \sin 2.5 + 2t$$

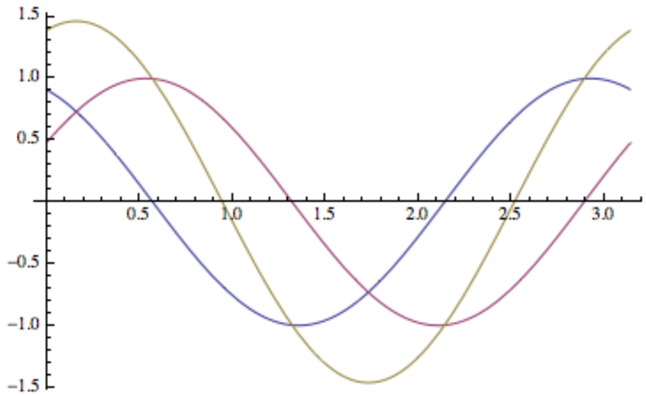
and

$$y_2(x,t) = \sin((1)(1) + 2t + 0) = \sin 1.0 + 2t.$$

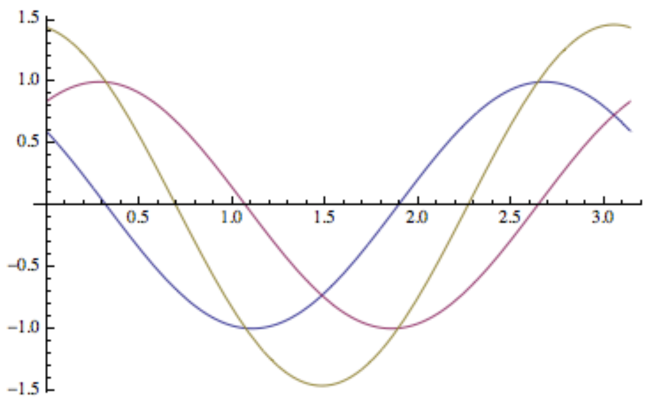
Hint 5: Plotting these functions at $x = 1$ m from $t = 0$ to π s results in the graph shown in the bottom portion of the figure.

```
Plot[{Sin[2 + 2 t], Sin[0.5 + 2 t], Sin[2 + 2 t] + Sin[0.5 + 2 t]}, {t, 0, Pi}]
```

3




```
Plot[{Sin[2.5 + 2 t], Sin[1 + 2 t], Sin[2.5 + 2 t] + Sin[1 + 2 t]}, {t, 0, Pi}]
```



Problem 9 (page 74 of text)

[Statement] Sketch the phasors for the waveforms of the previous problem (and their sum) at $x = 1$ m at times $t = 0.5$ s and $t = 1.0$ s.

Hint 1: The simplified phasor representation is an arrow with projection onto the vertical axis equal to the wave function's value.

Hint 2: The length of the phasor is the amplitude, $A = 1$ m.

Hint 3: The angle of the phasor above the horizontal x-axis is the argument of the sine. At $t = 0.5$ s and $x = 1$ m, the arguments are

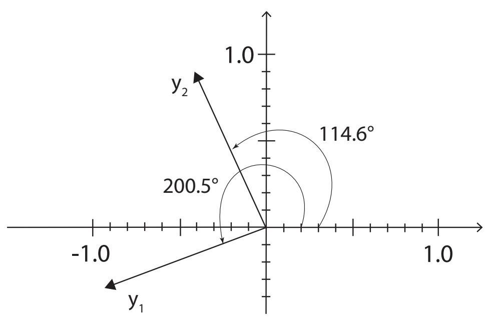
$$2.5 + 2(0.5) = 3.5 \text{ rad}$$

for y_1 and

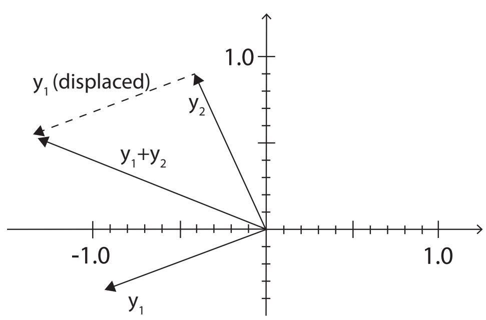
$$1.0 + 2(0.5) = 2.0 \text{ rad}$$

for y_2.

Hint 4: Plotting the two phasors for y_1 and y_2 looks like the first figure shown below.



Hint 5: To find the sum, add the two phasors graphically, with the result shown in the figure below.



Hint 6: Similarly, the arguments for the sines in y_1 and y_2 at $t = 1$ s are

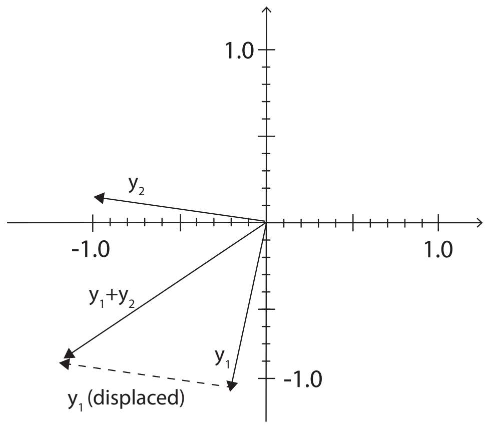
$$2.5 + 2(1) = 4.5 \text{ rad}$$

for y_1 and

$$1.0 + 2(1) = 3.0 \text{ rad}$$

for y_2.

Hint 7: The phasors and their sum at $t = 1$ are shown in the figure below.



Problem 10 (page 74 of text)

[Statement] Does the function $Ae^{i(kx-\omega t)}$ satisfy the advection equation as given in Eq. 2.27? What about the function $Ae^{i(kx+\omega t)}$?

Hint 1: The advection equation is $\frac{\partial y(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial y(x,t)}{\partial t}$. If $Ae^{i(kx-\omega t)}$ satisfies the equation, the left and right sides will be equal once the function is plugged in.

Hint 2: The partial derivative of $Ae^{i(kx-\omega t)}$ with respect to x is $(ik)Ae^{i(kx-\omega t)} = iky$, as determined in Problem 1.

Hint 3: The partial derivative of $Ae^{i(kx-\omega t)}$ with respect to t is $(-i\omega)Ae^{i(kx-\omega t)} = -i\omega y$, as determined in Problem 1.

Hint 4: Plugging into the advection equation gives

$$iky = -i\left(\frac{-\omega}{v}\right)y.$$

Everything cancels but

$$k = \frac{\omega}{v}.$$

Hint 5: Solving for velocity gives $v = \omega/k$, which is a valid velocity for a wave moving to the right (as expected).

Hint 6: The spatial derivative of $Ae^{i(kx+\omega t)}$ is the same as before, while the time derivative is positive instead of negative. Together, this means that

$$iky = i\left(\frac{-\omega}{v}\right)y,$$

or

$$k = -\frac{\omega}{v}.$$

and a velocity of $v = -\omega/k$. This is not a wave moving to the right, so it is not a good solution.

3

Wave Components click-throughs

Example 1 (page 78 of text)

[Statement] **Example:** *If the functions f and g in Eq. 3.10 both represent sine waves of amplitude A, how does the wavefunction $y(x,t)$ behave?*

Hint 1: To answer this question, write f and g as

$$f(x+vt) = A\sin(kx+\omega t)$$
$$g(x-vt) = A\sin(kx-\omega t)$$

(if you're concerned that v doesn't appear explicitly in the right side of these equations, recall from Chapter 1 that $kx - \omega t$ can be written as $k(x - \frac{\omega}{k}t)$, and $\frac{\omega}{k} = v$, where v is the phase velocity of the wave).

Hint 2: Inserting these expressions for f and g into the general solution for the wave equation (Eq. 3.10) gives

$$\begin{aligned} y &= f(x+vt) + g(x-vt) \\ &= A\sin(kx+\omega t) + A\sin(kx-\omega t). \end{aligned}$$

Hint 3: But

$$\sin(kx+\omega t) = \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)$$

and

$$\sin(kx-\omega t) = \sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t).$$

Hint 4: So

$$\begin{aligned} y &= A[\sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t)] + A[\sin(kx)\cos(\omega t) - \cos(kx)\sin(\omega t)] \\ &= A[\sin(kx)\cos(\omega t) + \sin(kx)\cos(\omega t) + \cos(kx)\sin(\omega t) - \cos(kx)\sin(\omega t)] \\ &= 2A\sin(kx)\cos(\omega t). \end{aligned}$$

Example 2 (page 83 of text)

[Statement] **Example:** *Find y(x,t) for a wave with this initial displacement condition:*

$$y(x,0) = I(x) = \begin{cases} 5\left(1 + \frac{x}{\frac{L}{2}}\right) & \text{for } \frac{L}{2} < x < 0 \\ 5\left(1 - \frac{x}{\frac{L}{2}}\right) & \text{for } 0 < x < \frac{L}{2} \\ 0 & \text{elsewhere} \end{cases}$$

and initial transverse velocity condition

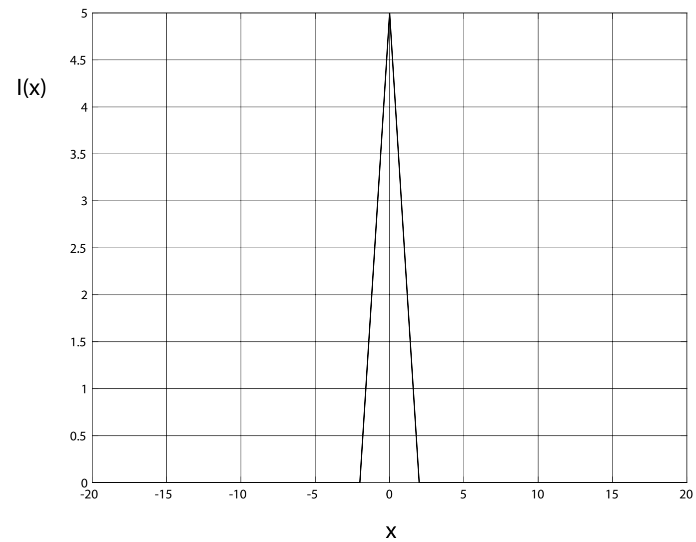
$$\frac{\partial y(x,t)}{\partial t}|_{t=0} = 0.$$

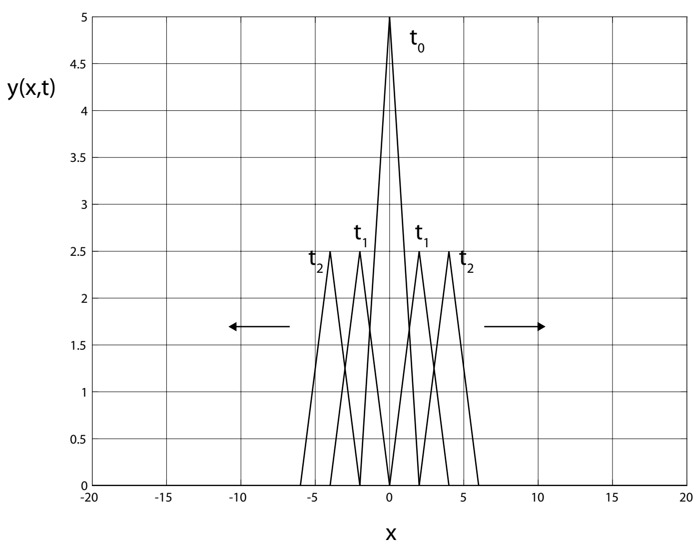
Hint 1: Since you're given the initial displacement (I) and transverse velocity (V) functions, you can use Eq. 3.16 to find $y(x,t)$. But it's often helpful to begin by plotting the initial displacement function, as in the figure below. In this case, the initial transverse velocity is zero, so there's no need to plot that function.

Hint 2: Now that you have an idea of what the initial displacement looks like, you're ready to use Eq. 3.16:

$$\begin{aligned} y(x,t) &= \frac{1}{2}I(x-vt) + \frac{1}{2}I(x+vt) + \frac{1}{2v}\int_{x-vt}^{x+vt} V(z)dz \\ &= \frac{1}{2}\left[I(x-vt) + I(x+vt)\right] + 0. \end{aligned}$$

This is just the initial shape of the wave ($I(x)$) scaled by 1/2 and propagating in both the negative- and positive-x directions while maintaining its shape over time, as you can see in the figure below. In this figure, the tall triangle centered on $x = 0$ is the sum of $\frac{1}{2}\left[I(x-vt)\right]$ and





$\frac{1}{2}[I(x+vt)]$ at time t=0, which is just $I(x)$. At a later time $t=t_1$, the wavefunction $I(x-vt)$ has propagated a distance vt_1 to the right (toward positive x), while the counterpropagating wavefunction $I(x+vt)$ has moved that same distance to the left (toward negative x), so the two component wavefunctions no longer overlap. As time progresses, the two component wavefunctions continue to move apart, as can be seen by the plots for $t = t_2$.

Example 3 (page 89 of text)

[Statement] **Example:** *Find the displacement y(x,t) produced by waves on a string fixed at both ends.*

Hint 1: Since the string is fixed at both ends, you know that the displacement $y(x,t)$ must be zero for all time at the locations corresponding to the ends of the string. If you define one end of the string to have value $x = 0$ and the other end to have value $x = L$ (where L is the length of the string), you know that $y(0,t) = 0$ and $y(L,t) = 0$. Separating $y(x,t)$ into the product of distance function $X(x) = A\cos(kx) + B\sin(kx)$ and time function $T(t)$ means that

$$\begin{aligned} y(0,t) = X(0)T(t) &= [A\cos(0) + B\sin(0)]\,T(t) = 0 \\ &[(A)(1) + (B)(0)]\,T(t) = 0. \end{aligned}$$

Hint 2: Since this must be true at all time (t), this means that the weighting coefficient A for the cosine term must equal zero. Applying the boundary condition at the other end of the string ($x = L$) is also useful:

$$\begin{aligned} y(L,t) = X(L)T(t) &= [A\cos(kL) + B\sin(kL)]\,T(t) = 0 \\ &[0\cos(kL) + B\sin(kL)]\,T(t) = 0. \end{aligned}$$

Hint 3: Once again invoking the fact that this must be true over all time, this can only mean that either $B = 0$ or $\sin(kL)$=0. Since $B = 0$ corresponds to the supremely boring case of no displacement anywhere on the string at any time (remember that you already know that $A = 0$), you'll have more fun if you consider the case for which B is non-zero and

$\sin(kL)$ is zero. You know that $k = 2\pi/\lambda$, so in this case

$$\begin{aligned}
\sin(kL) &= \sin\left(\frac{2\pi L}{\lambda}\right) = 0 \\
\frac{2\pi L}{\lambda} &= n\pi \\
\lambda &= \frac{2L}{n}
\end{aligned}$$

where n can be any positive integer (taking n to be zero or negative doesn't lead to any interesting physics).

Example 4 (page 108 of text)

[Statement] **Example:** *Verify the Fourier coefficients shown for the triangle wave in Fig. 3.16. Assume that the spatial period* ($2L$) *is 1 meter, and the units of* $X(x)$ *are also meters.*

Hint 1: If you followed the discussion about this triangle wave, you already know that the DC term (A_0) and the cosine coefficients (A_n) should be non-zero and that the sine coefficients (B_n) should all be zero (since this wave is an even function with non-zero average value). You can verify those conclusions using Eqs. 3.30, but first you have to figure out the period of X(x) and the equation for X(x).

Hint 2: You can read the period right off the graph: this waveform repeats itself with a period of one meter. Since the spatial period is represented as $2L$ in the Fourier series equations, this means that $L = 0.5$ meter.

Hint 3: To determine the equation for X(x), notice that this function is made up of straight lines, and the equation of a straight line is $y = mx+b$, where m is the slope of the line and b is the y-intercept (the value of y at the point at which the line crosses the y-axis).

Hint 4: You can choose to analyze any of the complete cycles shown on the graph, but in many cases you can save time and effort by selecting a cycle that's centered on $x = 0$ (you'll see why that's true later in this example). So instead of considering a cycle consisting of one of the triangles with the point at the top (such as the triangle between $x = 0$ and $x = 2L$), you can consider the "inverted triangle" (with the point at the bottom) between $x = -L$ and $x = L$.

Hint 5: The slope of the line between $x = -L = -0.5$ and $x = 0$ is -2 (because the "rise" is -1 and the "run" is 0.5, so the rise over the run is $\frac{-1}{0.5} = -2$) and the y-intercept is zero. So the equation for this portion of $X(x)$ is $X(x) = mx + b = -2x + 0$.

Hint 6: A similar analysis between $x = 0$ and $x = L = 0.5$ gives the equation $X(x) = mx + b = 2x + 0$. With these equations in hand, you can now plug $X(x)$ into the equation for A_0:

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} X(x)dx = \frac{1}{2(0.5)}\left[\int_{-0.5}^{0} -2xdx + \int_{0}^{0.5} 2xdx\right] \\
&= (1)\left[-2(\frac{x^2}{2})|_{-0.5}^{0} + 2(\frac{x^2}{2})|_{0}^{0.5}\right] = 0 - (-0.25) + 0.25 - 0 \\
&= 0.5
\end{aligned}$$

and into the equation for A_n:

$$\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} X(x)\cos{(\frac{n2\pi x}{2L})}dx \\
&= \frac{1}{0.5}\left[\int_{-0.5}^{0} -2x\cos{(2n\pi x)}dx + \int_{0}^{0.5} 2x\cos{(2n\pi x)}dx\right].
\end{aligned}$$

Hint 7: Using integration by parts (or looking up $\int x\cos{(ax)}dx$ in a table of integrals), you'll find that $\int x\cos{(ax)}dx = \frac{x}{a}\sin{(ax)} + \frac{1}{a^2}\cos{(ax)}$, so the equation for A_n becomes

$$\begin{aligned}
A_n = &\frac{-2}{0.5}\left[\frac{x}{2n\pi}\sin{(2n\pi x)}|_{-0.5}^{0} + \frac{1}{4n^2\pi^2}\cos{(2n\pi x)}|_{-0.5}^{0}\right] \\
&+ \frac{2}{0.5}\left[\frac{x}{2n\pi}\sin{(2n\pi x)}|_{0}^{0.5} + \frac{1}{4n^2\pi^2}\cos{(2n\pi x)}|_{0}^{0.5}\right] \\
= &\frac{-2}{0.5}\left[0 - \frac{-0.5}{2n\pi}\sin{(2n\pi(-0.5))} + \frac{1}{4n^2\pi^2}(1 - \cos{(2n\pi(-0.5))})\right] \\
&+ \frac{2}{0.5}\left[\frac{0.5}{2n\pi}\sin{(2n\pi(0.5))} - 0 + \frac{1}{4n^2\pi^2}(\cos{(2n\pi(0.5))} - 1)\right]
\end{aligned}$$

Hint 8: Recall that $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so

$$\begin{aligned} A_n &= \frac{-2}{0.5}\left[0 - 0 + \frac{1}{4n^2\pi^2}(1-(-1)^n)\right] \\ &\qquad + \frac{2}{0.5}\left[0 - 0 + \frac{1}{4n^2\pi^2}((-1)^n - 1)\right] \\ &= \frac{-4}{0.5}\left[\frac{1}{4n^2\pi^2}(1-(-1)^n)\right] = \left[\frac{-2}{n^2\pi^2}(1-(-1)^n)\right] \\ &= \frac{-4}{n^2\pi^2} \text{ for odd n.} \end{aligned}$$

Hint 9: Fortunately, determining the B_n coefficients for this waveform is much easier. Since

$$B_n = \frac{1}{L}\int_{-L}^{L} X(x)\sin\left(\frac{n2\pi x}{2L}\right)dx$$

you can see by inspection that B_n must be zero. What exactly is in that inspection? Well, you know that $X(x)$ is an even function, since it has the same values at $-x$ as it does at $+x$. You also know that the sine function is odd, since $\sin(-x) = -\sin(x)$, and the product of an even function (like $X(x)$) and an odd function (like the sine function) is odd. But when you integrate an odd function between limits that are symmetric about $x = 0$ (such as $\int_{-L}^{L}$), the result is zero. Hence you know that B_n must equal zero for all values of n. This is one reason why choosing the cycle between $x = -L$ and $x = L$ is advantageous in this case (another reason is that $\int_{-L}^{L}$ (even function)dx $= 2\int_0^L$ (even function)dx, and both $X(x)$ and the cosine function are even, so you could have simplified the calculation of the A_n coefficients as well).

Hint 10: So the Fourier coefficients for the triangle wave shown in Fig. 3.16 are indeed

$$A_0 = \frac{1}{2} \qquad A_n = \frac{-4}{\pi^2 n^2} \qquad B_n = 0$$

as expected from Fig. 3.16. If you'd like more practice at finding Fourier coefficients, you'll find additional problems like this at the end of this chapter, with full solutions on the book's website.

Example 5 (page 114 of text)

[Statement] **Example:** *Find the Fourier transform of a single rectangular distance-domain pulse $X(x)$ with height A over interval $2L$ centered on $x = 0$.*

Hint 1: Since the pulse is a distance-domain function, you can use Eq. 3.33 to transform $X(x)$ to $K(k)$.

Hint 2: Since $X(x)$ has amplitude A between positions $x = -L$ and $x = L$ and zero amplitude at all other times, this becomes

$$\begin{aligned}
K(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} X(x) e^{-ikx} dt = \int_{-L}^{L} A e^{-ikx} dt \\
&= \frac{1}{\sqrt{2\pi}} A \frac{1}{-ik} e^{-ikx}|_{-L}^{L} = \frac{1}{\sqrt{2\pi}} \frac{A}{-ik} \left[e^{-ikL} - e^{-ik(-L)} \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{2A}{k} \left[\frac{e^{-ikL} - e^{ikL}}{-2i} \right] = \frac{1}{\sqrt{2\pi}} \frac{2A}{k} \left[\frac{e^{ikL} - e^{-ikL}}{2i} \right].
\end{aligned}$$

Hint 3: Euler says the term in square brackets is equal to $\sin kL$, so

$$K(k) = \frac{1}{\sqrt{2\pi}} \frac{2A}{k} \sin kL$$

and multiplying by L/L makes this

$$K(k) = \frac{A(2L)}{\sqrt{2\pi}} \left[\frac{\sin kL}{kL} \right].$$

This explains the $\frac{\sin(x)}{x}$ shape of the wavenumber spectrum of the rectangular pulse shown in Fig. 3.22.

[Statement] **Problem 1** (page 124 of text)

[Statement] Show that the expression $C \sin(\omega t + \phi_0)$ is equivalent to $A \cos \omega t + B \sin \omega t$, and write equations for C and ϕ_0 in terms of A and B.

Hint 1: Apply the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

to the expression $C = \sin(\omega t + \phi_0)$.

Hint 2: Set $C = \sin(\omega t + \phi_0) = C \sin \omega t \cos \phi_0 + \cos \omega t \sin \phi_0$ equal to $A \cos \omega t + B \sin \omega t$.

Hint 3: If

$$C \sin \omega t \cos \phi_0 + \cos \omega t \sin \phi_0 = A \cos \omega t + B \sin \omega t$$

then it must be true that

$$A = C \sin \phi_0$$

and

$$B = C \cos \phi_0$$

which are the desired relationships of A and B to C and ϕ_0.

Hint 4: To find ϕ_0 in terms of A and B, divide this expression for A by the expression for B, which gives

$$\frac{A}{B} = \frac{C \sin \phi_0}{C \cos \phi_0} = \tan \phi_0$$

so

$$\arctan \phi_0 = \frac{A}{B}.$$

Hint 5: To find C in terms of A an B, square and add the expressions for A and B

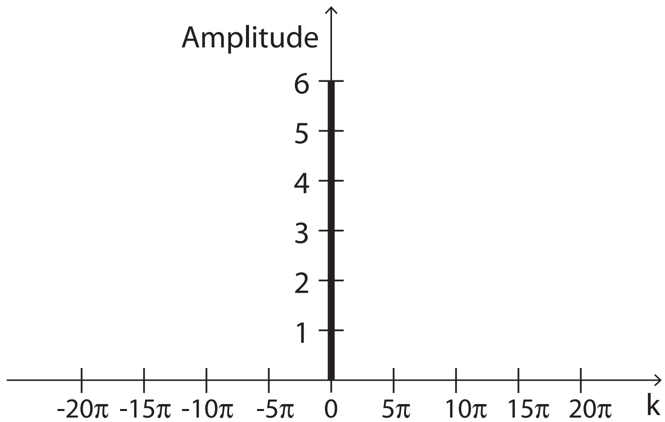
$$\begin{aligned} A^2 + B^2 &= (C \sin \phi_0)^2 + (C \cos \phi_0)^2 \\ &= C^2(\sin^2 \phi_0 + \cos^2 \phi_0) = C^2. \end{aligned}$$

So $C = \sqrt{A^2 + B^2}$.

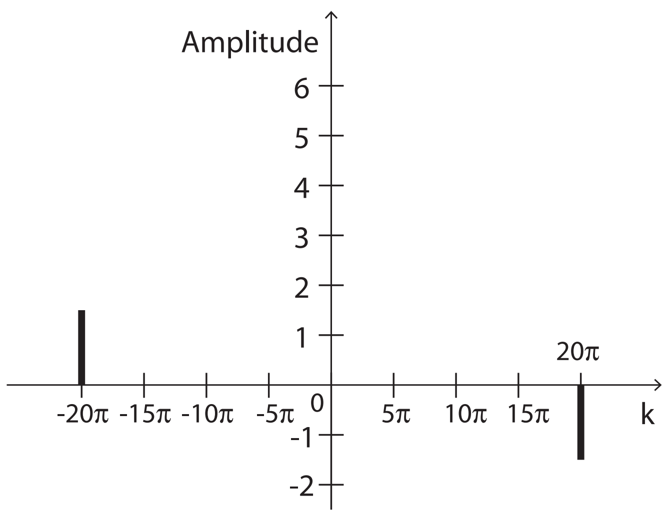
[Statement] **Problem 2** (page 124 of text)

[Statement] Sketch the two-sided wavenumber spectrum of the function $X(x) = 6 + 3\cos(20\pi x - \pi/2) - \sin(5\pi x) + 2\cos(10\pi x + \pi)$.

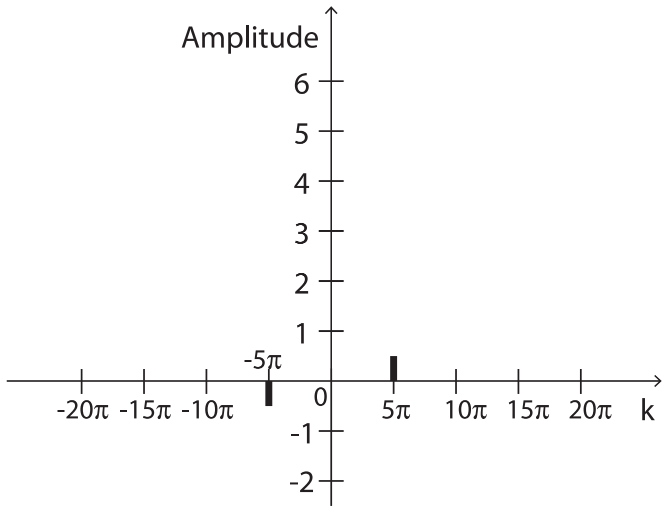
Hint 1: Look at each of the four terms of $X(x)$, and notice that the first term has no x-dependence. So it's the $k = 0$ term, and it has amplitude of 6. So its contribution to the spectrum of $X(x)$ looks like this:



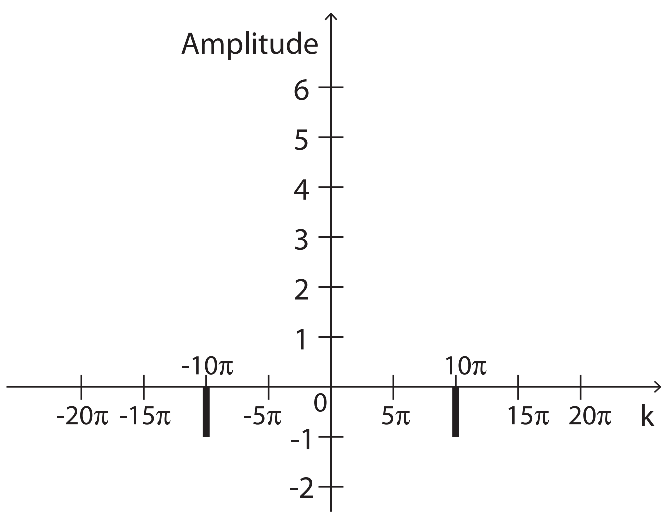
Hint 2: The second term $3\cos(20\pi x - \pi/2)$ is equivalent to $3\sin(20\pi x)$, since $\cos(\theta - \pi/2) = \sin\theta$. This term has wavenumber $k = 20\pi$ and amplitude of 3, so its positive and negative wavenumber components each have height of 1.5:



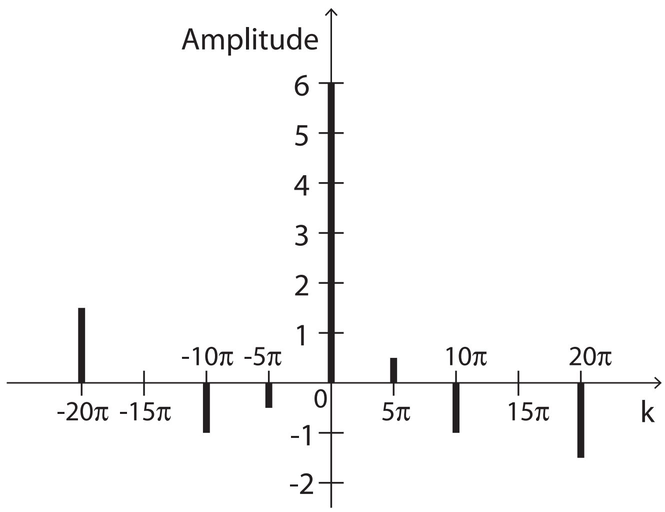
Hint 3: The third term $-\sin(5\pi x)$ is equivalent to $3\sin(20\pi x)$, has wavenumber $k = 5\pi$ and amplitude of -1, so its positive and negative wavenumber components each have height of 0.5:



Hint 4: The fourth term $2\cos(10\pi x + \pi)$ is equivalent to $-2\cos(10\pi x)$, since $\cos(\theta + \pi) = -\cos\theta$. This term has wavenumber $k = 10\pi$ and amplitude of -2, so its positive and negative wavenumber components each have height of 1.0:



Hint 5: Combining the contributions of all four terms, the spectrum of $X(x)$ is:



[Statement] **Problem 3** (page 124 of text)

[Statement] Find the Fourier series representation of a periodic function for which one period is given by $f(x) = x^2$ for x between $-L$ and $+L$.

Hint 1:To find the DC term A_0, use $A_0 = \frac{1}{2L}\int_{-L}^{L} X(x)dx$ with $X(x) = x^2$ between the limits of $-L$ and L.

Hint 2: Integrating gives

$$\begin{aligned}
A_0 &= \frac{1}{2L}\int_{-L}^{L} X(x)dx = \frac{1}{2L}\int_{-L}^{L} x^2 dx \\
&= \frac{1}{2L}\left(\frac{1}{3}x^3|_{-L}^{L}\right) = \frac{1}{2L}\left(\frac{1}{3}L^3 - \frac{1}{3}(-L)^3\right) \\
&= \frac{1}{2L}\left(\frac{2}{3}L^3\right) = \frac{L^2}{3}.
\end{aligned}$$

Hint 1:To find the cosine coefficients A_n, use $A_n = \frac{1}{L}\int_{-L}^{L} X(x)\cos\left(\frac{n2\pi x}{2L}\right)dx$ with $X(x) = x^2$ between the limits of $-L$ and L:

$$A_n = \frac{1}{L}\int_{-L}^{L} X(x)\cos\left(\frac{n2\pi x}{2L}\right)dx = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n2\pi x}{2L}\right)dx$$

Hint 2: This can be integrated with the help of

$$\int x^2\cos(ax)dx = \frac{2x}{a^2}\cos(ax) + \left(\frac{x^2}{a} - \frac{2}{a^3}\right)\sin(ax)$$

in which a represents $n\pi/L$ in this case.

Hint 3: Integrating gives

$$\begin{aligned} A_n &= \frac{1}{L}\left\{\frac{2L^2}{n^2\pi^2}x\cos\left(\frac{n\pi x}{L}\right) + \left[\frac{x^2L}{n\pi} - \frac{2L^3}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{L}\right)\right\}\Bigg|_{-L}^{L} \\ &= \frac{1}{L}\frac{2L^2}{n^2\pi^2}\left[(L)\cos\left(\frac{n\pi L}{L}\right) - (-L)\cos\left(\frac{n\pi(-L)}{L}\right)\right] + 0 \\ &= \frac{1}{L}\frac{2L^2}{n^2\pi^2}\left[2\cos\left(\frac{n\pi L}{L}\right)\right] = \frac{4L^2}{n^2\pi^2}\cos(n\pi) \end{aligned}$$

Hint 4: Since the function x^2 is even (because $(-x)^2 = (+x)^2$), the sine coefficients (B_n) must be zero for all n.

[Statement] **Problem 4** (page 124 of text)

[Statement] Verify the coefficients A_0, A_n, and B_n for the periodic triangle wave shown in Fig. 3.15.

Hint 1: Before you can use Eqs. 3.30 to find A_0, A_n, and B_n, it's necessary to write a function $X(x)$ that describes the waveform over one cycle.

Hint 2: Consider one cycle of the waveform shown in Fig. 3.15 between $x = -L$ and $x = L$, which consists of straight lines, each of which may be written as an equation of the form $y = mx + b$, where m is the slope of the line and b is the line's y-intercept.

Hint 3: The first segment of this cycle, between $x = -L$ and $x = -L/2$, has slope $m = -2/L$, as you can see by taking the rise (-1) divided by the run (L/2). The y-intercept for this segment is -2. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2}{L}x + (-2) = -2\left(\frac{x}{L} + 1\right).$$

Hint 4: The second segment of this cycle, between $x = -L/2$ and $x = +L/2$, has slope $m = 2/L$, which is the rise of 2 divided by the run of L; in this case the y-intercept is 0. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{2}{L}x + 0 = \frac{2}{L}x.$$

Hint 5: The third segment of this cycle, between $x = L/2$ and $x = L$, has the same slope as the first segment ($m = -2/L$), since the rise is again -1 and the run is L/2. But in this case the y-intercept is $+2$, so for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2}{L}x + (+2) = -2\left(\frac{x}{L} - 1\right).$$

Hint 6: With the equations for $X(x)$ for each of the segments in hand, you can now integrate $X(x)$ over the appropriate limits to find the Fourier coefficients. For the DC ($k = 0$) term, you can find A_0 by doing the integration $A_0 = \frac{1}{2L}\int_{-L}^{L} X(x)dx$, but you can save time by observing that the area under the $X(x)$ function over one cycle is zero, since there's exactly as much area under the horizontal axis as there is above the axis. Thus $A_0 = 0$ for this waveform.

Hint 7: For the Fourier cosine coefficients, you can find A_n by doing the integration $A_n = \frac{1}{L}\int_{-L}^{L} X(x)\cos\left(\frac{n2\pi x}{2L}\right)dx$, but once again you can save time by examining the function $X(x)$ before you integrate. In this case, notice that the function $X(x)$ is odd, so the integral between symmetric limits ($-L$ and L in this case) of the product of this function with the even cosine function must give zero (since multiplying an odd function by an even function gives an odd function, and the integral of an odd function between symmetric limits is zero). Thus $A_n = 0$ for this waveform.

Hint 8: For the Fourier sine coefficients, you can find B_n by doing the integration $B_n = \frac{1}{L}\int_{-L}^{L} X(x)\sin\left(\frac{n2\pi x}{2L}\right)dx$. Using the equations for each of the three segments of $X(x)$ (between $-L$ and $-L/2$, between $-L/2$ and $L/2$, and between $L/2$ and L) makes three integrals.

$$B_n = \frac{1}{L}\int_{-L}^{-L/2} -2\left(\frac{x}{L}+1\right)\sin\left(\frac{n2\pi x}{2L}\right)dx + \frac{1}{L}\int_{-L/2}^{L/2} \frac{2}{L}x\sin\left(\frac{n2\pi x}{2L}\right)dx$$
$$+ \frac{1}{L}\int_{L/2}^{L} -2\left(\frac{x}{L}-1\right)\sin\left(\frac{n2\pi x}{2L}\right)dx$$

Hint 9: The first of these integrals expands into two integrals:

$$\text{First Segment} = \frac{-2}{L}\left[\int_{-L}^{-L/2}\left(\frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx + \int_{-L}^{-L/2}(1)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$
$$= \frac{-2}{L^2}\left\{\frac{L^2\sin\left(\frac{n\pi x}{L}\right)}{n^2\pi^2} - \frac{Lx\cos\left(\frac{n\pi x}{L}\right)}{n\pi} - \frac{L^2\cos\left(\frac{n\pi x}{L}\right)}{n\pi}\right\}\Bigg|_{-L}^{-L/2}$$

in which the relations $\int x\sin(ax)dx = \frac{\sin(ax)}{a^2} - \frac{x\cos(ax)}{a}$ and $\int \sin(ax)dx = -\frac{\cos(ax)}{a}$ were used, with a representing $\frac{n\pi}{L}$ in this case.

Plugging in the limits gives these terms:

$$\text{First Segment} = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - 0 - \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{2\cos(n\pi)}{n\pi} + \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{2\cos(n\pi)}{n\pi}$$
$$= \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$

Hint 10: The integral of the second segment (between $-L/2$ and $L/2$) gives:

$$\text{Second Segment} = \frac{2}{L^2}\left\{\frac{L^2\sin\left(\frac{n\pi x}{L}\right)}{n^2\pi^2} - \frac{Lx\cos\left(\frac{n\pi x}{L}\right)}{n\pi}\right\}\Bigg|_{-L/2}^{L/2}$$
$$= \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi}.$$

Hint 11: Like the first segment, the integral of the third segment (between $L/2$ and L) expands into two integrals:

$$\text{Third Segment} = \frac{-2}{L}\left[\int_{L/2}^{L}\left(\frac{x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx - \int_{L/2}^{L}(1)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$
$$= \frac{-2}{L^2}\left\{\frac{L^2\sin\left(\frac{n\pi x}{L}\right)}{n^2\pi^2} - \frac{Lx\cos\left(\frac{n\pi x}{L}\right)}{n\pi} + \frac{L^2\cos\left(\frac{n\pi x}{L}\right)}{n\pi}\right\}\Bigg|_{L/2}^{L}$$

Plugging in the limits gives these:

$$\text{Third Segment} = 0 + \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} + \frac{\cos(n\pi)}{n\pi} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{2\cos(n\pi)}{n\pi} + \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$
$$= \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$

Hint 12: Adding the terms from all three segments gives

$$B_n = \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{2\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \frac{2\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi}$$
$$= \frac{8\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}.$$

[Statement] **Problem 5** (page 124 of text)

[Statement] If a string (fixed at both ends) is plucked rather than struck (non-zero initial displacement, zero initial velocity), show that the displacement at position x and time t is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right).$$

Hint 1: For a plucked string, the initial displacement of each segment of the string is specified by some function of x, and the initial velocity of each segment is zero.

Hint 2: As in the case of the struck string discussed in Section 3.2, begin by writing the displacement as the product of a spatial function $X(x)$ and a temporal function $T(t)$:$y(x,t) = X(x)T(t)$.

Hint 3: Write the time function $T(t)$ as

$$T(t) = C\cos(kvt) + D\sin(kvt) = C\cos\left(\frac{2\pi}{\lambda}vt\right) + D\sin\left(\frac{2\pi}{\lambda}vt\right)$$

and take the derivative of $T(t)$ with respect to time:

$$\frac{\partial T(t)}{\partial t} = -\frac{2\pi}{\lambda}vC\sin\left(\frac{2\pi}{\lambda}vt\right) + \frac{2\pi}{\lambda}vD\cos\left(\frac{2\pi}{\lambda}vt\right).$$

Hint 4: Since the initial velocity of each string segment must be zero, set this expression for $\frac{\partial T(t)}{\partial t}$ equal to zero at time $t = 0$:

$$-\frac{2\pi}{\lambda}vC\sin\left(\frac{2\pi}{\lambda}vt\right) + \frac{2\pi}{\lambda}vD\cos\left(\frac{2\pi}{\lambda}vt\right) = 0$$

which, at $t = 0$, gives

$$\frac{2\pi}{\lambda}vD\cos\left(\frac{2\pi}{\lambda}v(0)\right) = 0$$

which means $D = 0$.

Hint 5: Thus

$$T(t) = C\cos(kvt) + D\sin(kvt) = C\cos\left(\frac{2\pi}{\lambda}vt\right) + 0.$$

Hint 6: Using $\lambda = 2L/n$ for a string fixed at both ends, this is

$$T(t) = C\cos\left(\frac{n\pi}{L}vt\right) + 0.$$

Hint 7: This means that the general solution for $T(t)$ is the sum of the cosine terms for each value of n:

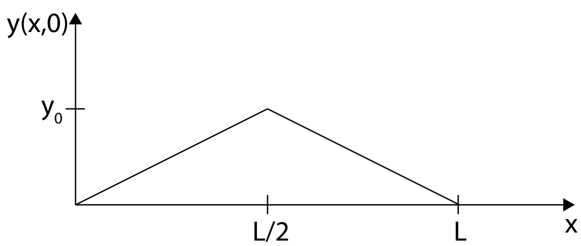
$$T(t) = \sum_{n=1}^{\infty} C_n\cos\left(\frac{n\pi}{L}vt\right).$$

Hint 8: Combining the expressions for $X(x)$ an $T(t)$ and absorbing the C_n weighting coefficients into B_n makes the solution for displacement

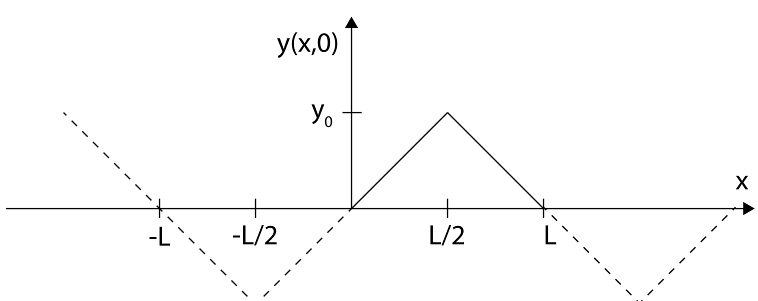
$$y(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right).$$

[Statement] **Problem 6** (page 124 of text)

[Statement] Find the B_n coefficients for the plucked string of the previous problem if the initial displacement is given by the function shown below.



Hint 1: To make the displacement into a periodic function, extend the function as shown in the figure below. This is the odd extension of the function, since $y(-x,t) = -y(x,t)$, which means that the A_n coefficients will be zero and the B_n coefficients will be non-zero. You could choose to form an even extension by making $y(-x,t) = y(x,t)$, in which case the B_n coefficients would be zero and the A_n coefficients (and A_0) would be non-zero.



Hint 2: This is now similar to the wavefunction in Problem 4, with the exception that the height of the wavefunction is y_0 in this case (rather

than one). Thus the Fourier coefficients A_0 and A_n are zero, and you can find the B_n coefficients using the same approach as in Problem 4.

Hint 3: In this case, the first segment of this cycle, between $x = -L$ and $x = -L/2$, has slope $m = -2y_0/L$, as you can see by taking the rise $(-y_0)$ divided by the run $(L/2)$. The y-intercept for this segment is $-2y_0$. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2y_0}{L}x + (-2y_0) = -2y_0\left(\frac{x}{L} + 1\right).$$

Hint 4: The second segment of this cycle, between $x = -L/2$ and $x = +L/2$, has slope $m = 2y_0/L$, which is the rise of $2y_0$ divided by the run of L; in this case the y-intercept is 0. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{2y_0}{L}x + 0 = \frac{2y_0}{L}x.$$

Hint 5: The third segment of this cycle, between $x = L/2$ and $x = L$, has the same slope as the first segment $(m = -2y_0/L)$, since the rise is again $-y_0$ and the run is $L/2$. But in this case the y-intercept is $+2y_0$, so for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2y_0}{L}x + (+2y_0) = -2y_0\left(\frac{x}{L} - 1\right).$$

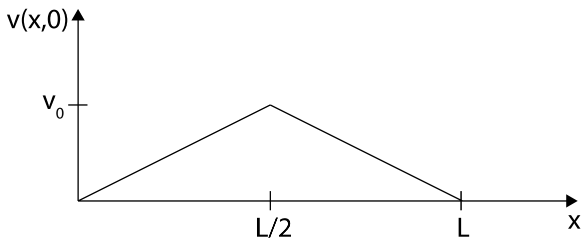
Hint 6: Using these equations the segments of $X(x)$, the integrals for B_n are identical to those of Problem 4 with an additional factor of y_0 in the numerator.

Hint 7: The resulting Fourier sine coefficients differ from those of Problem 4 only by the factor y_0:

$$B_n = \frac{8y_0 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}.$$

[Statement] **Problem 7** (page 124 of text)

[Statement] Find the B_n coefficients for a hammered string with initial displacement of zero and initial velocity given by the function shown below.



Hint 1: As described in Section 3.2, for a hammered or struck string initially at equilibrium, one boundary condition specifies that the initial displacement $y(x,0)$=0 at each location at time $t = 0$, and another boundary condition specifies that the initial transverse velocity $\frac{\partial y(x,t)}{\partial t}$ equals the imparted velocity v_0 at time $t = 0$. The solution for the displacement $y(x,t)$ is

$$y(x,t) = X(x)T(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi vt}{L}\right). \tag{3.1}$$

Hint 2: In this case, the initial velocity function $v(x)$ is given by the graph, and this must equal the time derivative of the displacement $y(x,t)$ at time $t = 0$. Taking the time derivative of $y(x,t)$ gives

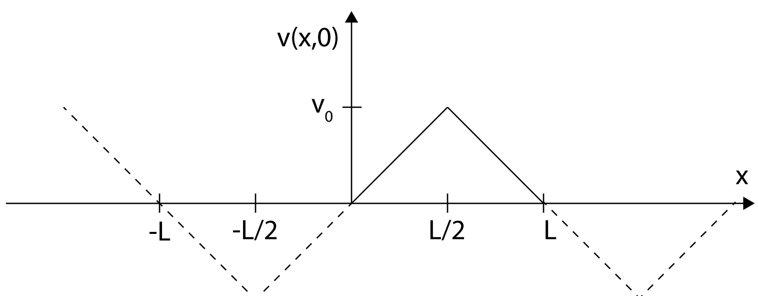
$$\frac{\partial y(x,t)}{\partial t} = X(x)\frac{\partial T(t)}{\partial t} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)\left(\frac{n\pi v}{L}\right).$$

which, at time $t = 0$, is

$$\begin{aligned} v_0 &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \cos(0) \left(\frac{n\pi v}{L}\right) \\ &= \sum_{n=1}^{\infty} B_n \left(\frac{n\pi v}{L}\right) \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Hint 3: Comparing this expression for v_0 to the expression for the spatial component $(X(x))$ of $y(x,t)$ in Problem 5 ($y(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$), you can see that the only difference is the presence of the factor $n\pi v/L$ in the expression for v_0. That means you can apply Fourier analysis to this velocity function in the same way you applied Fourier analysis to the displacement function in Problem 6.

Hint 4: Begin by making the initial velocity into a periodic function by extending the function as shown in the figure below. As in Problem 6, this is the odd extension of the function, since $v(-x,0) = -v(x,0)$, which means that the A_n coefficients will be zero and the B_n coefficients will be non-zero.



Hint 5: This is now similar to the wavefunction in Problem 6, with the exception that the height of the wavefunction is v_0 in this case (rather than y_0). Thus the Fourier coefficients A_0 and A_n are zero, and you can find the B_n coefficients using the same approach as in Problem 6.

Hint 6: In this case, the first segment of this cycle, between $x = -L$ and $x = -L/2$, has slope $m = -2v_0/L$, as you can see by taking the rise $(-v_0)$ divided by the run $(L/2)$. The y-intercept for this segment is $-2v_0$. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2v_0}{L}x + (-2v_0) = -2v_0\left(\frac{x}{L} + 1\right).$$

Hint 7: The second segment of this cycle, between $x = -L/2$ and $x = +L/2$, has slope $m = 2v_0/L$, which is the rise of $2v_0$ divided by the run of L; in this case the y-intercept is 0. Thus for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{2v_0}{L}x + 0 = \frac{2v_0}{L}x.$$

Hint 8: The third segment of this cycle, between $x = L/2$ and $x = L$, has the same slope as the first segment ($m = -2v_0/L$), since the rise is again $-v_0$ and the run is $L/2$. But in this case the y-intercept is $+2v_0$, so for this segment, the equation for $X(x)$ is

$$X(x) = mx + b = \frac{-2v_0}{L}x + (+2v_0) = -2y_0\left(\frac{x}{L} - 1\right).$$

Hint 9: Using these equations the segments of $X(x)$, the integrals for B_n are identical to those of Problem 6 with v_0 in place of y_0 in the numerator.

Hint 10: The resulting Fourier sine coefficients differ from those of Problem 6 only by the factor v_0:

$$B_n = \frac{8v_0 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}.$$

Hint 11: Since these are the Fourier sine coefficients for the velocity function, the displacement-function Fourier coefficients can be found from

$$B_n(\text{velocity}) = \frac{8v_0 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} = B_n(\text{displacement})\left(\frac{n\pi v}{L}\right)$$

or

$$\begin{aligned} B_n(\text{displacement}) &= \frac{B_n(\text{velocity})}{\left(\frac{n\pi v}{L}\right)} = \frac{8v_0 \sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2}\left(\frac{L}{n\pi v}\right) \\ &= \frac{8v_0 L \sin\left(\frac{n\pi}{2}\right)}{n^3\pi^3 v}. \end{aligned}$$

[Statement] **Problem 8** (page 124 of text)

[Statement] Find the Fourier Transform of the Gaussian function $T(t) = \sqrt{\frac{\alpha}{\pi}}e^{-\alpha t^2}$.

Hint 1: You can use Eq. 3.35 to find the Fourier Transform of a time-domain function:

$$F(f) = \int_{-\infty}^{\infty} T(t)e^{-i\left(2\pi\frac{t}{T}\right)}dt = \int_{-\infty}^{\infty} T(t)e^{-i(2\pi ft)}dt.$$

Hint 2: In this case, $T(t) = \sqrt{\frac{\alpha}{\pi}}e^{-\alpha t^2}$, so

$$F(f) = \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{\pi}}e^{-\alpha t^2}e^{-i(2\pi ft)}dt = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} e^{-\alpha t^2}e^{-i(2\pi ft)}dt$$

Hint 3: You can integrate this expression by parts or use an integral table; here's a helpful identity:

$$\int_{-\infty}^{\infty} e^{-ax^2}e^{-2bx}dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{a}} \qquad a > 0$$

Hint 4: Applying this with $x = t$, $a = \alpha$ and $b = i\pi f$ gives

$$\begin{aligned} F(f) &= \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{\pi}{\alpha}}e^{\frac{(-i\pi f)^2}{\alpha}} \\ &= e^{-\frac{\pi^2}{\alpha}f^2} \end{aligned}$$

which is also a Gaussian function. Thus the Fourier Transform of a Gaussian function in one domain is a Gaussian function in the transform domain.

[Statement] **Problem 9** (page 124 of text)

[Statement] Show that the complex-exponential version of the Fourier series (Eq. 3.31) is equivalent to the version using sines and cosines (Eq. 3.25).

Hint 1: To show that the complex-exponential version of the Fourier Series

$$X(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n2\pi x}{2L}\right)}.$$

is equivalent to the version using sines and cosines

$$X(x) = A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\frac{n2\pi x}{2L} + B_n \sin\frac{n2\pi x}{2L}\right].$$

begin by expanding the sine and cosine terms using the Euler relations:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Hint 2: Using these Euler relations with $\theta = \frac{n2\pi x}{2L}$ makes $X(x)$

$$\begin{aligned} X(x) &= A_0 + \sum_{n=1}^{\infty}\left[A_n \frac{e^{i\frac{n2\pi x}{2L}} + e^{-i\frac{n2\pi x}{2L}}}{2} + B_n \frac{e^{i\frac{n2\pi x}{2L}} - e^{-i\frac{n2\pi x}{2L}}}{2i}\right] \\ &= A_0 + \sum_{n=1}^{\infty}\left[A_n \frac{e^{i\frac{n2\pi x}{2L}} + e^{-i\frac{n2\pi x}{2L}}}{2} - iB_n \frac{e^{i\frac{n2\pi x}{2L}} - e^{-i\frac{n2\pi x}{2L}}}{2}\right] \end{aligned}$$

since $\frac{1}{i} = -i$.

Hint 3: Gathering terms gives

$$X(x) = A_0 + \sum_{n=1}^{\infty}\left[\frac{A_n - iB_n}{2}\right]\left[e^{i\frac{n2\pi x}{2L}}\right] + \sum_{n=1}^{\infty}\left[\frac{A_n + iB_n}{2}\right]\left[e^{-i\frac{n2\pi x}{2L}}\right]$$

Hint 4: Setting these two terms equal to the complex-exponential version of the Fourier series gives

$$\begin{aligned} X(x) &= A_0 + \sum_{n=1}^{\infty}\left[\frac{A_n - iB_n}{2}\right]\left[e^{i\frac{n2\pi x}{2L}}\right] + \sum_{n=1}^{\infty}\left[\frac{A_n + iB_n}{2}\right]\left[e^{-i\frac{n2\pi x}{2L}}\right] \\ &= \sum_{n=-\infty}^{\infty} C_n e^{i\left(\frac{n2\pi x}{2L}\right)} \end{aligned}$$

which is true if $C_n = (A_n - iB_n)/2$ for positive n and $C_n = (A_n + iB_n)/2$ for negative n.

[Statement] **Problem 10** (page 124 of text)

[Statement] Under certain conditions, the dispersion relation for deep-water waves is $\omega = \sqrt{gk}$, in which g is the acceleration of gravity. Compare the group velocity to the phase velocity for this type of wave.

Hint 1: You can find the group velocity using $v_{group} = \frac{d\omega}{dk}$ and the phase velocity using $v_{phase} = \frac{\omega}{k}$.

Hint 2: In this case $\omega = \sqrt{gk}$, so the group velocity is

$$v_{group} = \frac{d\omega}{dk} = \frac{d(gk)^{\frac{1}{2}}}{dk} = \frac{1}{2}(gk)^{-\frac{1}{2}}(g) = \frac{1}{2}\frac{g}{\sqrt{gk}} = \frac{1}{2}\sqrt{\frac{g}{k}}$$

Hint 3: For $\omega = \sqrt{gk}$ the phase velocity is

$$v_{phase} = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}$$

which is twice the group velocity.

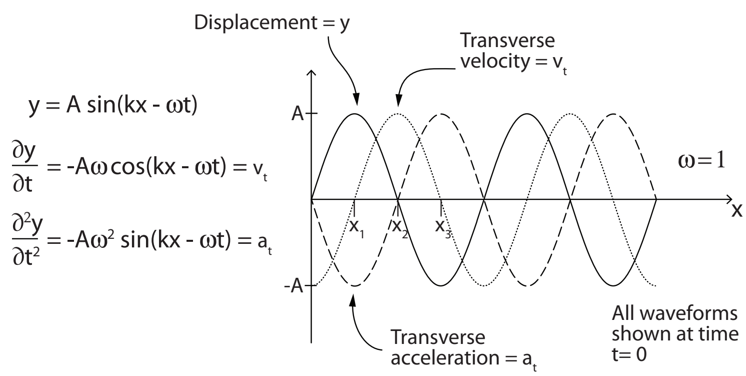
4
The Mechanical Wave Equation click-throughs

Example 1 (page 133 of text)

[Statement] **Example:** *Compare the displacement, velocity, and acceleration for a transverse harmonic wave on a string.*

Hint 1: If the displacement $y(x,t)$ is given by $A\sin(kx-\omega t)$, the transverse velocity of any segment of the string is given by $v_t = \frac{\partial y}{\partial t} = -A\omega\cos(kx-\omega t)$.

Hint 2: The transverse acceleration is $a_t = \frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(kx-\omega t)$. Note that this wave is moving in the positive x-direction, since the sign of the kx term is opposite to the sign of the ωt term.



Hint 3: Plotting the displacement, transverse velocity, and transverse

acceleration on the same graph, as in figure shown above, reveals some interesting aspects of wave behavior. This figure is a snapshot of y, v_t, and a_t at time t=0, and the angular frequency has been taken as $\omega = 1$ in order to scale all three waveforms to the same vertical size.

Example 2 (page 141 of text)

[Statement] **Example:** *Determine the speed of sound in air.*

Hint 1: Sound is a type of pressure wave, so you can use Eq. 4.12 to determine the speed of sound in air, if you know the values of the bulk modulus and density of air. Since the air pressure may be more readily available than bulk modulus in the region of interest, you may find it helpful to write this equation in a form that explicitly includes pressure.

Hint 2: To do that, use the definition of bulk modulus (Eq. 4.7) to write Eq. 4.12 as

$$v = \sqrt{\frac{K}{\rho_0}} = \sqrt{\frac{\frac{dP}{\frac{d\rho}{\rho_0}}}{\rho_0}} = \sqrt{\frac{dP}{d\rho}}. \tag{4.13}$$

Hint 3: The quantity $\frac{dP}{d\rho}$ can be related to the equilibrium pressure (P_0) and density (ρ_0) using the adiabatic gas law. Since an adiabatic process is one in which energy does not flow between a system and its environment by heat, using the adiabatic law means that we're assuming that the regions of compression and rarefaction produced by the sound wave will not lose or gain energy by heating as the wave oscillates. That's a good assumption for sound waves in air under typical conditions, because the flow of energy by conduction (molecules colliding and transferring kinetic energy) occurs over distances comparable to the mean free path (the average distance molecules travel between collisions). That distance is several orders of magnitude smaller than the distance between regions of compression and rarefaction (that is, half a wavelength) in sound waves. So the squeezing and stretching of the air by the wave produces regions of slightly higher and slightly lower temperature, and the molecules do not move far enough to restore equilibrium before the wave causes the compressed regions to rarefy and the rarified regions to compress. Thus the wave action may indeed be considered to be an adiabatic process[1].

[1] When Newton first calculated the speed of sound in his great *Principia*, he

Hint 4: To apply the adiabatic gas law, write the relationship between pressures (P) and volume (V) as

$$PV^{\gamma} = \text{constant} \tag{4.14}$$

in which γ represents the ratio of specific heats at constant pressure and constant volume and has a value of approximately 1.4 for air under typical conditions.

Hint 5: Since volume is inversely proportional to density ρ, Eq. 4.14 can be written as

$$P = (\text{constant})\rho^{\gamma}$$

so

$$\frac{dP}{d\rho} = (\text{constant})\gamma\rho^{\gamma-1} = \gamma\frac{(\text{constant})\rho^{\gamma}}{\rho}.$$

Hint 6: But $(\text{constant})\rho^{\gamma} = P$, so this is

$$\frac{dP}{d\rho} = \gamma\frac{P}{\rho}.$$

Hint 7: Inserting this into Eq. 4.13 gives

$$v = \sqrt{\gamma\frac{P}{\rho}}.$$

For typical values of air of $P = 1\times10^5$ Pa and $\rho = 1.2$ kg/m^3, this yields a value for the speed of sound of

$$v = \sqrt{1.4\frac{1\times10^5}{1.2}} = 342 \text{ m/s}$$

which is very close to the measured value.

Example 3 (page 147 of text)

[Statement] **Example:** *What are the kinetic, potential, and total mechanical energy of a segment of string of length dx with wavefunction $y(x,t) = A\sin(kx-\omega t)$?*

instead used a constant-temperature law (Boyle's Law), which caused him to underestimate the speed of sound by about 15%.

Hint 1: For this wavefunction, the transverse velocity is $v_t = \frac{\partial y}{\partial t} = -A\omega \cos(kx - \omega t)$, so by Eq. 4.15 the kinetic energy (KE) is

$$\text{KE}_{\text{segment}} = \frac{1}{2}(\mu dx)\left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)dx \tag{4.20}$$

Hint 2: The slope of the wavefunction is $\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t)$, so by Eq. 4.16 the potential energy (PE) is

$$\text{PE}_{\text{segment}} = T\left[\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx\right] = T\left[\frac{1}{2}A^2k^2\cos^2(kx - \omega t)dx\right].$$

Hint 3: The tension (T) can be eliminated from this equation using the relationships $v_{phase} = \sqrt{\frac{T}{\mu}}$ and $v_{phase} = \frac{\omega}{k}$, which can be combined to give $T = \mu\frac{\omega^2}{k^2}$. Thus

$$\text{PE}_{\text{segment}} = \left(\mu\frac{\omega^2}{k^2}\right)\frac{1}{2}A^2k^2\cos^2(kx - \omega t)dx$$

or

$$\text{PE}_{\text{segment}} = \frac{1}{2}\mu A^2\omega^2\cos^2(kx - \omega t)dx. \tag{4.21}$$

Hint 4: If you compare Eq. 4.21 with Eq. 4.20, you'll see that the segment's kinetic and potential energies are identical. Adding these expressions together gives the total energy density:

$$\text{ME}_{\text{segment}} = \mu A^2\omega^2\cos^2(kx - \omega t)dx \tag{4.22}$$

Example 4 (page 150 of text)

[Statement] **Example:** *Find the power in a transverse mechanical wave with wavefunction $y(x,t) = A\sin kx - \omega t$.*

Hint 1: As discussed earlier in this section, for this type of harmonic wave the transverse velocity is $v_{phase} = \sqrt{\frac{T}{\mu}}$, which can be combined with the expression $v_{phase} = \frac{\omega}{k}$ to give $T = \mu\frac{\omega^2}{k^2}$. Thus

$$\begin{aligned} P = (\sqrt{\mu T})v_t^2 &= \left[\sqrt{\mu\left(\mu\frac{\omega^2}{k^2}\right)}\right]v_t^2 \\ &= \mu\frac{\omega}{k}v_t^2. \end{aligned}$$

Hint 2: But for this wave $v_t = -\omega A \cos(kx - \omega t)$, so

$$\begin{aligned} P &= \mu\frac{\omega}{k}[-\omega A \cos(kx-\omega t)]^2 \\ &= \mu\frac{\omega^3}{k}[A^2\cos^2(kx-\omega t)]. \end{aligned}$$

Hint 3: To find the average power, recall that the average value of $\cos^2$ over many cycles is 1/2, so the average power in a harmonic wave of amplitude A is

$$P_{avg} = \mu\frac{\omega^3}{k}\left[A^2\left(\frac{1}{2}\right)\right] = \frac{1}{2}\mu A^2\omega^2\left(\frac{\omega}{k}\right)$$

or

$$P_{avg} = \frac{1}{2}\mu A^2\omega^2 v_{phase} = \frac{1}{2}ZA^2\omega^2. \tag{4.24}$$

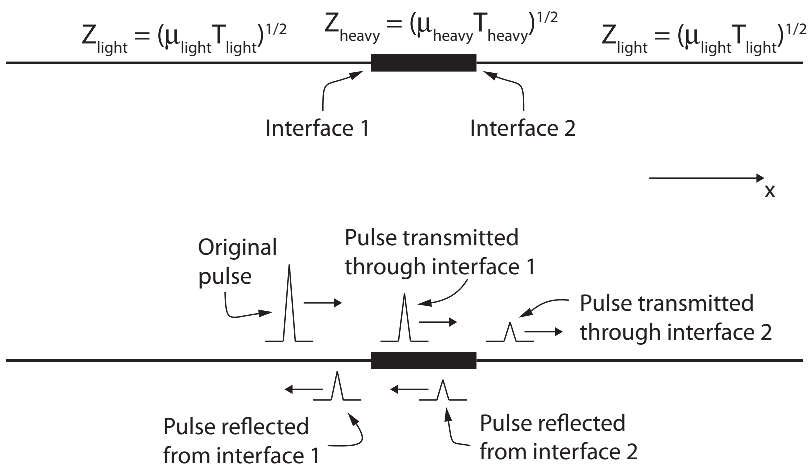
We've written this in several forms to emphasize the relationships between the string parameters of linear mass density μ, phase velocity v_{phase}, and impedance Z (which is $\sqrt{\mu T} = \sqrt{\mu^2\frac{\omega^2}{k^2}} = \mu v_{phase}$).

Example 5 (page 159 of text)

[Statement] **Example:** *Consider a transverse pulse with maximum displacement of 2 cm propagating in the positive-x direction on a string with mass density of 0.15 g/cm and tension of 10N. What happens if the pulse enounters a short section of string with twice the mass density and the same tension?*

Hint 1: A sketch of this situation is shown in the figure below. As you can see in the top portion of the figure, there are two interfaces between the two strings. At the first (left) interface, a pulse traveling in the positive-x direction will be going from a medium with impedance Z_{light} into a medium with impedance Z_{heavy}. So for the rightward-moving pulse at the left interface, $Z_1 = Z_{light}$ and $Z_2 = Z_{heavy}$.

As you can see in the lower portion of the figure, some fraction of the pulse will be reflected (leftward) from the left interface, and the remainder of the pulse will be transmitted (rightward) through the first interface. After propagating through the heavy section of string, that transmitted pulse will encounter the second (right) interface. At that interface it will be going from a medium with impedance Z_{heavy} into a medium with impedance Z_{light}. So for a rightward moving pulse at the right interface $Z_1 = Z_{heavy}$ and $Z_2 = Z_{light}$. As happened at the left



interface, some portion of the pulse will be reflected (leftward) from the second interface, and another portion will be transmitted (rightward) through that interface.

Hint 2: To determine the amplitude of the pulse transmitted through each interface, you can use Eq. 4.28 with the appropriate values of impedances Z_1 and Z_2.

Hint 3: You can use Eq. 4.26 to find the impedances after converting the linear mass density to SI units (0.15 g/cm = 0.015 kg/m):

$$Z_1 = \sqrt{\mu_{light} T_{light}} = \sqrt{(0.015 kg/m)(10N)} = 0.387 \text{ kg/s}$$
$$Z_2 = \sqrt{\mu_{heavy} T_{heavy}} = \sqrt{2(0.015 kg/m)(10N)} = 0.548 \text{ kg/s}.$$

So the transmission coefficient at the left interface is

$$t = \frac{2Z_1}{Z_1 + Z_2} = \frac{(2)(0.387)}{0.387 + 0.548} = 0.83.$$

Hint 4: Thus in propagating from the light string to the heavy string, the amplitude of the pulse is reduced to 83% of its original value. That reduced-amplitude pulse propagates rightward and is further reduced in amplitude at the second (right) interface. In that case, the heavy string is the medium in which the incident wave propagates, and the light string is the medium of the transmitted wave. Since Z_1 refers to the medium in which the incoming and reflected waves propagate and Z_2 refers to the medium in which the transmitted wave propagates, for this interface the impedances are $Z_1 = 0.548$ kg/s and $Z_2 = 0.387$ kg/s. This makes the transmission coefficient at the right interface

$$t = \frac{2Z_1}{Z_1 + Z_2} = \frac{(2)(0.548)}{0.548 + 0.387} = 1.2$$

which means that in propagating past both interfaces, the amplitude of the pulse is reduced by a factor of 0.83 times 1.2, so the final amplitude is about 97% of its original value of 2 cm.

Problem 1 (page 163 of text)

[Statement] Show that the expression $\sqrt{\frac{T}{\mu}}$ in Eq. 4.6 has dimensions of velocity.

Hint 1: The units of tension are N and the units of linear density are kg/m.

Hint 2: 1 N = kg m/s^2 (which you can determine from $F = ma$, if you don't have it memorized).

Hint 3: So, together the units are

$$\begin{aligned}\sqrt{\frac{\mathrm{N}}{\mathrm{kg/m}}} &= \sqrt{\frac{\mathrm{kg\ m}}{\mathrm{s}^2} \cdot \frac{\mathrm{m}}{\mathrm{kg}}} \\ &= \sqrt{\frac{\mathrm{m}^2}{\mathrm{s}^2}} \\ &= \frac{\mathrm{m}}{\mathrm{s}}\end{aligned}$$

which are the units of velocity.

Problem 2 (page 163 of text)

[Statement] If a string with length of 2 meters and mass of 1 gram is tensioned with a hanging mass of 1 kg, what is the phase velocity of transverse waves on the string?

Hint 1: To find the phase velocity, you need to know the tension and the linear density, since $v = \sqrt{\frac{T}{\mu}}$.

Hint 2: The tension is being applied by a hanging mass, so its value is the weight of the mass: $T = mg = (1 \text{ kg})(9.8 \text{ m/s}^2) = 9.8$ N. The linear density is the mass of the string divided by its length: $\mu = M/L = 1 \times 10^{-3}$ kg/2 m $= 0.5 \times 10^{-3}$ kg/m.

Hint 3: The phase velocity is

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{9.8}{0.5 \times 10^{-3}}} = 99 \text{ m/s}.$$

Problem 3 (page 163 of text)

[Statement] Show that the expression $\sqrt{\frac{K}{\rho}}$ in Eq. 4.12 has dimensions of velocity.

Hint 1: The bulk modulus K has units of Pascals, which is the same as N/m^2 (you can figure this out from pressure = force/area, if you don't have it memorized). The units of ρ, a volume density, are kg/m^3.

Hint 2: The units are

$$\begin{aligned}\sqrt{\frac{\text{N/m}^2}{\text{kg/m}^3}} &= \sqrt{\frac{\text{kg m/s}^2/\text{m}^2}{\text{kg/m}^3}} \\ &= \sqrt{\frac{\text{kg m}}{\text{s}^2\text{m}^2} \cdot \frac{\text{m}^3}{\text{kg}}} \\ &= \sqrt{\frac{\text{m}^2}{\text{s}^2}} \\ &= \frac{\text{m}}{\text{s}},\end{aligned}$$

the units of velocity.

Problem 4 (page 163 of text)

[Statement] What is the phase speed of pressure waves in an 8 m^3 steel cube, which has a bulk modulus of approximately 150 GPa and mass of 63,200 kg?

Hint 1: Using Eq. 4.12, the velocity is

$$v = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{150 \times 10^9 \text{ Pa}}{63200 \text{ kg}/8 \text{ m}^3}} = 545 \text{ m/s},$$

where the volume density is the total mass divided by its volume.

Hint 2: You can compare this velocity to the speed of waves in air, which is 342 m/s: 545/342 = 1.59. The speed of sound in steel is roughly one and a half times faster than the speed of sound in air.

Problem 5 (page 163 of text)

[Statement] A transverse harmonic wave with amplitude of 5 cm and wavelength of 30 cm propagates on a string of length 70 cm and mass of 0.1 gram. If the string is tensioned by a hanging mass of 0.3 kg, what are the kinetic, potential and total mechanical energy densities of the wave?

Hint 1: The energy densities KE_{segment}, PE_{segment}, and ME_{segment} are given in Eqs. 4.20, 4.21, and 4.22 respectively:

$$\begin{aligned}
\text{KE}_{\text{segment}} &= \frac{1}{2}\mu A^2\omega^2 \cos^2(kx - \omega t)dx \\
\text{PE}_{\text{segment}} &= \frac{1}{2}\mu A^2\omega^2 \cos^2(kx - \omega t)dx \\
\text{ME}_{\text{segment}} &= \mu A^2\omega^2 \cos^2(kx - \omega t)dx.
\end{aligned}$$

You know A (the amplitude), λ (the wavelength), M (the mass of the string), L (the length of the string), and m (the hanging mass), so you still need μ (the linear density), ω (the angular frequency), and k (the wavenumber).

Hint 2: The wavenumber k can be found from

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.3 \text{ m}} = 20.9 \text{ m}^{-1},$$

remembering to convert the wavelength into meters. The linear density μ is found from

$$\mu = \frac{M}{L} = \frac{0.1 \times 10^{-3} \text{ kg}}{0.7 \text{ m}} = 1.42 \times 10^{-4} \text{ kg/m}.$$

Hint 3: To find ω, you can use the phase velocity equations $v = \sqrt{T/\mu}$

and $v = \omega/k$, which combine to give

$$\omega = k\sqrt{\frac{T}{\mu}} = (20.9\ \text{m}^{-1})\sqrt{\frac{2.94\ \text{N}}{1.42\times 10^{-4}\ \text{kg/m}}} = 3000\ \text{rad/s}$$

and

$$\omega^2 = k^2\frac{T}{\mu} = 9\times 10^6\ \text{rad}^2/\text{s}^2,$$

where the tension T is created by the weight of the hanging mass, so $T = mg =$(0.3 kg) (9.8 m/s^2) = 2.94 N.

Hint 4: Plugging in these values gives

$$\begin{aligned}
\text{KE}_{\text{segment}} &= \frac{1}{2}(1.42\times 10^{-4}\ \text{kg/m})(0.05\ \text{m})^2(9\times 10^6\ \text{rad}^2/\text{s}^2)\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx\\
&= 1.61\ \text{kg m/s}^2\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx\\
\text{PE}_{\text{segment}} &= \frac{1}{2}(1.42\times 10^{-4}\ \text{kg/m})(0.05\ \text{m})^2(9\times 10^6\ \text{rad}^2/\text{s}^2)\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx\\
&= 1.61\ \text{kg m/s}^2\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx\\
\text{ME}_{\text{segment}} &= (1.42\times 10^{-4}\ \text{kg/m})(0.05\ \text{m})^2(9\times 10^6\ \text{rad}^2/\text{s}^2)\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx\\
&= 3.22\ \text{kg m/s}^2\cos^2\left[(20.9\ \text{m}^{-1})x - (3000\ \text{rad/s})t\right]dx
\end{aligned}$$

The units kg m/s^2 are the equivalent of N = (N m)/m = J/m, an energy density.

Problem 6 (page 163 of text)

[Statement] How much power is carried by the wave in the previous problem, and what is the maximum transverse velocity of the string?

Hint 1: According to Eq. 4.24, the average power of a harmonic wave is

$$P_{avg} = \frac{1}{2}ZA^2\omega^2.$$

Hint 2: The impedance Z is equal to $\sqrt{\mu T}$, which makes the average power

$$\begin{aligned}
P_{avg} &= \frac{1}{2}\sqrt{\mu T}A^2\omega^2\\
&= \frac{1}{2}\sqrt{(1.42\times 10^{-4}\ \text{kg/m})(2.94\ \text{N})}(0.05\ \text{m})^2(9\times 10^6\ \text{rad}^2/\text{s}^2)\\
&= 230\ \text{J/s}.
\end{aligned}$$

Hint 3: To find the transverse velocity, use

$$P = (\sqrt{\mu T})v_t^2.$$

Solving for v_t gives

$$\begin{aligned} v_t &= \sqrt{\frac{P_{avg}}{\sqrt{\mu T}}} \\ &= \sqrt{\frac{230\ \mathrm{J/s}}{\sqrt{(1.42 \times 10^{-4}\ \mathrm{kg/m})(2.94\ \mathrm{N})}}} \\ &= 106\ \mathrm{m/s}. \end{aligned}$$

Problem 7 (page 163 of text)

[Statement] Consider two strings. String A is 20 cm long and has mass of 12 milligrams, while String B is 30 cm long and has mass of 25 milligrams. If each string is put under tension by the same hanging mass, how do the phase velocities of two transverse waves and impedances of the two strings compare?

Hint 1: The phase velocities are determined by $v = \sqrt{T/\mu}$; comparing v_A to v_B gives

$$\frac{v_A}{v_B} = \frac{\sqrt{T_A/\mu_A}}{\sqrt{T_B/\mu_B}}.$$

Hint 2: The strings are under the same tension, so $T_A = T_B$. That simplifies the expression to

$$\begin{aligned} \frac{v_A}{v_B} &= \sqrt{\frac{\mu_B}{\mu_A}} \\ &= \sqrt{\frac{M_B/L_B}{M_A/L_A}} \\ &= \sqrt{\frac{25/30}{12/20}} \\ &= 1.18, \end{aligned}$$

where you can leave the masses in mg and lengths in cm because the units mg/cm of the linear densities cancel in their ratio.

Hint 3: Because $Z = \sqrt{\mu T}$, the ratio of impedances is

$$\frac{Z_A}{Z_B} = \frac{\sqrt{T_A \mu_A}}{\sqrt{T_B \mu_B}}.$$

Hint 4: Once again the tensions will cancel out, leaving

$$\frac{Z_A}{Z_B} = \sqrt{\frac{\mu_A}{\mu_B}}.$$

This is the reciprocal of the previous result: $1.18^{-1} = 0.85$. The lower density String A has waves with higher velocities and has a lower impedance.

Problem 8 (page 163 of text)

[Statement] If a short segment of the light string of Problem 7 is inserted into the heavy string of that problem, find the amplitude reflection coefficients for waves at both interfaces (light-to-heavy and heavy-to-light).

Hint 1: The amplitude reflection coefficient r is given in Eq. 4.27:

$$r = \frac{Z_1 - Z_2}{Z_1 + Z_2}.$$

Hint 2: Because $Z = \sqrt{\mu T}$, this equation can be rewritten as

$$r = \frac{\sqrt{\mu_1 T_1} - \sqrt{\mu_2 T_2}}{\sqrt{\mu_1 T_1} + \sqrt{\mu_2 T_2}}.$$

The tension in both strings is still the same, so $T_1 = T_2$ and cancels out in the ratio:

$$r = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}.$$

Hint 3: For the light-to-heavy interface, String 1 is String A and String 2 is String B, which gives

$$\begin{aligned}
r_{\text{L to H}} &= \frac{\sqrt{\mu_A} - \sqrt{\mu_B}}{\sqrt{\mu_A} + \sqrt{\mu_B}} \\
&= \frac{\sqrt{M_A/L_A} - \sqrt{M_B/L_B}}{\sqrt{M_A/L_A} + \sqrt{M_B/L_B}} \\
&= \frac{\sqrt{12/20} - \sqrt{25/30}}{\sqrt{12/20} + \sqrt{25/30}} \\
&= -0.0819
\end{aligned}$$

which indicates that the reflected wave from this interface has a small amplitude and is inverted.

Hint 4: For the heavy-to-light interface, String 1 is String B and String 2 is String A, which gives

$$\begin{aligned} r_{\text{H to L}} &= \frac{\sqrt{\mu_B} - \sqrt{\mu_A}}{\sqrt{\mu_B} + \sqrt{\mu_A}} \\ &= \frac{\sqrt{M_B/L_B} - \sqrt{M_A/L_A}}{\sqrt{M_A/L_A} + \sqrt{M_B/L_B}} \\ &= \frac{\sqrt{25/30} - \sqrt{12/20}}{\sqrt{12/20} + \sqrt{25/30}} \\ &= +0.0819 \end{aligned}$$

which indicates that the reflected wave from this interface has a small amplitude and is not inverted.

Problem 9 (page 163 of text)

[Statement] Find the amplitude transmission coefficients for waves at both interfaces (light-to-heavy and heavy-to-light) in the previous problem.

Hint 1: The amplitude transmission coefficient is given by Eq. 4.28:

$$t = \frac{2Z_1}{Z_1 + Z_2}.$$

Hint 2: Because $Z = \sqrt{\mu T}$, this equation can be rewritten as

$$t = \frac{2\sqrt{\mu_1 T_1}}{\sqrt{\mu_1 T_1} + \sqrt{\mu_2 T_2}}.$$

The tension in both strings is still the same, so $T_1 = T_2$ and cancels out in the ratio:

$$t = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}.$$

Hint 3: For the light-to-heavy interface, String 1 is String A and String

2 is String B, which gives

$$\begin{aligned} t_{\text{L to H}} &= \frac{2\sqrt{\mu_A}}{\sqrt{\mu_A}+\sqrt{\mu_B}} \\ &= \frac{2\sqrt{M_A/L_A}}{\sqrt{M_A/L_A}+\sqrt{M_B/L_B}} \\ &= \frac{2\sqrt{12/20}}{\sqrt{12/20}+\sqrt{25/30}} \\ &= 0.918 \end{aligned}$$

Hint 4: For the heavy-to-light interface, String 1 is String B and String 2 is String A, which gives

$$\begin{aligned} t_{\text{H to L}} &= \frac{2\sqrt{\mu_B}}{\sqrt{\mu_B}+\sqrt{\mu_A}} \\ &= \frac{2\sqrt{M_B/L_B}}{\sqrt{M_A/L_A}+\sqrt{M_B/L_B}} \\ &= \frac{2\sqrt{25/30}}{\sqrt{12/20}+\sqrt{25/30}} \\ &= 1.082 \end{aligned}$$

which indicates that the transmitted wave has an amplitude of (1.082)(0.918) = 0.993 of the original wave's amplitude.

Problem 10 (page 163 of text)

[Statement] Verify that the power of the transmitted and reflected waves in the previous two problems add up to power of the incoming wave.

Hint 1: If the sum of the power reflection coefficient R and the power transmission coefficient T for each interface is 1, then the power is the same for outgoing and incoming waves.

Hint 2: The power reflection coefficient R is given in Eq. 4.30:

$$R = r^2$$

The power transmission coefficient T is given in Eq. 4.29

$$T = \left(\frac{Z_2}{Z_1}\right)t^2 = \left(\sqrt{\frac{\mu_2}{\mu_1}}\right)t^2,$$

since $Z = \sqrt{\mu T}$ and the tensions cancel. Their sum is

$$r^2 + \left(\sqrt{\frac{\mu_2}{\mu_1}}\right) t^2.$$

Hint 3: Plugging in numbers for the light-to-heavy interface, where 1 is A and 2 is B, gives

$$\begin{aligned}
&r^2_{\text{L to H}} + \left(\sqrt{\frac{\mu_B}{\mu_A}}\right) t^2_{\text{L to H}} \\
&= (-0.0819)^2 + \left(\sqrt{\frac{25/30}{12/20}}\right) (0.918)^2 \\
&= 0.9999 \approx 1
\end{aligned}$$

Hint 4: Plugging in numbers for the heavy-to-light interface, where 1 is B and 2 is A, gives

$$\begin{aligned}
&r^2_{\text{H to L}} + \left(\sqrt{\frac{\mu_A}{\mu_B}}\right) t^2_{\text{H to L}} \\
&= (0.0819)^2 + \left(\sqrt{\frac{12/20}{25/30}}\right) (1.082)^2 \\
&= 0.9999 \approx 1
\end{aligned}$$

Hint 5: This problem can also be solved in the general case, using the impedance forms for the amplitude coefficients:

$$\begin{aligned}
r^2 + \left(\frac{Z_2}{Z_1}\right) t^2 &= \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 + \left(\frac{Z_2}{Z_1}\right)\left(\frac{2Z_1}{Z_1 + Z_2}\right)^2 \\
&= \left(\frac{1}{Z_1 + Z_2}\right)^2 \left(Z_1^2 - 2Z_1Z_2 + Z_2^2 + \frac{4Z_2Z_1^2}{Z_1}\right) \\
&= \left(\frac{1}{Z_1 + Z_2}\right)^2 \left(Z_1^2 + 2Z_1Z_2 + Z_2^2\right) \\
&= \left(\frac{Z_1 + Z_2}{Z_1 + Z_2}\right)^2 \\
&= 1
\end{aligned}$$

5

The Electromagnetic Wave Equation click-throughs

Example 1 (page 175 of text)

[Statement]**Example:** *If an electromagnetic plane wave is propagating along the positive z-direction and its electric field at a certain location points along the positive x-axis, in what direction does the wave's magnetic field point at that same location?*

Hint 1: If the electric field points along the positive x-axis, you know that E_{0x} is positive and $E_{0y} = 0$. That means that the magnetic field must have a non-zero component along the positive y-axis, since $B_{0y} = E_{0x}$ and E_{0x} is positive.

Hint 2: The fact that $E_{0y} = 0$ means that B_{0x} (which equals $-E_{0y}$ must also be zero.

Hint 3: And if B_{0y} is positive and B_{0x} is zero, then $\vec{B}$ must point entirely along the positive y-axis at this location.

Example 2 (page 183 of text)

[Statement]**Example:** *At the surface of the Earth, the average power density of sunlight on a clear day is approximately 1300 W/m^2. Find the average magnitude of the electric and magetic fields in sunlight.*

Hint 1: To find the average electric field strength, solve Eq. 5.22 for

$|\vec{E}|_{\text{avg}}$:

$$|\vec{S}|_{\text{avg}} = \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{E}_0|^2$$
$$|\vec{E}_0| = \sqrt{\frac{2|\vec{S}|_{\text{avg}}}{\sqrt{\frac{\epsilon_0}{\mu_0}}}} = \sqrt{2|\vec{S}|_{\text{avg}}\sqrt{\frac{\mu_0}{\epsilon_0}}}.$$

Hint 2: Plugging in values gives

$$|\vec{E}_0| = \sqrt{(2)1300\ \text{W/m}^2\sqrt{\frac{(4\pi \times 10^{-7}\ \text{H/m})}{(8.8541878 \times 10^{-12}\ \text{F/m})}}} \approx 990\ \text{V/m}.$$

Hint 3: Once you know the electric field magnitude, you can use Eq. 5.15 to find the magnitude of the magnetic field:

$$|\vec{B}_0| = \frac{|\vec{E}_0|}{c} = \frac{990\ \text{V/m}}{3 \times 10^8\ \text{m/s}} \approx 3.3 \times 10^{-6}\ \text{T}.$$

Example 3 (page 185 of text)

[Statement]**Example:** *Use the definition of the Poynting vector from Eq. 5.26 to find the vector power density $\vec{S}$ of an electromagnetic plane wave propagating along the positive z-axis.*

Hint 1: From Fig. 5.5 and Eq. 5.26, the Poynting vector is

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$
$$= \frac{1}{\mu_0}|\vec{E}||\vec{B}|\sin\theta\hat{k}.$$

Hint 2: Since $\vec{E}$ is perpendicular to $\vec{B}$, θ is 90°.

Hint 3: You also know that $|\vec{B}| = |\vec{E}|/c$, so this can be written as

$$\begin{aligned}\vec{S} &= \frac{1}{\mu_0}|\vec{E}||\vec{B}|\sin 90^\circ \hat{k} = \frac{1}{\mu_0}|\vec{E}||\vec{B}|\hat{k} \\ &= \frac{1}{\mu_0}|E|\frac{|\vec{E}|}{c}\hat{k} = \frac{1}{\mu_0}|\vec{E}|\frac{|\vec{E}|}{\sqrt{\frac{1}{\mu_0\epsilon_0}}}\hat{k} \\ &= \frac{\sqrt{\mu_0\epsilon_0}}{\mu_0}|\vec{E}|^2\hat{k} = \sqrt{\frac{\epsilon_0}{\mu_0}}|\vec{E}|^2|\hat{k} = \frac{|\vec{E}|^2}{Z_0}\hat{k}\end{aligned}$$

as expected from Eq. 5.24.

Problem 1 (page 186 of text)

[Statement] If the electric field in a certain region is given by $\vec{E} = 3x^2y\hat{\imath} - 2xyz^2\hat{\jmath} + x^3y^2z^2\hat{k}$ in SI units, what is the electric charge density at the point x=2, y=3, z=1?

Hint 1: To find the electric charge density (ρ) if you know the vector electric field $\vec{E}$, solve $\vec{\nabla} \circ \vec{E} = \rho/\epsilon_0$ for ρ.

Hint 2: $\rho = (\epsilon_0)\vec{\nabla} \circ \vec{E}$.

Hint 3: The divergence of $\vec{E} = 3x^2y\hat{\imath} - 2xyz^2\hat{\jmath} + x^3y^2z^2\hat{k}$ is

$$\vec{\nabla} \circ \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 6xy - 2xz^2 + 2x^3y^2z.$$

Hint 4: Thus

$$\rho = (\epsilon_0)\vec{\nabla} \circ \vec{E} = (\epsilon_0)(6xy - 2xz^2 + 2x^3y^2z).$$

Hint 5: Plugging in values gives

$$\rho = (\epsilon_0)\vec{\nabla}\circ\vec{E} = (8.85\times10^{-12})[6(2)(3)-2(2)(1^2)+2(2^3)(3)^2(1)] = 1.56\times10^{-9}\ \text{C/m}^3.$$

Problem 2 (page 186 of text)

[Statement] If the magnetostatic field in a certain region is given by

$\vec{B} = 3x^2y^2z^2\hat{\imath} + xy^3z^2\hat{\jmath} - 3xy^2z^3\hat{k}$ in SI units, what is the magnitude of the electric current density at the point x=1, y=4, z=2?

Hint 1: To find the electric current density if you know the magnetostatic field, solve $\vec{\nabla} \times \vec{B} = \mu_0\vec{J}$ for $\vec{J}$.

Hint 2: $\vec{J} = \frac{\vec{\nabla}\times\vec{B}}{\mu_0}$.

Hint 3: The curl of $\vec{B}$ is

$$\vec{\nabla} \times \vec{B} = \hat{\imath}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{k}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right).$$

Hint 4: Taking the partial derivatives of $\vec{B} = 3x^2y^2z^2\hat{\imath}+xy^3z^2\hat{\jmath}-3xy^2z^3\hat{k}$ gives

$$\vec{\nabla} \times \vec{B} = \hat{\imath}\left(-6xyz^3 - 2xy^3z\right) + \hat{\jmath}\left(6x^2y^2z - 3y^2z^3\right) + \hat{k}\left(y^3z^2 - 6x^2yz^2\right).$$

Hint 5: Plugging in values gives the curl of $\vec{B}$ at the point x=1, y=4, z=2:

$$\begin{aligned}\vec{\nabla} \times \vec{B} &= \hat{\imath}\left[-6(1)(4)(2^3) - 2(1)(4^3)(2)\right] + \hat{\jmath}\left[6(1^2)(4^2)(2) - 3(4^2)(2^3)\right] \\ &\quad + \hat{k}\left[(4^3)(2^2) - 6(1^2)(4)(2^2)\right] \\ &= \hat{\imath}\,[-448] + \hat{\jmath}\,[-192] + \hat{k}\,[160] \text{ T/m}.\end{aligned}$$

Hint 6: Thus the magnitude of the curl of $\vec{B}$ is

$$|\vec{\nabla} \times \vec{B}| = \sqrt{(-448)^2 + (-192)^2 + (160)^2} = 513 \text{ T/m}$$

Hint 7: Dividing by the magnetic permeability μ_0 gives the magnitude of the electric current density $\vec{J}$:

$$|\vec{J}| = \frac{|\vec{\nabla} \times \vec{B}|}{\mu_0} = \frac{513}{4\pi \times 10^{-7}} = 4.08 \times 10^8 \text{ A/m}^3.$$

Problem 3 (page 186 of text)

[Statement] If the magnetic field at a certain location is changing with time according to the equation $\vec{B} = 3t^2\hat{\imath} + t\hat{\jmath}$ in SI units, what are the magnitude and direction of the curl of the induced electric field at that location at time $t = 2$ seconds?

Hint 1: To find the curl of the induced electric field, use $\vec{\nabla}\times\vec{E} = -\partial\vec{B}/\partial t$.

Hint 2: The time derivative of $\vec{B} = 3t^2\hat{\imath} + t\hat{\jmath}$ is

$$\frac{\partial\vec{B}}{\partial t} = \frac{\partial(3t^2\hat{\imath} + t\hat{\jmath})}{\partial t} = 6t\hat{\imath} + 1\hat{\jmath}.$$

Hint 3: Thus the curl of $\vec{E}$ at time $t = 2$ seconds is

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -6(2)\hat{\imath} - 1\hat{\jmath} = [-12\hat{\imath} - 1\hat{\jmath}]\ \mathrm{V/m^2}.$$

Hint 4: The magnitude of the vector curl of $\vec{E}$ at $t = 2$ seconds is

$$|\vec{\nabla}\times\vec{E}| = \sqrt{(-12)^2 + (-1)^2} = 12.04\mathrm{V/m^2}.$$

Hint 5: The angle of the vector curl of $\vec{E}$ is

$$\theta = \arctan\left(\frac{-1}{-12}\right) = 4.8^\circ$$

and, since the denominator of the arctan argument is negative, adding 180° to the answer gives the angle measured anti-clockwise from the positive x-axis as

$$\theta = 4.8^\circ + 180^\circ = 184.8^\circ.$$

Problem 4 (page 186 of text)

[Statement] Show that the x-component of Faraday's Law for a plane wave propagating in the positive z-direction leads to the equation $E_{0y} =$

$-cB_{0x}$.

Hint 1: The relation between the curl of the electric field and the time derivative of the magnetic field is $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$.

Hint 2: The x-component of this equation is

$$\left(\vec{\nabla} \times \vec{E}\right)_x = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -\frac{\partial B_x}{\partial t}.$$

Hint 3: For a plane wave propagating in the positive z-direction, the components of the electric and magnetic field may be written as

$$\begin{aligned}
E_x &= E_{0x} \sin{(kz - \omega t)} \\
E_y &= E_{0y} \sin{(kz - \omega t)} \\
E_z &= 0 \\
B_x &= B_{0x} \sin{(kz - \omega t)} \\
B_x &= B_{0y} \sin{(kz - \omega t)} \\
B_z &= 0.
\end{aligned}$$

Hint 4: Thus the x-component of the curl of the electric field is

$$\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = 0 - \frac{\partial [E_{0y} \sin{(kz - \omega t)}]}{\partial z} = -kE_{0y} \cos{(kz - \omega t)}.$$

Hint 5: The x-component of the time derivative of the magnetic field is

$$-\frac{\partial B_x}{\partial t} = -\frac{\partial [B_{0x} \sin{(kz - \omega t)}]}{\partial t} = -[-\omega B_{0x} \cos{(kz - \omega t)}].$$

Hint 6: Thus

$$\begin{aligned}
-kE_{0y} \cos{(kz - \omega t)} &= \omega B_{0x} \cos{(kz - \omega t)} \\
-kE_{0y} &= \omega B_{0x}
\end{aligned}$$

so

$$E_{0y} = -\frac{\omega}{k} B_{0x} = -cB_{0x}$$

Problem 5 (page 186 of text)

[Statement] Show that the y-component of Faraday's Law for a plane wave propagating in the positive z-direction leads to the equation $E_{0x} = cB_{0y}$.

Hint 1: The relation between the curl of the electric field and the time derivative of the magnetic field is $\vec{\nabla} \times \vec{E} = -\partial \vec{B}/\partial t$.

Hint 2: The y-component of this equation is

$$\left(\vec{\nabla} \times \vec{E}\right)_y = \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -\frac{\partial B_y}{\partial t}.$$

Hint 3: For a plane wave propagating in the positive z-direction, the components of the electric and magnetic field may be written as

$$\begin{aligned}
E_x &= E_{0x} \sin(kz - \omega t) \\
E_y &= E_{0y} \sin(kz - \omega t) \\
E_z &= 0 \\
B_x &= B_{0x} \sin(kz - \omega t) \\
B_x &= B_{0y} \sin(kz - \omega t) \\
B_z &= 0.
\end{aligned}$$

Hint 4: Thus the y-component of the curl of the electric field is

$$\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = \frac{\partial[E_{0x} \sin(kz - \omega t)]}{\partial z} - 0 = kE_{0x} \cos(kz - \omega t).$$

Hint 5: The y-component of the time derivative of the magnetic field is

$$-\frac{\partial B_y}{\partial t} = -\frac{\partial[B_{0y} \sin(kz - \omega t)]}{\partial t} = -[-\omega B_{0y} \cos(kz - \omega t)].$$

Hint 6: Thus

$$kE_{0x} \cos{(kz - \omega t)} = \omega B_{0y} \cos{(kz - \omega t)}$$
$$kE_{0x} = \omega B_{0y}$$

so

$$E_{0x} = \frac{\omega}{k} B_{0y} = cB_{0y}$$

Problem 6 (page 186 of text)

[Statement] Show that the two Eqs. 5.13 and 5.14 mean that $\vec{E}$ and $\vec{B}$ are perpendicular to one another.

Hint 1: One way to show that two vectors are perpendicular to one another is to show that the dot product between the two vectors is zero. To do that, begin by writing $\vec{E_0}$ and $\vec{B_0}$ as

$$\vec{E_0} = E_{0x}\hat{\imath} + E_{0y}\hat{\jmath}$$
$$\vec{B_0} = B_{0x}\hat{\imath} + B_{0y}\hat{\jmath}.$$

Hint 2: Using the expressions for E_{0x} and E_{0y} from Eqs. 5.13 and 5.14 gives

$$\vec{B_0} = -\frac{E_{0y}}{c}\hat{\imath} + \frac{E_0 x}{c}\hat{\jmath}.$$

Hint 3: Now form the dot product between $\vec{E}$ and $\vec{B}$:

$$\vec{E} \circ \vec{B} = \vec{E_0} \circ \vec{B_0} = (E_{0x}\hat{\imath} + E_{0y}\hat{\jmath}) \circ \left(-\frac{E_{0y}}{c}\hat{\imath} + \frac{E_0 x}{c}\hat{\jmath}\right)$$
$$= -\frac{E_{0x}E_{0y}}{c} + \frac{E_{0y}E_{0x}}{c} = 0$$

and zero dot product means that these two vectors are perpendicular to one another.

Problem 7 (page 186 of text)

[Statement] Show that $\sqrt{\frac{1}{\mu_0\epsilon_0}}$ has units of meters per second.

Hint 1: The SI units of the electric permittivity ϵ_0 are farads per meter (F/m), and farads represent coulombs per volt (C/V).

Hint 2: Since volts represent newtons time meters over coulombs (Nm/C), coulombs per volt (C/V) are C/(Nm/C)=C^2/(Nm). So the units of ϵ_0 are equivalent to $\text{C}^2/(\text{Nm}^2)$.

Hint 3: The SI units of the magnetic permeability μ_0 are henries per meter (H/m), and henries represent volts times seconds squared over coulombs (Vs^2/C).

Hint 4: Again using the fact that volts represent newtons time meters over coulombs (Nm/C), volts times seconds squared over coulombs (Vs^2/C) are Nms^2/C^2. So the units of μ_0 are equivalent to $\text{Nms}^2/\text{C}^2\text{m}=\text{Ns}^2/\text{C}^2$.

Hint 5: Multiplying the units of ϵ_0 by the units of μ_0 gives

$$\epsilon_0\mu_0 \text{ (units)} = \left(\frac{C^2}{Nm^2}\right)\left(\frac{Ns^2}{C^2}\right) = \frac{s^2}{m^2}$$

Hint 6: Thus the units of $\sqrt{\frac{1}{\mu_0\epsilon_0}}$ are

$$\sqrt{\frac{1}{\epsilon_0\mu_0}} \text{ (units)} = \sqrt{\frac{1}{\frac{s^2}{m^2}}} = \frac{m}{s}$$

Problem 8 (page 186 of text)

[Statement] According to the inverse-square law, the power density of an electromagnetic wave transmitted by an isotropic source (that is, a source that radiates equally in all directions) is given by the equation

$$|\vec{S}| = \frac{P_{\text{transmitted}}}{4\pi r^2}$$

where $P_{\text{transmitted}}$ is the transmitted power and r is the distance from the source to the receiver. Find the magnitude of the electric and magnetic fields produced by a 1,000-watt radio transmitter at a distance of 20 km.

Hint 1: To find the magnitude of the electric and magnetic field, begin by finding the magnitude of the Poynting vector $\vec{S}$, which is related to the field magnitudes through the equations of Section 5.5.

Hint 2: The magnitude of the Poynting vector at a distance of 20 km (2×10^4 m) from a 1,000-W transmitter is

$$|\vec{S}| = \frac{P_{\text{transmitted}}}{4\pi r^2} = \frac{1000}{4\pi(2 \times 10^4)^2} = 1.99 \times 10^{-7}\ \text{W/m}^2.$$

Hint 3: Solving Eq. 5.24 for the magnitude of the electric field gives

$$|\vec{E}| = \sqrt{Z_0|\vec{S}|}$$

where Z_0 represents the impedance of free space.

Hint 4: Plugging in the value for the magnitude of the Poynting vector and the impedance of free space (377Ω) gives

$$|\vec{E}| = \sqrt{Z_0|\vec{S}|} = \sqrt{(377)(1.99 \times 10^{-7})} = 8.66 \times 10^{-3}\ \text{V/m}$$

Hint 5: Once you know the magnitude of the electric field $\vec{E}$, you can find the magnitude of the magnetic field $\vec{B}$ using Eq. 5.15:

$$|\vec{B}| = \frac{|\vec{E}|}{c} = \frac{8.66 \times 10^{-3}}{3 \times 10^8} = 2.89 \times 10^{-11}\ \text{T}.$$

Problem 9 (page 186 of text)

[Statement] If vector $\vec{A} = 8\hat{\imath} + 3\hat{\jmath} + 6\hat{k}$ and vector $\vec{B} = 12\hat{\imath} - 7\hat{\jmath} + 4\hat{k}$, what are the magnitude and direction of the vector cross-product $\vec{A} \times \vec{B}$?

Hint 1: The vector cross-product between vectors $\vec{A}$ and $\vec{B}$ can be found using

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

or

$$\begin{aligned}\vec{A} \times \vec{B} =& (A_y B_z - A_z B_y)\hat{\imath} \\ &+ (A_z B_x - A_x B_z)\hat{\jmath} \\ &+ (A_x B_y - A_y B_x)\hat{k}.\end{aligned}$$

Hint 2: Inserting the values for the components of $\vec{A}$ and $\vec{B}$ gives

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 8 & 3 & 6 \\ 12 & -7 & 4 \end{vmatrix}$$

or

$$\begin{aligned}\vec{A} \times \vec{B} =& [(3)(4) - (6)(-7)]\hat{\imath} \\ &+ [(6)(12) - (8)(4)]\hat{\jmath} \\ &+ [(8)(-7) - (3)(12)]\hat{k} \\ &= 54\hat{\imath} + 40\hat{\jmath} - 92\hat{k}.\end{aligned}$$

Hint 3: The magnitude of the vector $\vec{A} \times \vec{B}$ is

$$|\vec{A} \times \vec{B}| = \sqrt{(54)^2 + (40)^2 + (-92)^2} = 113.93$$

Hint 4: To find the angle (θ_x) of the vector $\vec{A} \times \vec{B}$ with the x-axis, find the dot product of the unit vector $\hat{\imath}$ with $\vec{A} \times \vec{B}$ (which gives x-component of the vector $\vec{A} \times \vec{B}$) and then use

$$\begin{aligned}\theta_x &= \arccos\left[\frac{\hat{\imath} \circ (\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|}\right] \\ &= \arccos\left[\frac{54}{113.93}\right] = 61.7^\circ\end{aligned}$$

Hint 5: To find the angle (θ_y) of the vector $\vec{A} \times \vec{B}$ with the y-axis, find the dot product of the unit vector $\hat{\jmath}$ with $\vec{A} \times \vec{B}$ (which gives y-component of the vector $\vec{A} \times \vec{B}$) and then use

$$\theta_y = \arccos\left[\frac{\hat{\jmath} \circ (\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|}\right]$$
$$= \arccos\left[\frac{40}{113.93}\right] = 69.4^\circ$$

Hint 6: To find the angle (θ_z) of the vector $\vec{A}\times\vec{B}$ with the z-axis, find the dot product of the unit vector $\hat{k}$ with $\vec{A} \times \vec{B}$ (which gives z-component of the vector $\vec{A} \times \vec{B}$) and then use

$$\theta_z = \arccos\left[\frac{\hat{k} \circ (\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|}\right]$$
$$= \arccos\left[\frac{-92}{113.93}\right] = 143.9^\circ$$

Problem 10 (page 186 of text)

[Statement] In certain plasmas (such as the Earth's ionosphere), the dispersion relation of electromagnetic waves is $\omega^2 = c^2k^2 + \omega_p^2$, where c is the speed of light and ω_p is the natural "plasma frequency" that depend on the particle concentration. Find the phase velocity (ω/k) and the group velocity ($d\omega/dk$) of electromagnetic waves in such a plasma, and show that their product equals the square of the speed of light.

Hint 1: You can find the group velocity using $v_{group} = \frac{d\omega}{dk}$ and the phase velocity using $v_{phase} = \frac{\omega}{k}$.

Hint 2: In this case $\omega = \sqrt{c^2k^2 + \omega_p^2}$, so the group velocity is

$$v_{group} = \frac{d\omega}{dk} = \frac{d[(c^2k^2 + \omega_p^2)^{\frac{1}{2}}]}{dk}$$
$$= \frac{1}{2}\frac{(2c^2k)}{\sqrt{c^2k^2 + \omega_p^2}} = \frac{c^2}{\sqrt{c^2 + \frac{\omega_p^2}{k^2}}}.$$

Hint 3: For $\omega = \sqrt{c^2k^2 + \omega_p^2}$ the phase velocity is

$$v_{phase} = \frac{\omega}{k} = \frac{\sqrt{c^2k^2 + \omega_p^2}}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}.$$

Hint 4: Multiplying the group velocity by the phase velocity gives

$$v_{group} \times v_{phase} = \left[\frac{c^2}{\sqrt{c^2 + \frac{\omega_p^2}{k^2}}}\right]\left[\sqrt{c^2 + \frac{\omega_p^2}{k^2}}\right] = c^2.$$

6

The Quantum Wave Equation click-throughs

Example 1 (page 193 of text)

[Statement] **Example:** *What is the de Broglie wavelength of a 75-kg human walking at a speed of 1.5 m/s ?*

Hint 1: The human in this example has momentum $p = 75\ \text{kg}\times 1.5\ \text{m/s} = 113\ \text{kg m/s}$, which gives a de Broglie wavelength of

$$\lambda = \frac{6.626 \times 10^{-34}\ \text{Js}}{113\ \text{kg m/s}} = 5.9 \times 10^{-36}\ \text{m}. \tag{6.4}$$

Hint 2: This is not only billions of times smaller than the spacing between atoms in a typical solid, but billions of times smaller than the protons and neutrons that make up the atoms' nuclei. Hence an object with the mass of a human is not a good candidate for demonstrating the wave behavior of matter. However, for a very small mass with very low velocity, the de Broglie wavelength can be large enough to be measured.

Example 2 (page 193 of text)

[Statement] **Example:** *What is the de Broglie wavelength of an electron that has passed through a potential difference of 50 volts?*

Hint 1: After passing through a potential difference of 50 volts, an electron has 50 electron volts (eV) of energy. One eV is 1.6×10^{-19} J, so the electron's energy in SI units is

$$50\ \text{eV}\ \frac{1.6 \times 10^{-19}\ \text{J}}{1\ \text{eV}} = 8 \times 10^{-18}\ \text{J}. \tag{6.5}$$

Hint 2: To relate this energy to the electron's momentum, you can use the classical expression for kinetic energy

$$KE = \frac{1}{2}mv^2 \tag{6.6}$$

and then multiply and divide the right side by the mass:

$$KE = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{p^2}{2m}. \tag{6.7}$$

Hint 3: In this case, the electron's energy (E) is all kinetic, so $E = KE$, and the momentum is:

$$p = \sqrt{2mE}. \tag{6.8}$$

Applying this to the electron with 8 $\times 10^{-18}$ J of energy, the momentum is

$$\begin{aligned} p &= \sqrt{2 \times 9.1 \times 10^{-31} \text{ kg} \times 8 \times 10^{-18} \text{ J}} \\ &= 3.8 \times 10^{-24} \text{ kg m/s}. \end{aligned}$$

Hint 4: Putting this result into de Broglie's equation (Eq. 6.3) gives a wavelength of

$$\lambda = \frac{6.626 \times 10^{-34} \text{ Js}}{3.8 \times 10^{-24} \text{ kg m/s}} = 1.7 \times 10^{-10} \text{ m}, \tag{6.9}$$

or 0.17 nanometers. This is similar to the spacing between atoms in a crystal array, so such an array can be used to experimentally determine a moving electron's wavelength.

Example 3 (page 200 of text)

[Statement] **Example:** *What is the time-independent Schrödinger equation for a free particle?*

Hint 1: In this context, "free" means that the particle is free of the influence of external forces, and since force is the gradient of potential energy, a free particle travels in a region of constant potential energy. Since the reference location of zero potential energy is arbitrary, you can set $V = 0$ in the Schrödinger equation for a free particle. Thus Eq. 6.22 becomes

$$E\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} \tag{6.23}$$

or

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\Psi. \tag{6.24}$$

Hint 2: Since the total energy of a free particle equals the particle's kinetic energy, you can set $E = \frac{p^2}{2m}$:

$$\frac{\partial^2 \Psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\Psi. \tag{6.25}$$

Example 4 (page 206 of text)

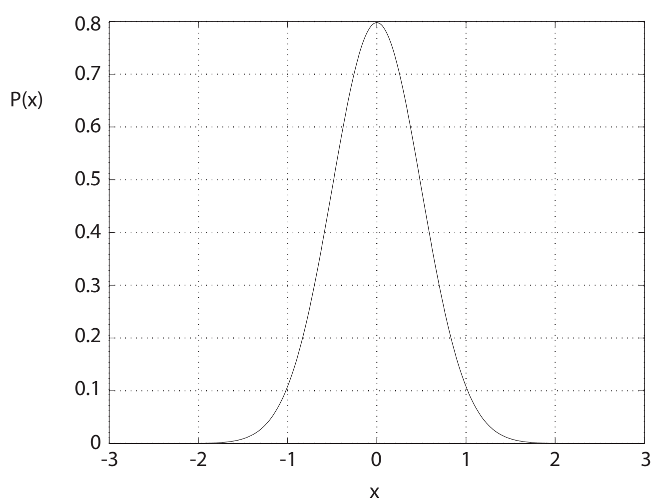
[Statement] **Example:** *Determine the probability of finding a particle at a given location if the particle's wavefunction is defined as*

$$\psi(x) = \left(\frac{0.2}{\pi}\right)^{1/4} e^{-0.1x^2} e^{ikx}$$

Hint 1: In this case, the width constant a is 0.1, which makes the probability density

$$\begin{aligned}\psi^*(x)\psi(x) &= \left[\left(\frac{0.2}{\pi}\right)^{1/4} e^{-0.1x^2} e^{-ikx}\right]\left[\left(\frac{0.2}{\pi}\right)^{1/4} e^{-0.1x^2} e^{ikx}\right] \\ &= \left(\frac{0.2}{\pi}\right)^{1/2} e^{-0.2x^2}\end{aligned}$$

which is a Gaussian distribution, as shown in the figure below.



Hint 2: To find the probability of the particle with this wave function being located in a particular spot, you have to integrate the density around that place. In this example, the likelihood of finding the particle at $x = 1$ m, give or take 0.1 m, is

$$\begin{aligned}\mathcal{P}(1 \pm 0.1) &= \left(\frac{0.2}{\pi}\right)^{1/2} \int_{0.9}^{1.1} e^{-0.2x^2} dx \\ &= 0.041,\end{aligned}$$

or 4.1%.

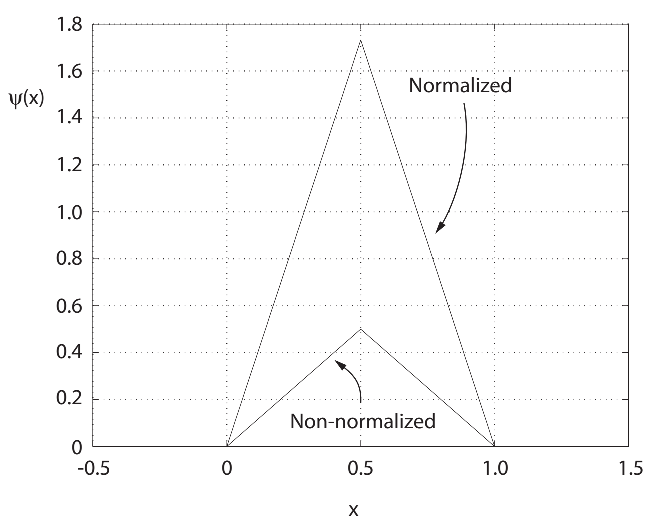
Hint 3: You can check the normalization of this function by integrating over all space:

$$\mathcal{P}_{\text{all space}} = \left(\frac{0.2}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-0.2x^2} dx = 1.$$

So the probability of finding this particle somewhere in space is indeed 100%.

Example 5 (page 207 of text)

[Statement] **Example:** *Normalize the triangular pulse wave function in the figure shown below.*



Hint 1: The equation for this triangular pulse can be written as

$$\psi(x) = \begin{cases} Ax & 0 \le x \le 0.5 \\ A(1-x) & 0.5 \le x \le 1 \\ 0 & \text{else.} \end{cases}$$

Hint 2: The equation can be plugged into the probability density integral:

$$\mathcal{P}_{\text{all space}} = 1 = \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx.$$

Thus

$$1 = \int_0^{0.5} (A^*x)(Ax)dx + \int_{0.5}^{1} \left(A^*(1-x)\right)\left(A(1-x)\right)dx.$$

Hint 3: Pulling out A^*A from each integral leaves

$$1 = A^*A\left(\int_0^{0.5} x^2 dx + \int_{0.5}^{1} (1-x)^2 dx\right)$$
$$1 = A^*A\left(\frac{1}{24} + \frac{1}{24}\right).$$

In this case, all factors in the equation are real, so $A^*A = A^2$.

Hint 4: Solving for the normalization constant A gives

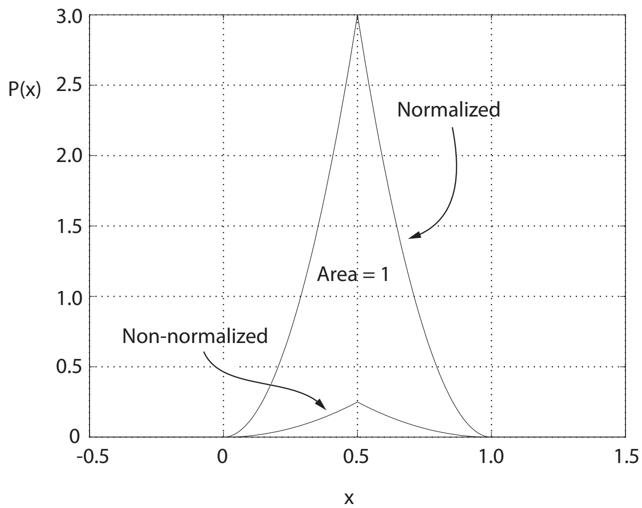
$$A^2 = 12$$
$$A = \sqrt{12}$$

Hint 5: This figure shows the probability density before and after normalization. As desired, the area under the normalized probability density is one, but the shape of both the normalized wave function and the normalized probability density haven't changed from the non-normalized functions; only their scale has changed.

Problem 1 (page 216 of text)

[Statement] Find the de Broglie wavelength of a water molecule, with mass 2.99×10^{-26} kg, traveling at 640 m/s (a likely speed at room temperature).

Hint 1: The water molecule has a momentum of $mv = (2.99\times10^{-26}\text{ kg})(640\text{ m/s}) = 1.91\times10^{-23}$ kg m/s.



Hint 2: At that momentum, the molecule's de Broglie wavelength is

$$\lambda = \frac{6.626 \times 10^{-34} \text{ Js}}{1.91 \times 10^{-23} \text{ kg m/s}} = 3.46 \times 10^{-11} \text{ m}.$$

Hint 3: This result is around 0.3 Å, less than the (approximate) radius of a hydrogen atom. So, it's often reasonable to treat water vapor like a collection of classical particles.

Problem 2 (page 216 of text)

[Statement] What is the de Broglie wavelength of a proton, with mass 1.67×10^{-27} kg, when it has an energy of 15 MeV?

Hint 1: The momentum is related to the energy by $p = \sqrt{2mE}$.

Hint 2: One eV is 1.6×10^{-19} J, so the proton's energy in SI units is

$$15 \times 10^6 \text{ eV} \; \frac{1.6 \times 10^{-19} \text{ J}}{1 \text{ eV}} = 2.4 \times 10^{-12} \text{ J}.$$

Hint 3: Therefore, the momentum is

$$\begin{aligned} p &= \sqrt{2 \times 1.67 \times 10^{-27} \text{ kg} \times 2.4 \times 10^{-12} \text{ J}} \\ &= 8.95 \times 10^{-20} \text{ kg m/s}. \end{aligned}$$

Hint 4: Putting this result into de Broglie's equation gives a wavelength of

$$\lambda = \frac{6.626 \times 10^{-34} \text{ Js}}{8.95 \times 10^{-20} \text{ kg m/s}} = 7.4 \times 10^{-15} \text{ m}.$$

Hint 5: Compare this result to the example in Section 6.2. There, the electron's energy was 10^{-6} and its mass was 10^{-4} of the proton's. You'd expect the electron's de Broglie wavelength, which is proportional to $1/\sqrt{mE}$, to be $1/\sqrt{10^{-10}} = 10^5$ the size. Comparing orders of magnitude shows that the electron's wavelength is roughly 10^5 the wavelength of this proton's.

Problem 3 (page 216 of text)

[Statement] The spread in measured positions of an ensemble of electrons is 1 micron. What is the best case spread in the measured momenta of a similar ensemble?

Hint 1: Heisenberg's uncertainty principle says that $\Delta x \Delta p \geq \hbar/2$. So,

$$\Delta p \geq \frac{\hbar}{2\Delta x}.$$

Hint 2: $\hbar$ is the reduced Planck's constant, $h/(2\pi)$. Plugging values in gives

$$\Delta p \geq \frac{6.626 \times 10^{-34} \text{ Js}}{4\pi(1 \times 10^{-6} \text{ m})} = 5.3 \times 10^{-29} \text{ kg m/s}.$$

A relatively wide spread in position (1×10^{-6} m) results in a narrow spread in momentum (5.3×10^{-29} kg m/s).

Problem 4 (page 216 of text)

[Statement] Normalize the wavefunction $\psi(x) = xe^{-x^2/2}$ over all space.

Hint 1: Include a normalization constant, A, in the wavefunction: $\psi(x) = Axe^{-x^2/2}$.

Hint 2: Plug the wave function into the probability density integral:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx \\
&= \int_{-\infty}^{\infty} (Axe^{-x^2/2})(Axe^{-x^2/2})dx \\
&= A^2 \int_{-\infty}^{\infty} x^2 e^{-x^2} dx
\end{aligned}$$

Hint 3: The integral $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$ is equal to $\sqrt{\pi}/2$ (which you can determine either via mathematical software or by looking it up in an integral table).

Hint 4: The equation now reduces to

$$1 = A^2 \frac{\sqrt{\pi}}{2}$$

or

$$A^2 = \frac{2}{\sqrt{\pi}}.$$

Solving for A gives $(2/\sqrt{\pi})^{1/2}$.

Problem 5 (page 216 of text)

[Statement] Normalize the wavefunction $\psi(x) = \sin(15x)$ when $0 \leq x \leq \pi/5$, zero elsewhere.

Hint 1: Include a normalization constant, A, in the wavefunction: $\psi(x) = A\sin(15x)$.

Hint 2: Plug the wave function into the probability density integral:

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx \\
&= \int_{0}^{\pi/5} (A\sin(15x))(A\sin(15x))dx \\
&= A^2 \int_{0}^{\pi/5} \sin^2(15x)dx
\end{aligned}$$

where the limits of integration have been changed to 0 and $\pi/5$ since the function, and integral, is zero everywhere else.

Hint 3: The integral $\int_0^{\pi/5} \sin^2(15x)dx$ is equal to $\pi/10$ (which you can determine either via mathematical software or by looking it up in an integral table). If you want to do it by hand, you can use the trigonometric identity

$$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax)).$$

The integral becomes

$$\int_0^{\pi/5} \frac{1}{2}(1 - \cos(30x))dx$$

which separates into two integrals,

$$\int_0^{\pi/5} \frac{1}{2}dx - \int_0^{\pi/5} \frac{1}{2}\cos(30x)dx.$$

The first integral is $\frac{1}{2}x|_0^{\pi/5} = \pi/10 - 0 = \pi/10$. The second integral gives, ignoring constants, $\sin(6\pi) - \sin(0) = 0$.

Hint 4: The probability equation now reduces to

$$1 = A^2\frac{\pi}{10}$$

or

$$A^2 = \frac{10}{\pi}.$$

Solving for A gives $\sqrt{10/\pi}$.

Problem 6 (page 216 of text)

[Statement] Determine the probability of finding the particle of the previous problem between 0.1 and 0.2 meters.

Hint 1: The probability of finding a particle between two points is given by

$$\mathcal{P}_{ab} = \int_a^b \psi^*(x)\psi(x)dx.$$

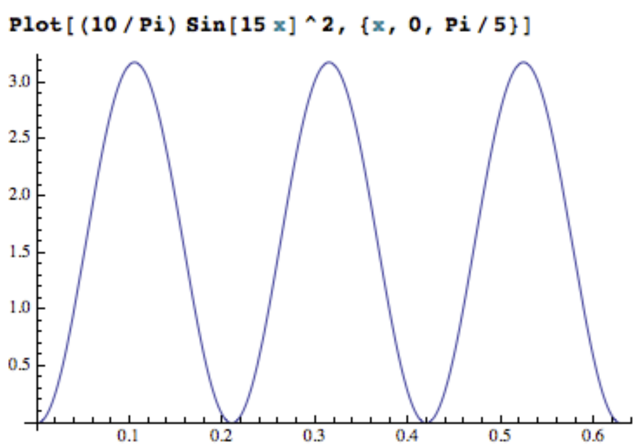
Hint 2: Plugging in the normalized wavefunction gives

$$\mathcal{P} = \int_{0.1}^{0.2} (\sqrt{\frac{10}{\pi}} \sin(15x))(\sqrt{\frac{10}{\pi}} \sin(15x))dx$$
$$= \frac{10}{\pi} \int_{0.1}^{0.2} \sin^2(15x)dx,$$

which can be solved as in Problem 5.

Hint 3: The value of the integral is 0.057. Multiplying by $10/\pi$ gives a probability of 0.18, or 18%.

Hint 4: Is this a reasonable probability? Looking at the graph of the wave function squared (shown below) can help you see that it is. The area from 0.1 to 0.2 represents roughly half of one of three bumps, or 1/6 the total probability. 18% is very close to 1/6 = 17%.



Problem 7 (page 216 of text)

[Statement] Show that the wavenumber distribution $\phi(k)$ for the wavefunction in Eq. 6.39 is $\phi(k) = \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} e^{\frac{\sigma_x^2}{2}(k_0-k)^2}$.

Hint 1: The Fourier transform finds the wavenumber distribution for a given wavefunction:

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{-ikx} dx.$$

Plugging in the wave packet function from Eq. 6.39 gives

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} e^{-x^2/(2\sigma_x^2)} e^{ik_0 x} e^{-ikx} dx,$$

which simplifies to

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} \int_{-\infty}^{\infty} e^{-x^2/(2\sigma_x^2)} e^{i\Delta kx} dx,$$

where $\Delta k \equiv k_0 - k$ for ease of notation.

Hint 2: The previous equation can be solved with mathematical software, but if you want to see how to solve it by hand, read on. Concentrate on the arguments of the exponentials–you need to convert them to a simple form that will be easy to integrate.

$$\begin{aligned} e^{-x^2/(2\sigma_x^2)} e^{i\Delta kx} &= \exp\left(-\frac{x^2}{2\sigma_x^2} + i\Delta kx\right) \\ &= \exp\left(-(ax^2 - bx)\right) \end{aligned}$$

where $a = 1/(2\sigma_x^2)$ and $b = i\Delta k$.

Hint 3: If you complete the square for the term $ax^2 - bx$, you'll be able to write the argument as $(rk - s)^2 - t^2$.

Hint 4: Expand $(rk - s)^2 - t^2$ into $r^2k^2 - 2rsk + s^2 - t^2$. This must be equivalent to $ax^2 - bx$, so you can match terms:

$$\begin{aligned} r^2 &= a \\ r &= \sqrt{a}, \end{aligned}$$

$$\begin{aligned} 2rs &= b \\ s &= \frac{b}{2r} \\ s &= \frac{b}{2\sqrt{a}}, \end{aligned}$$

and

$$\begin{aligned} s^2 - t^2 &= 0 \\ t^2 &= s^2 \\ t^2 &= \frac{b^2}{4a}. \end{aligned}$$

Hint 5: Rewriting the argument in terms of a and b gives:

$$\left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a} = \left(\frac{1}{\sqrt{2}\sigma_x}x - \frac{\sigma_x i\Delta k}{\sqrt{2}}\right)^2 + \frac{\Delta k^2\sigma_x^2}{2}$$

(It's a good idea to check that this form is equivalent to the original.) Replacing the a and b form in the integral (for simplicity) gives

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{\pi\sigma_x^2}\right)^{1/4}\int_{-\infty}^{\infty} \exp\left[-\left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a}\right] dx$$

Hint 6: Now that the square has been completed, u substitution will finish bringing the integral into its integrable form. Set $u = \left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)$, which means that $du = \sqrt{a}dx$ or $dx = (1/\sqrt{a})du$:

$$\frac{1}{\sqrt{2\pi a}}\left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty} e^{-u^2} du$$

Hint 7: The integral can now be found in a table:

$$\int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi}.$$

The solution is therefore, replacing the values for a and b,

$$\phi(k) = \frac{\sqrt{\pi}}{\sqrt{2\pi(1/(2\sigma_x^2))}}\left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} e^{-\frac{(i\Delta k)^2}{4(1/(2\sigma_x^2))}}$$

Hint 8: Algebraic simplification leads to

$$\begin{aligned} \phi(k) &= \sigma_x\left(\frac{1}{\pi\sigma_x^2}\right)^{1/4} e^{-\frac{(i\Delta k)^2\sigma_x^2}{2}} \\ &= \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} e^{-\frac{i(k_0-k)^2\sigma_x^2}{2}} \end{aligned}$$

as expected.

Problem 8 (page 216 of text)

[Statement] a. Determine the probability of finding a particle with wavenumber between 6.1×10^4 rad/m and 6.3×10^4 rad/m if it has the wavenumber distribution of the previous problem with the values $k_0 = 6.2 \times 10^4$ rad/m and $\sigma_x = 250$ microns.

Hint 1: The probability of finding a particle with a wavenumber between two values is given by

$$\mathcal{P}_{ab} = \int_a^b \phi^*(k)\phi(k)dk.$$

Hint 2: Plugging in the wavenumber distribution gives

$$\begin{aligned}\mathcal{P} &= \int_a^b \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} e^{\frac{\sigma_x^2}{2}(k_0-k)^2} \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} e^{\frac{\sigma_x^2}{2}(k_0-k)^2} dk. \\ &= \left(\frac{\sigma_x^2}{\pi}\right)^{1/2} \int_a^b e^{\sigma_x^2(k_0-k)^2} dk.\end{aligned}$$

Hint 3: With numbers plugged in, this becomes

$$\left(\frac{(240 \times 10^{-6})^2}{\pi}\right)^{1/2} \int_{6.1\times10^4}^{6.3\times10^4} e^{(240\times10^{-6})^2(6.2\times10^4-k)^2} dk.$$

which is equal to 0.288, or 29%.

[Statement] b. Compare to the result if $\sigma_x = 400$ microns.

Hint 1: Replace σ_x with its new value to get

$$\left(\frac{(400 \times 10^{-6})^2}{\pi}\right)^{1/2} \int_{6.1\times10^4}^{6.3\times10^4} e^{(400\times10^{-6})^2(6.2\times10^4-k)^2} dk.$$

which is equal to 0.476, or 48%.

Hint 2: Is this reasonable? With a wider spread in position (larger σ_x), you'd expect the distribution in k to become narrower; for a normalized function, this means that the probability of finding a wavenumber near k_0 becomes bigger.

Problem 9 (page 216 of text)

[Statement] Show that inserting a Gaussian wave packet (Eq. 6.40) into Eq. 6.44 leads to the expression for $\Psi(x,t)$ given in Eq. 6.45.

Hint 1: Inserting $\phi(k) = \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} e^{\frac{\sigma_x^2}{2}(k_0-k)^2}$ into $\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i[kx-\omega(k)t]} dk$ gives

$$\Psi(x,t) = \left(\frac{\sigma_x^2}{4\pi^3}\right)^{1/4} \int_{-\infty}^{\infty} e^{\frac{\sigma_x^2}{2}(k_0-k)^2} e^{i[kx-\omega(k)t]} dk,$$

where $\omega(k) = \frac{\hbar k^2}{2m}$.

Hint 2: As in Problem 7, focus on the arguments of the exponentials in order to simplify the problem:

$$\exp\left[-\frac{\sigma_x^2}{2}(k_0-k)^2 + i(kx - \frac{\hbar k^2}{2m}t)\right].$$

Unlike Problem 7, here it's easiest to do a u substitution before completing the square. The first term has $k_0 - k$, although to switch variables from k it's easier to use $u = k - k_0$ (and, handily, in that first term the negative sign squares away, so this choice will be simplest to work with). This also means that $k = u + k_0$:

$$\exp\left[-\frac{\sigma_x^2}{2}u^2 + i\left((u+k_0)x - (u^2 + 2uk_0 + k_0^2)\frac{\hbar}{2m}t\right)\right]$$

Hint 3: Collecting powers of u gives

$$\exp\left[u^2\left(-\frac{\sigma_x^2}{2} - \frac{i\hbar t}{2m}\right) + u\left(ix - \frac{i\hbar k_0 t}{m}\right) + \left(ixk_0 - \frac{i\hbar k_0^2 t}{2m}\right)\right].$$

This can be expressed as

$$\exp\left[-Au^2 - Bu + C\right],$$

with

$$A = \frac{\sigma_x^2}{2} + \frac{i\hbar t}{2m},$$
$$B = -ix + \frac{i\hbar k_0 t}{m},$$
$$C = ixk_0 - \frac{i\hbar k_0^2 t}{2m}.$$

Hint 4: The integral now has the form

$$\Psi(x,t) = \left(\frac{\sigma_x^2}{4\pi^3}\right)^{1/4} \int_{-\infty}^{\infty} \exp\left[-Au^2 - Bu + C\right] du,$$

where the earlier substitution $u = k - k_0$ makes $du = dk$ and retains the $\pm\infty$ integration limits. The constant term can be pulled out of the integral as e^C, which, because $\omega_0 = \frac{\hbar k_0^2}{2m}$, is equal to $\exp\left[i(k_0 x - \omega_0 t)\right]$:

$$\Psi(x,t) = \left(\frac{\sigma_x^2}{4\pi^3}\right)^{1/4} \exp\left[i(k_0 x - \omega_0 t)\right] \int_{-\infty}^{\infty} \exp\left[-Au^2 - Bu\right] du.$$

Hint 5: To solve the integral $\int_{-\infty}^{\infty} \exp\left[-Au^2 - Bu\right] du$, you can either complete the square (as in Problem 7), or you can look up this form in an integral table:

$$\int_{-\infty}^{\infty} \exp\left[-Au^2 - Bu\right] du = \sqrt{\frac{\pi}{A}} e^{B^2/(4A)}$$

which is consistent with the result in Problem 7.

Hint 6: The final result is

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sigma_x^2}{\pi}\right)^{1/4} \exp\left[i(k_0 x - \omega_0 t)\right] \sqrt{\frac{\pi}{A}} \exp\left[B^2/(4A)\right].$$

Rewritten in terms of the original variables gives

$$\Psi(x,t) = \left(\frac{\sigma_x^2}{4\pi^3}\right)^{1/4} \exp\left[i(k_0 x - \omega_0 t)\right] \sqrt{\frac{\pi}{\frac{\sigma_x^2}{2} + \frac{i\hbar t}{2m}}} \exp\left[\frac{(-ix + \frac{i\hbar k_0 t}{m})^2}{4(\frac{\sigma_x^2}{2} + \frac{i\hbar t}{2m})}\right].$$

Hint 7: Pulling a factor of $(-i)^2 = -1$ from $(-ix + \frac{i\hbar k_0 t}{m})^2$ makes the expression into

$$\Psi(x,t) = \left(\frac{\sigma_x^2}{4\pi^3}\right)^{1/4} \exp\left[i(k_0 x - \omega_0 t)\right] \sqrt{\frac{\pi}{\frac{\sigma_x^2}{2} + \frac{i\hbar t}{2m}}} \exp\left[-\frac{(x - \frac{\hbar k_0 t}{m})^2}{4(\frac{\sigma_x^2}{2} + \frac{i\hbar t}{2m})}\right].$$

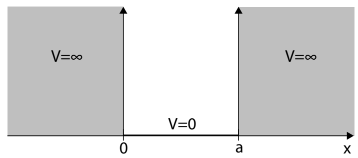
which matches Eq. 6.45.

Problem 10 (page 216 of text)

[Statement] A very different situation than the free particle is the trapped one: a particle in a potential well. The simplest case is the particle in a box: an infinitely deep, constant potential well as shown below.

[Statement] a. In this case, the wavefunction does not penetrate the side walls and $\psi(0) = \psi(a) = 0$. Show that $\psi(x) = \sin(n\pi x/a)$ satisfies both Eq. 6.21 and the boundary conditions. What values may n have?

Hint 1: Plug $\psi(x) = \sin(n\pi x/a)$ into $E\Psi = \frac{-\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$. Because the



potential is zero between 0 and a, the $V\Psi$ term disappears. You're left with

$$E \sin\left(\frac{n\pi x}{a}\right) = \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{a}\right).$$

Hint 2: Find the partial derivatives of $\sin(n\pi x/a)$:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi x}{a}\right) &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\sin\left(\frac{n\pi x}{a}\right)\right)\\ &= \frac{\partial}{\partial x}\left(\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)\right)\\ &= -\left(\frac{n\pi}{a}\right)^2\sin\left(\frac{n\pi x}{a}\right)\end{aligned}$$

Hint 3: Plug this result back into the Schrödinger equation:

$$E \sin\left(\frac{n\pi x}{a}\right) = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\sin\left(\frac{n\pi x}{a}\right),$$

where the two negatives on the right side have canceled. At this point the sines also cancel, leaving

$$E = \frac{\hbar^2 n^2 \pi^2}{2ma^2}.$$

This seems like a valid energy; it has the right units

$$\frac{\mathrm{J}^2\ \mathrm{s}^2}{\mathrm{kg\ m}^2} = \frac{\mathrm{J}^2}{\mathrm{kg\ m}^2/\mathrm{s}^2} = \frac{\mathrm{J}^2}{\mathrm{J}} = \mathrm{J}$$

and is a constant value for a given n and a.

Hint 4: To see if the wavefunction satisfies the boundary conditions, plug in 0 and a to see if $\psi(x) = 0$:

$$\begin{aligned}\sin(0) &= 0\\ \sin(n\pi a/a) &= \sin(n\pi).\end{aligned}$$

The latter equals zero only when n is an integer. In other words, the valid solutions to the particle in the box have discrete values of energy, corresponding to integer n. These quantized states are one reason for the name "quantum mechanics."

[Statement] b. Normalize $\psi(x)$.

Hint 1: Neither the Schrödinger equation nor the boundary conditions set the normalization constant. Instead, you (as always) must impose the fact that the sum of all probability must be one.

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx \\
&= \int_0^a (A \sin\left(\frac{n\pi x}{a}\right))(A \sin\left(\frac{n\pi x}{a}\right))dx \\
&= A^2 \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx
\end{aligned}
$$

Hint 2: Using the methods of Problem 5, the integral is equal to $a/2$. So, the probability equation now reduces to

$$1 = A^2 \frac{a}{2}$$

or

$$A^2 = \frac{2}{a}.$$

Solving for A gives $\sqrt{2/a}$.

[Statement] c. Plot several wavefunctions with the three smallest n values and compare to Fig. 3.5 for standing waves on a string.

Hint: Set $a = 1$ for easier comparison with Fig. 3.5. The plots look like standing waves, each with their own energy.

```
Plot[{Sqrt[2] Sin[Pi x], Sqrt[2] Sin[2 Pi x], Sqrt[2] Sin[3 Pi x]}, {x, 0, 1}]
```