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Chapter 1

CT

Supplement: The Generated Wave

Brevity compelled us to omit from the printed text a derivation of Equation 2.11, which describes the generated electromagnetic wave in terms of the induced nonlinear polarization. In this supplement, we present a derivation of this Equation.

We begin with Maxwell's equations:

$$\nabla \cdot \mathbf{D} = \rho \tag{1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.3}$$

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \tag{1.4}$$

and the constitutive relationships:

$$\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P} \tag{1.5}$$

$$\mathbf{B} = \mu(\mathbf{H} + \mathbf{M}) \tag{1.6}$$

where $\mathbf{E}$ is the electric field vector, $\mathbf{D}$ is the displacement vector, $\mathbf{B}$ is the magnetic flux density vector, $\mathbf{H}$ is the magnetic field vector, $\mathbf{j}$ is the current density vector, $\mathbf{M}$

is the magnetization vector, ρ is the charge density, ϵ_o is the permittivity of free space, and μ is the magnetic permeability. In a non-magnetic material, we have $\mathbf{M} = 0$, and $\mu = \mu_o$. If there are no free charges, we have $\rho = 0$. These conditions will be true for most nonlinear materials of interest here. The current density $\mathbf{j}$ is given by $\mathbf{j} = \sigma : \mathbf{E}$. The current density $\mathbf{j}$ and the electric field $\mathbf{E}$ are both vectors, but nothing says that they must point in the same direction. Thus they are related by the tensor conductivity σ, where : indicates the product of the tensor σ with the vector $\mathbf{E}$ to give the vector $\mathbf{j}$. Even if the material has negligible intrinsic conductivity, this formalism is still often used and an effective conductivity adopted to account for losses due to scattering and absorption.

Taking the curl of Eq. 1.3 and using Eq. 1.6 and Eq. 1.4 on the right hand side gives:

$$\nabla \times \nabla \times \mathbf{E} + \mu_o \frac{\partial\left(\sigma : \mathbf{E}\right)}{\partial t} + \mu_o \frac{\partial^2 \mathbf{D}}{\partial t^2} = 0 \tag{1.7}$$

Replacing $\mathbf{D}$ using Eq. 1.5 and recalling that $\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL}$ gives:

$$\nabla \times \nabla \times \mathbf{E} + \mu_o \frac{\partial\left(\sigma : \mathbf{E}\right)}{\partial t} + \mu_o \epsilon_o \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_o \frac{\partial^2 \mathbf{P}_L}{\partial t^2} = -\mu_o \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2} \tag{1.8}$$

Since $\mathbf{P}_L = \epsilon_o \chi^{(1)} \mathbf{E}$, and using $\epsilon^{(1)} = 1 + \chi^{(1)}$, we can write:

$$\nabla \times \nabla \times \mathbf{E} + \mu_o \frac{\partial\left(\sigma : \mathbf{E}\right)}{\partial t} + \mu_o \epsilon_o \frac{\partial^2 [\epsilon^{(1)} : \mathbf{E}]}{\partial t^2} = -\mu_o \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2} \tag{1.9}$$

Here, $\epsilon^{(1)} : \mathbf{E}$ is a vector obtained by the action of the tensor $\epsilon^{(1)}$ on the vector $\mathbf{E}$. Normally, we choose to work in a rectangular coordinate system in which $\epsilon^{(1)}$ and σ can be represented by diagonal matrices; thus we have:

$$\epsilon^{(1)} : \mathbf{E} = \begin{bmatrix} \epsilon^{(1)}_{xx} & 0 & 0 \\ 0 & \epsilon^{(1)}_{yy} & 0 \\ 0 & 0 & \epsilon^{(1)}_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{1.10}$$

$$\sigma : \mathbf{E} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{1.11}$$

where $\mathbf{E} = \widehat{\mathbf{x}}E_x + \widehat{\mathbf{y}}E_y + \widehat{\mathbf{z}}E_z$, and $\widehat{\mathbf{x}}$, $\widehat{\mathbf{y}}$, and $\widehat{\mathbf{z}}$ are unit vectors along the x, y, and z directions respectively.

To further simplify this expression, we will make use of the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$. We will also consider plane waves, for which $\nabla \cdot \mathbf{E} = 0$ identically. Then we can write:

$$\nabla^2\mathbf{E} - \mu_o \frac{\partial\left[\sigma : \mathbf{E}\right]}{\partial t} - \mu_o\epsilon_o \frac{\partial^2[\epsilon^{(1)} : \mathbf{E}]}{\partial t^2} = \mu_o \frac{\partial^2\mathbf{P}_{NL}}{\partial t^2} \tag{1.12}$$

In light of Eqs. 1.10 and 1.11, we see that Eq. 1.12 can be expressed as three equations, one for each scalar component of $\mathbf{E}$. For example, the equation for the z-component E_z is:

$$\nabla^2 E_z\left(\mathbf{r}, t\right) - \mu_o \frac{\partial\left[\sigma : \mathbf{E}\left(\mathbf{r}, t\right)\right]}{\partial t} - \mu_o\epsilon_o \frac{\partial^2[\epsilon_{zz}^{(1)} E_z\left(\mathbf{r}, t\right)]}{\partial t^2} = \mu_o \frac{\partial^2 P_{NL,z}\left(\mathbf{r}, t\right)}{\partial t^2} \tag{1.13}$$

where $\mathbf{r}$ is a vector specifying position. If we consider a plane wave propagating along the $x-$direction and polarized along the $z-$direction, we arrive at the even simpler form:

$$\frac{\partial^2 E_z(x,t)}{\partial x^2} - \mu_o\sigma_{zz} \frac{\partial E_z\left(x,t\right)}{\partial t} - \mu_o\epsilon_o\epsilon_{zz}^{(1)} \frac{\partial^2 E_z(x,t)}{\partial t^2} = \mu_o \frac{\partial^2 P_{NL,z}(x,t)}{\partial t^2} \tag{1.14}$$

If $P_{NL,z} = 0$, we have a homogeneous wave equation for propagation in a lossy anisotropic medium, a solution of which is $E_z(x,t) = E_o e^{-\alpha x}\cos(\omega t - kx + \phi)$, where

$$k^2 - \alpha^2 = \frac{\omega^2}{c_o^2} n_z^2 \tag{1.15}$$

$$\alpha = \frac{\omega\mu_o\sigma_{zz}}{2k} \tag{1.16}$$

where k is the propagation constant of the wave, α is the attenuation coefficient, $c_o = 1/\sqrt{\mu_o\epsilon_o}$ is the speed of light in vacuum and $n_z = \sqrt{\epsilon_{zz}^{(1)}}$ is the refractive index of the nonlinear medium.

The attenuation coefficient α accounts for the decrease in amplitude of the electric field due to loss as the wave propagates through the nonlinear material. In many cases, α will be sufficiently small that it can be neglected. In this case, Eq. 1.15 simplifies to the more familiar relationship $k = \omega n_z/c$.

If $P_{NL,z} \neq 0$, the term on the right-hand side of Eq. 1.14 acts as a source to drive the generation of $E_z(x,t)$, and we are faced with finding the solution of this driven equation.

In Chapter 2 of "Compact Blue-Green Lasers" we stated that the nonlinear polarization is defined in the frequency domain. That is, since $\chi^2(\omega)$ is explicitly a frequency-domain concept, we need to find the frequency components generated by the second-order interaction of the applied electric field, and multiply in the frequency domain by $\epsilon_o\chi^2(\omega)$ in order to determine the frequency components present in the driving polarization. We could convert this frequency-domain polarization $\mathcal{P}_{NL}(\omega)$ back into the time domain using an inverse Fourier transform, in order to solve Eq. 1.14. Alternatively, we could convert Eq. 1.14 to a frequency domain equation, solve for the frequency components present in the generated second-harmonic or sum-frequency field, then transform that result back to the time domain. We will pursue this latter course, since it provides an approach to the problem that is useful when considering non-monochromatic waves (such as those emitted by a multi-longitudinal-mode laser (treated in Section 2.2.6 of "CBGL") or such as short pulses (treated in another chapter of the supplement).

We will therefore express $E_z(x,t)$ and $P_{NL,z}(x,t)$ in Eq. 1.14 in terms of their inverse Fourier transforms. We can write the electric field as:

$$E_g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}_g(x,\omega)e^{j\omega t}\, d\omega \tag{1.17}$$

Since we have assumed that the generated wave will be $z-$polarized, the z subscript no longer carries useful information, and we have replaced it by the g subscript to distinguish the generated electric field from the applied electric fields. It is convenient to write $E_g(x,t)$ in a form that emphasizes our expectation that the output will consist of travelling waves:

$$E_g(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widetilde{\mathcal{E}}_g(x,\omega)e^{j[\omega t - k_g(\omega)x]}\, d\omega \tag{1.18}$$

Here, we have allowed for the possibility that the amplitude $\widetilde{\mathcal{E}}_g$ will grow as the wave propagates along the x-axis. Similarly, we can write the polarization in terms of an inverse Fourier transform:

$$P(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathcal{P}(x,\omega)e^{j\omega t}\,d\omega \tag{1.19}$$

Substituting Equations 18 and 19 into Equation 14, and performing the differentiations, we arrive at the expression:

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\frac{\partial^2\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x^2} - 2jk_g(\omega)\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} - \left\{\left[k_g^2(\omega) - \omega^2\mu_o\epsilon_o n_g^2(\omega)\right] - j\omega\mu_o\sigma\right\}\widetilde{\mathcal{E}}_g(x,\omega)\right]e^{j[\omega t - k_g(\omega)x]}\,d\omega \tag{1.20}$$

$$= \frac{\mu_o}{2\pi}\int\limits_{-\infty}^{\infty}\left(-\omega^2\right)\mathcal{P}_{NL}(x,\omega)e^{j\omega t}\,d\omega \tag{1.21}$$

Consider the first two terms in the integrand on the left-hand side of the equation, which we can re-write as $\frac{\partial}{\partial x}\left[\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} - j\frac{4\pi}{\lambda_g}\widetilde{\mathcal{E}}_g(x,\omega)\right]$, where λ_g is the generated wavelength in the nonlinear material. We can re-write this again as $\frac{1}{\lambda_g}\frac{\partial}{\partial x}\left[\lambda_g\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} - 4j\pi\widetilde{\mathcal{E}}_g(x,\omega)\right]$. The quantity $\left|\lambda_g\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x}\right|$ is just the change in the magnitude of $\widetilde{\mathcal{E}}_g$ that occurs over a distance of one wavelength, and we assume this change to be small in comparison to the magnitude of the electric field. If this assumption (usually called the "Slowly-Varying Envelope Approximation") is valid, then we see that the first term in the brackets can be neglected in comparison to the second. The term involving the second derivative then disappears from the integrand, and we are left only a first derivative with which to contend. If the attenuation is negligible ($\alpha \approx 0$), we can further simplify the integrand by noting from our undriven solution that $k_g^2(\omega) - \omega^2\mu_o\epsilon_o n_g^2(\omega) \approx 0$. Thus, we are left with the equation

$$\frac{\partial\widetilde{\mathcal{E}}_g(x,\omega)}{\partial x} = \frac{\mu_o\omega^2}{2jk_g(\omega)}\mathcal{P}_{NL}(x,\omega)e^{jk_g(\omega)x} = \frac{-jk_g(\omega)}{2\epsilon_o n_g^2(\omega)}\mathcal{P}_{NL}(x,\omega)e^{jk_g(\omega)x} \tag{1.22}$$

This equation is the main result of this section. Section 2.2 of the text showed how to find the polarization induced in the nonlinear material from the applied fields. This equation tells us how to determine the freely-propagating electromagnetic wave generated by that polarization. Once we know the polarization (here, in its frequency-domain representation), we can insert it into this equation, and integrate over the length of the crystal to determine the generated output wave, $\widetilde{\mathcal{E}}_g(x, \omega)$. Again, $\widetilde{\mathcal{E}}_g(x, \omega)$ is a frequency-domain representation of the generated field. If we desire to know the time-domain behavior of this generated field, we can take the inverse Fourier transform (Eq. 1.17) to find the time-domain representation $E_g(x, t)$. If, as is often the case, we only want to know the power contained in this field, we can determine it from either the time- or frequency-domain representations. How to do so is the subject of another chapter of the supplement.