Exercises to the
PRIMER ON THEORETICAL SOIL MECHANICS

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1 Exercises

1. Determine the velocity field within a soil specimen of height H that is compressed by a piston moving downwards with velocity V
   (a) in an oedometer test.
   (b) in an undrained triaxial test
   (c) in a drained triaxial test.
2. Given the vectors $\mathbf{a}$ and $\mathbf{b}$ and the orthogonal matrix $\mathbf{Q}$, show that $\mathbf{a} \cdot \mathbf{Q}\mathbf{b} = \mathbf{Q}^T\mathbf{a} \cdot \mathbf{b}$.
3. To obtain the density of a material, we measure the mass m and the volume V. The obtained mean values are: $\bar{m} = 2.651{,}2$ g and $\bar{V} = 1.003{,}7$ cm^3. The standard deviations were obtained as : $s_m = 4.5$ g, $s_V = 2.3$ cm^3. Calculate the standard deviation of the density $\rho = m/V$.
4. Determine the matrix $\mathbf{Q}$ that rotates the vector $\mathbf{v}$ $(\mathbf{v}' = \mathbf{Q}\mathbf{v})$ by the angle θ about the axis given by the vector $\mathbf{k}$.
5. Using cartesian coordinates in a 2D or 3D space we can express rigid body translation by the fact that all points of the considered body have the same velocity. Now we consider the motion of rigid tectonic plates ('cratons') on the surface of the earth. Are there translations such that all points of a tectonic plate have the same velocity?
6. (a) Show that shear stresses must occur in a sloping terrain.
   (b) At which angle intersect the principal stress trajectories the surface of a cohesionless soil inclined by the angle $\beta < \phi$?
   (c) Express the stresses σ_A and σ_B (Fig. 1) in dependence of $\gamma \cdot d$.
   (d) Calculate the maximum inclination of a cohesionless slope, which is partially immersed in water.
   (e) Plot heuristically the principal stress trajectories in a triangular earth dam and show the distribution of vertical stress at the bottom edge (hint: use the equations of Lamé-Maxwell). Show that it deviates from the intuitively assumed triangular distribution of Fig. 2

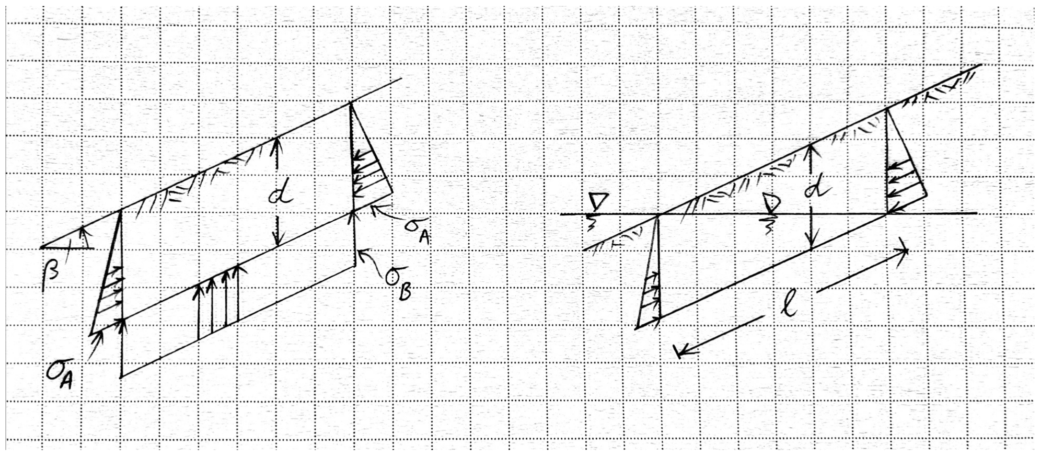

Figure 1: Stresses in a slope.

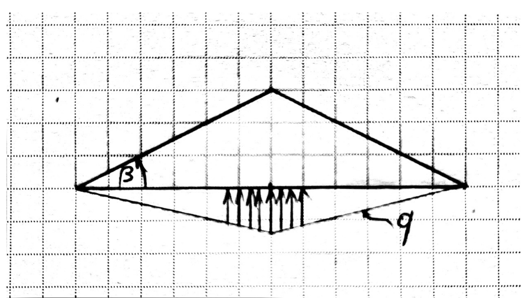

Figure 2: Intuitively assumed triangular distribution of pressure q.

7. The tensile force in a rod is 60 N. Its cross section area is 2 cm^2. The rod has the direction of the vector $(1,\ 1,\ 2)$. Express the stress tensor in the rod as a matrix in a cartesian system of coordinates.

8. Silt particles float in a lake so that the suspension has a specific weight of 11 kN/m^3. They sink with a velocity of 1 cm per 100 years. The bottom of the lake is silt with specific weight of 20 kN/m^3. Determine the velocity with which the bottom of the lake rises.
The specific weights of water and silt particles are $\gamma^w = 10$ and $\gamma^s = 27$ kg/m^3, respectively.

9. In a one-lane road cars move with a velocity of 70 km/h. Their density is $\rho_1 = 3$ cars pro 100 m. At an accident the cars have to stop and form a queue, the density of the stopped cars is $\rho_2 = 16$ cars pro 100 m. Determine the backwards propagation velocity u of the end of the queue.

10. A suspension in water consists of particles floating in water and has a specific weight greater that water (γ_w), e.g. a bentonite slurry can have the specific weight of $\gamma_B = 11$ kN/m^3. According to the principle of Archimedes, however, a body can only float if its specific weight equals the one of water. How can we explain this contradiction?

11. Starting from Hooke's law $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$ determine the inverse relation $\varepsilon_{ij} = f(\sigma_{ij})$. Write Hooke's law for a volume preserving deformation.

12. A linear-elastic rod is compressed by the strain ε_z in axial direction z.

    (a) What is the strain ε_r in radial direction r?

    (b) What is the stiffness $d\sigma_z/d\varepsilon_z$?

    (c) Calculate the stiffness for the case that the lateral expansion/contraction is inhibited.

    (d) Which is the value of ν for incompressible materials?

13. (a) Calculate with barodesy the value of K_0 in dependence of φ_c. K_0 is defined as the ratio of lateral (radial) to vertical (axial) stress: $K_0 = \sigma_r/\sigma_z$.

    (b) How does this value change if the soil sample is placed in a cylindrical ring with inner radius $r = 4$ cm, thickness $t = 3$ mm, made from steel ($E = 210 \cdot 10^6$ kN/m^2)?

14. Using the barodetic constitutive equation calculate for Hostun sand

    (a) the earth pressure coefficient K_0

    (b) the critical friction angle φ_c

    (c) the ratio $K_1 := \sigma_3/\sigma_1$ for undrained plane ($\varepsilon_{33} \equiv 0$) compression in direction 1.

15. The dilatancy at the peak of a triaxial test compression test with Karlsruhe sand conducted at the lateral stress of $\sigma_3 = 400$ kPa reads $\delta = 0.15$. Calculate the peak stress $\sigma_{1,max}$.

16. The constitutive equation of barodesy (equation 16.6 of the book) can also be written as $\dot{\mathbf{T}} = \mathbf{HD}$. Derive the expression for the stiffness matrix $\mathbf{H}$.

17. Express tr$\mathbf{A}^4$ using invariants of $\mathbf{A}$, $\mathbf{A}^2$, $\mathbf{A}^3$.

18. Show that the matrix exponential $\exp \mathbf{A}$ can be represented as $a\mathbf{1} + b\mathbf{A} + c\mathbf{A}^2$.

19. The following constitutive equations are given:

    (a) $\dot{\sigma} = c_1\dot{\varepsilon}$

    (b) $\sigma = c_2\dot{\varepsilon}$

    (c) $\dot{\sigma} = c_3\varepsilon$

    Which of these constitutive equations describes creep?
    Which of these constitutive equations describes relaxation?

20. A one-dimensional constitutive equation is $\dot{\sigma} = c(a - \sigma)\dot{\varepsilon}$. The loading programme is: $\sigma(t = 0) = 0$ und $\dot{\varepsilon} = $ const > 0 für $t > 0$.

    (a) What is the yield stress?

    (b) What is the initial stiffness $\frac{d\sigma}{d\varepsilon}(t = 0)$?

21. A column of radius $r_0 = 30$ cm and height $h_0 = 3$ m consists of dry sand ($\varphi = 34°, \gamma = 18$ kN/m^3) jacketed into a rubber membrane of thickness $d_0 = 3$ cm and tensile strength of $\sigma_u = 11,000$ kN/m^2 obtained at an elongation of 180%. Calculate the bearing capacity (maximum vertical load F) of this column assuming that the deformation of rubber is volume preserving.

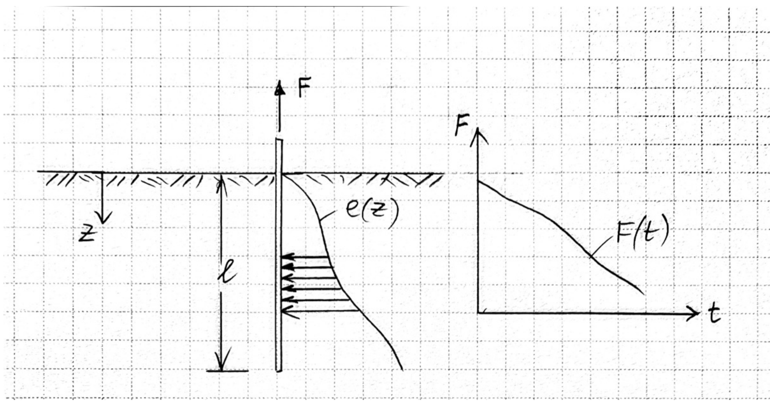

Figure 3: Pull out of a vertical strip embedded in soil.

22. The differential equation 19.31 of the book describes the process of consolidation in time but does not predict the related amount of compression of the considered soil layer. Which quantity determines the compression?

23. In the course of an undrained triaxial compression a sand sample gets completely liquefied. Subsequently the drainage is opened. What happens then?

24. A method to measure the horizontal stress in soil is to pull out vertical embedded metallic strips and measure the related force (Fig. 24). Assuming that the strip does not influence the stress state in the soil, determine how the stress distribution $e(z)$ can be inferred form the measured force $F(t)$.

25. Show that the flow of water in the narrow gap between two parallel vertical glas plates (so-called Hele-Shaw apparatus) is governed by the Darcy equation.

26. A standpipe is filled with water, see Fig. 4, and subsequently the water runs out into the soil. Find an approximate relation between the rate of descent v_0 in the standpipe and the permeability of the soil. Hint: Assume a central velocity field in the soil.

27. A cylindrical sample of Hostun sand carries the stress $\sigma_1 = 600$ kPa, $\sigma_2 = \sigma_3 = 300$ kPa. Calculate the stress increments $\Delta\sigma_1$ and $\Delta\sigma_2$ resulting from the strain $\Delta\varepsilon_1 = \Delta\varepsilon_2 = \Delta\varepsilon_3 = 0.001$. Set $e = e_c = 0.70$.

28. An ice sheet of thickness $d = 1$ m and $\gamma_{ice} = 9.2$ kN/m^3 is a floating in water. Determine its immersed depth b and the subgrade reaction modulus k.

29. Consider the constitutive equation

$$\dot{\mathbf{T}} = \mathbf{h}(\mathbf{T}, \mathbf{D}, \ldots) + b\left(\sqrt{c^2\mathbf{1} + \mathbf{D}^2}\right)^{-1}\dot{\mathbf{D}} \,,$$

where $\mathbf{h}(\mathbf{T}, \mathbf{D}, \ldots)$ is a rate independent constitutive equation. Using this equation calculate

(a) the stress jump that results from a jump

$$\mathbf{D}_1 = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}_2 = \begin{pmatrix} d_2 & 0 & 0 \\ 0 & -d_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

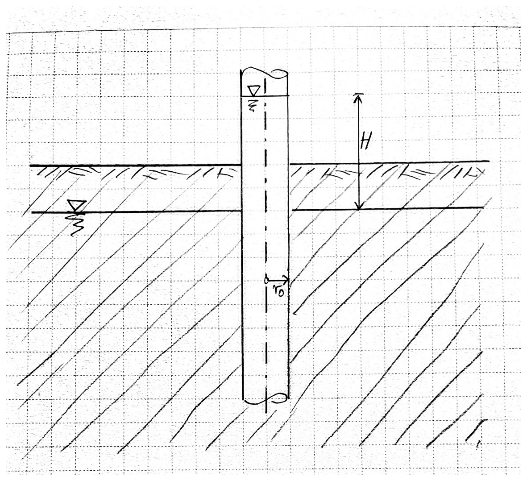

Figure 4: Standpipe in soil.

(b) the stress jump that results from a jump

$$\mathbf{D}_1 = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rightarrow \quad \mathbf{D}_2 = \mathbf{0}$$

30. A conical hole (radius $R = 20$ cm, depth $h = 15$ cm, is excavated into dry sand ($k = 10^{-5}$ m/s) and filled with water.

    (a) How long does it take until the water is entirely run out into the soil? (Neglect effects due to capillarity).

    (b) How long does it take until the water is entirely run out into the soil if the experiment is carried out on the surface of the moon with an identical sand ($g_{moon} = 1.63$ m/s^2)? (neglect any effects due to temperature, evaporation etc.)

31. A 4 m thick layer of water saturated silt is loaded by $\sigma_0 = 100$ kPa and consolidates. Its thickness is reduced by $s_0 = 30$ cm. Which energy is spent to deform the soil and which energy is spent to overcome viscosity at squeezing out water?

32. The following questions refer to Figure 19.3 of the book (tidal deformation and tidal locking). In this figure the gravity-induced deformation of the earth is *assumed* as affine deformation transforming a sphere into an ellipsoid. For simplicity we consider now plane deformation of a cylinder with initially circular cross section. The cylinder can be deformed to an ellipse of equal area if we squeeze it into an elliptical stencil. Subsequently this stencil is rotated.

    (a) Express the deformation gradient $\mathbf{F}$, the velocity field and the velocity gradient $\mathbf{L}$ for the case of tidal deformation.

    (b) Calculate the tangential velocity of the surface of the 'earth'.

    (c) Is the deformation imposed by a rotating stencil homogeneous? If it is homogeneous and if the 'earth' is linearly elastic, what is the related energy consumption?

(d) Express the deformation gradient **F** for the case of tidal locking.

33. A triaxial specimen of a soil with $\varphi = 30°$ is tested for compression and extension at a lateral stress of $\sigma_2 = 200$ kPa. Calculate $\sigma_{1,max}$ (for compression) and $\sigma_{1,min}$ (for extension) assuming homogeneous deformation. The friction angle is here assumed to be independent of stress.

2 Solutions

Exercise 1:

(a): $v_x = v_y = 0, v_z = -(V/H) \cdot z$;

(b): $v_z = -(V/H) \cdot z, v_x = v_y = -v_z/2$,

(c): $v_z = -(V/H) \cdot z, v_x = v_y$ depend on the dilatancy δ.

Exercise 2: The scalar quantity $\mathbf{a} \cdot \mathbf{Q}\mathbf{b}$ does not change if both vectors, $\mathbf{A}$ and $\mathbf{b}$, are rotated by $\mathbf{Q}^T$:

$$\mathbf{a} \cdot \mathbf{Q}\mathbf{b} = (\mathbf{Q}^T\mathbf{a}) \cdot (\mathbf{Q}^T\mathbf{Q}\mathbf{b}) = \mathbf{Q}^T\mathbf{a} \cdot \mathbf{b} \qquad \text{q.e.d.} \tag{1}$$

Exercise 3: One obtains from equation $s_y = \sqrt{\sum \left(\frac{\partial y}{\partial x_i} s_i \right)^2}$ with $\rho = m/V, \partial\rho/\partial m = 1/V, \partial\rho/\partial V = -m/V^2$:

$$s_\rho = \sqrt{\frac{1}{V^2} s_m^2 + \frac{m^2}{V^4} s_V^2} = \sqrt{\frac{1}{1,003.7^2} 4.5^2 + \frac{2,651.2^2}{1,003.7^4} 2.3^2} = 0.0075 \text{ g/cm}^3$$

Exercise 4: We use the formula of Rodriguez, $\mathbf{v}' = \mathbf{v}\cos\theta + (\mathbf{k} \times \mathbf{v})\sin\theta + \mathbf{k}(\mathbf{k} \cdot \mathbf{v})(1 - \cos\theta)$, and write using the permutation symbol ϵ_{pqi}:

$$\begin{aligned} v'_i &= v_i \cos\theta + \epsilon_{pqi} k_p v_q \sin\theta + k_i (k_p v_p)(1 - \cos\theta) && (2)\\ &= v_j \delta_{ij} \cos\theta + \epsilon_{pqi} k_p v_j \delta_{qj} \sin\theta + k_i (k_p v_j \delta_{pj})(1 - \cos\theta) && (3)\\ &= \left[\delta_{ij} \cos\theta + \epsilon_{pqi} k_p \delta_{qj} \sin\theta + k_i (k_p \delta_{pj})(1 - \cos\theta)\right] v_j \,. && (4) \end{aligned}$$

Hence

$$\begin{aligned} Q_{ij} &= \delta_{ij} \cos\theta + \epsilon_{pqi} k_p \delta_{qj} \sin\theta + k_i k_p \delta_{pj} (1 - \cos\theta) && (5)\\ &= \delta_{ij} \cos\theta + \epsilon_{pji} k_p \sin\theta + k_i k_j (1 - \cos\theta) \,. && (6) \end{aligned}$$

Exercise 5: Rigid body motions on the surface of a sphere are rotations about an axis passing through the centre of the sphere. Consider the special case that the entire surface of the sphere (earth) is rigid. It becomes then clear that different points thereupon (e.g. London and Sydney) will have different velocities.

Exercise 6: (a) If the stress were hydrostatic, only normal stresses would act on the intersecting surfaces of the triangle shown in Figure 5. Equilibrium of the horizontal forces would not be possible.

(b) We consider an infinite slope (every slope can be considered as such near to its surface) and note that the force acting upon a cut parallel to the soil surface is vertical. Interestingly, the stress state in the slope is not unique but depends on the mobilised friction angle $\varphi_m, \quad \beta \leq \varphi_m \leq \varphi$. Figure 6 shows a Mohr circle tangent to the line $\tau = \sigma \tan\varphi_m$. The pole is obtained by the intersection of this circle with the line $\tau = \sigma \tan\beta$. We consider the angle $180^\circ - \beta - (180^\circ - \alpha) = \alpha - \beta$. It is a central angle, hence the peripheral angle reads $(\alpha - \beta)/2$, and this is the inclination of the one family

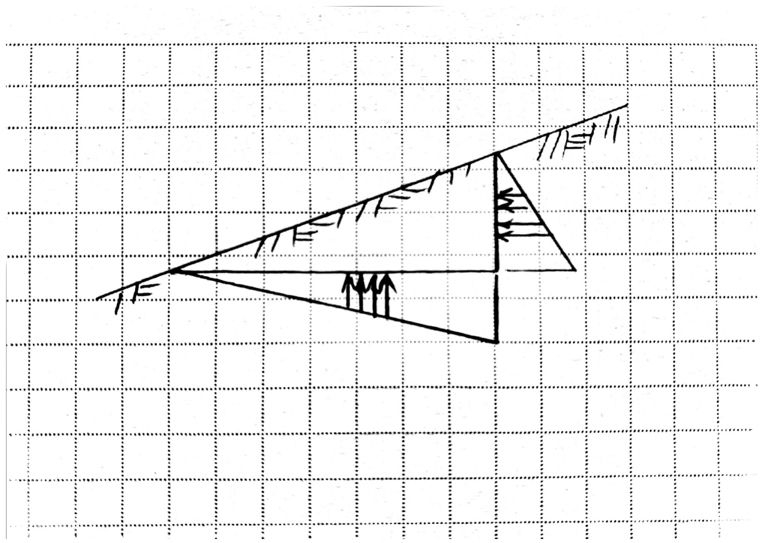

Figure 5: Hydrostatic stresses acting upon a triangular section of a slope.

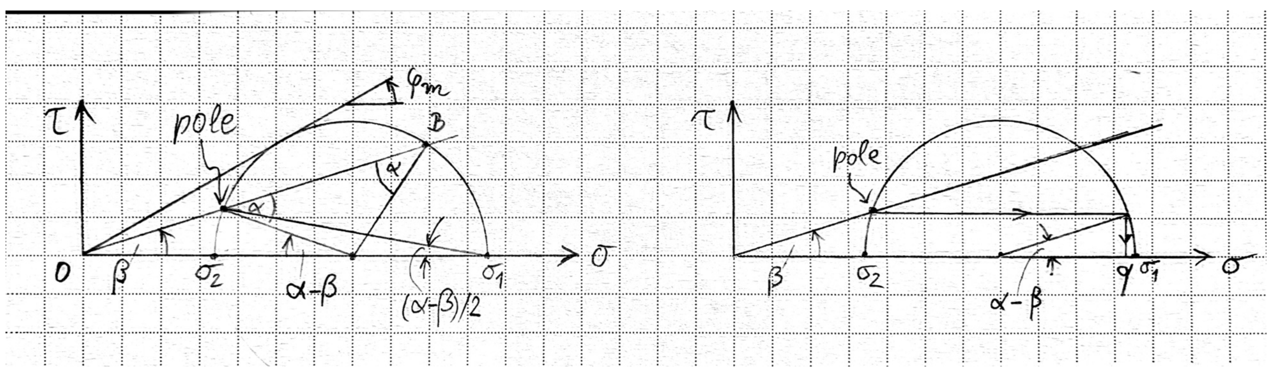

Figure 6: Mohr circle representation of the stress state in a slope.

of principal stress trajectories. The other family is perpendicular to this one. Now we apply the sinus theorem:

$$\frac{\sin\beta}{(\sigma_1-\sigma_2)/2} = \frac{\sin(180^\circ-\alpha)}{(\sigma_1+\sigma_2)/2} \rightsquigarrow \sin\alpha = \frac{\sigma_1+\sigma_2}{\sigma_1-\sigma_2}\,\sin\beta = \frac{\sin\beta}{\sin\varphi_m} \tag{7}$$

Hence, the inclination of the one family of trajectories reads

$$\frac{1}{2}\left[\arcsin\left(\frac{\sin\beta}{\sin\varphi_m}\right) - \beta\right]\ . \tag{8}$$

(c) Note that σ_A equals the distance between the origin O of the τ-σ-diagram and the pole. Similarly,

$\sigma_B = \gamma \cdot d \cdot \cos\beta$ equals the distance OB in Figure 6. Application of the sinus theorem yields:

$$\frac{(\sigma_1+\sigma_2)/2}{\sin\alpha} = \frac{\sigma_B}{\sin(\alpha+\beta)} \tag{9}$$

$$\frac{(\sigma_1+\sigma_2)/2}{\sin\alpha} = \frac{\sigma_A}{\sin(\alpha-\beta)} \rightsquigarrow \sigma_A = \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} \cdot \cos\beta \cdot \gamma \cdot d = K_A \cdot \gamma \cdot d \tag{10}$$

(11)

with

$$K_A := \frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} \cdot \cos\beta\ . \tag{12}$$

(d) Equilibrating the forces acting in the direction of the slope yields:

$$\frac{1}{2}\, K_A\, (\gamma-\gamma')\, d^2 + \frac{1}{2}\, \frac{d^2}{\tan\beta}\, (\gamma+\gamma')\ \sin\beta = \frac{1}{2}\, \frac{d^2}{\tan\beta}\, (\gamma+\gamma')\ \cos\beta\ \tan\varphi \tag{13}$$

i.e.

$$\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)} \cdot \frac{\gamma-\gamma'}{\gamma+\gamma'} + 1 = \frac{\tan\varphi}{\tan\beta}\quad . \tag{14}$$

This is an implicit equation for $\beta = \beta_{max} = f(\varphi)$. For $\varphi = 30°$ and $(\gamma-\gamma')/(\gamma+\gamma') = 1/3$ we obtain numerically the solution $\beta = \beta_{max} = 25.85°$.

(e) We refer to Fig. 7. In the regions ABF and DEF both families of the principal stress trajectories are straight lines with the inclinations obtained in (d). In the region BFD the one family of trajectories is a fan of straight lines centred at point F. The other trajectories are segments of the corresponding circles.

The distribution of vertical pressure q in the sections AB and DE can be obtained as follows. At a point in a horizontal distance x from the left edge there are contributions from σ_2 acting along the trajectory inclined by $\delta := (\alpha-\beta)/2$ and from σ_1 acting along the trajectory inclined by $90° - \delta$. From the Lamé-Maxwell equations it can be seen that $\sigma_2 = \gamma \cdot a$ and $\sigma_1 = \gamma \cdot b$ (see Fig. 7 below). I.e., the integral of a principal stress along a straight trajectory equals $\gamma \cdot \Delta h$, where Δh is the difference of elevation between initial and end points of the trajectory.

Application of the sinus theorem yields

$$a = x \cdot \frac{\sin\beta\sin\delta}{\sin(\beta+\delta)}, \quad b = x \cdot \frac{\sin\beta\cos\delta}{\cos(\beta+\delta)}\ . \tag{15}$$

From Mohr's circle (see Fig. 6, right) it follows

$$q = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2} \cdot \cos(2\delta) \tag{16}$$

E.g., for $\beta = 30°, \varphi_m = 35°$ we obtain from Equation 7:

$$\alpha = \arcsin\frac{\sin 30°}{\sin 35°} = 60.66° \rightsquigarrow \delta = (60.66° - 30°)/2 = 15.33° \tag{17}$$

From equations 15 and 16 follows

$$a = 0.1859 \cdot x, \quad b = 0.6859 \cdot x \quad \rightsquigarrow q = 0.6509 \cdot \gamma\, x\ . \tag{18}$$

Thus, q is larger than the value $\gamma \cdot x \cdot \tan 30^\circ = 0.5774 \cdot \gamma \cdot x$, which corresponds to the triangular distribution of Figure 2. To preserve the global equilibrium of vertical forces, the stresses in the central region BD (Fig. 7) must be smaller than the ones corresponding to the triangular distribution.

Now we estimate the vertical stress acting at point C. We consider the principal stress trajectory FC. At a depth z below F, this trajectory is crossed by a curved trajectory (see Fig. 8) with curvature $1/z$. We consider equilibrium of vertical stresses using the equations of Lamé-Maxwell (cf. Equation 9.24 of the book). With y being the length of FG we have $y = z/\sin\left(\frac{\alpha+\beta}{2}\right)$ and

$$x = y \cdot \sin\beta = z \cdot \frac{\sin\beta}{\sin\left(\frac{\alpha+\beta}{2}\right)} = z \cdot k \tag{19}$$

The horizontal principal stress at z is $\sigma_2 = \gamma \cdot (z - x) = \gamma \cdot z \cdot (1-k)$. According to the Lamé-Maxwell equation, vertical equilibrium reads

$$\frac{d\sigma_1}{dz} = \gamma - \frac{\sigma_1 - \sigma_2}{z} = \gamma - \frac{\sigma_1 - \gamma \cdot z \cdot (1-k)}{z} = \gamma \cdot (2-k) - \frac{\sigma_1}{z}\ . \tag{20}$$

The solution of this differential equation is

$$\sigma_1 = \gamma \cdot \frac{2-k}{2} \cdot z\ . \tag{21}$$

With $k = \frac{\sin\beta}{\sin\left(\frac{\alpha+\beta}{2}\right)} = \frac{\sin 30^\circ}{\sin 45.33^\circ} = 0.7031$ we have $(2-k)/2 = 0.6485 < 1$. I.e., σ_1 at point C is less than the value corresponding to the triangular distribution. The total stress distribution is shown in Fig. 8 right.

Exercise 7: With $t = 60/2 = 30$ N/cm^2, $\mathbf{n} = \frac{1}{\sqrt{6}}(1,\ 1,\ 2)$ and the unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in the Cartesian directions, the stress tensor reads $\mathbf{T} = t\mathbf{n} \otimes \mathbf{n}$. The first column of $\mathbf{T}$ is given by the vector $\mathbf{t}_1 = \mathbf{Ti}$. Similarly, $\mathbf{t}_2 = \mathbf{Tj}$ and $\mathbf{t}_3 = \mathbf{Tk}$. Hence, $\mathbf{t}_1 = n_1\mathbf{n}$, $\mathbf{t}_2 = n_2\mathbf{n}$ and $\mathbf{t}_3 = n_3\mathbf{n}$, i.e.

$$\mathbf{T} = \frac{30}{\sqrt{6}} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

Exercise 8: The volume fractions of grains and water are $\alpha_s = 1 - n$ and $\alpha_w = n$, respectively, where n is the porosity. From $\gamma = \gamma_s + \gamma_w = (1-n)\gamma^s + n\gamma^w$ we obtain

$$n = \frac{\gamma^s - \gamma}{\gamma^s - \gamma^w}\ ,$$

hence $n_1 = 0.94$ and $n_2 = 0.41$. With $\gamma_s = \alpha_s \gamma^s$ we obtain:

$$\gamma_1^s = (1-0.94) \cdot 27 = 1.62\,\text{kN/m}^3, \quad \gamma_2^s = (1-0.41) \cdot 27 = 15.93\,\text{kN/m}^3$$

From the jump relation for conservation of mass it follows
$\gamma_1(u - v_1) = \gamma_2 u \rightsquigarrow u = \frac{\gamma_1}{\gamma_1 - \gamma_2} v_1 = \frac{1.62}{1.62 - 15.93} \cdot 1 = -0.118$ cm in 100 years.
The minus sign indicates that u is directed upwards.

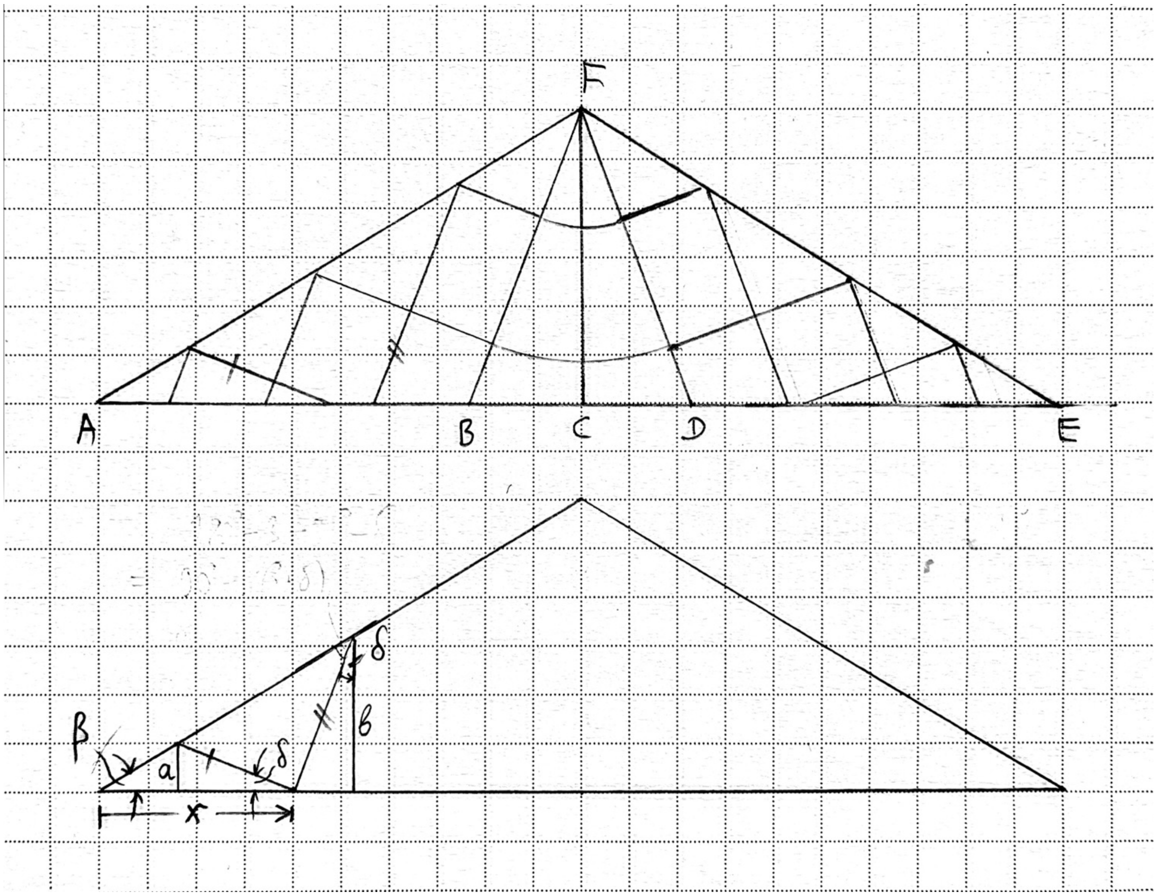

Figure 7: Principal stress trajectories in a pile of sand.

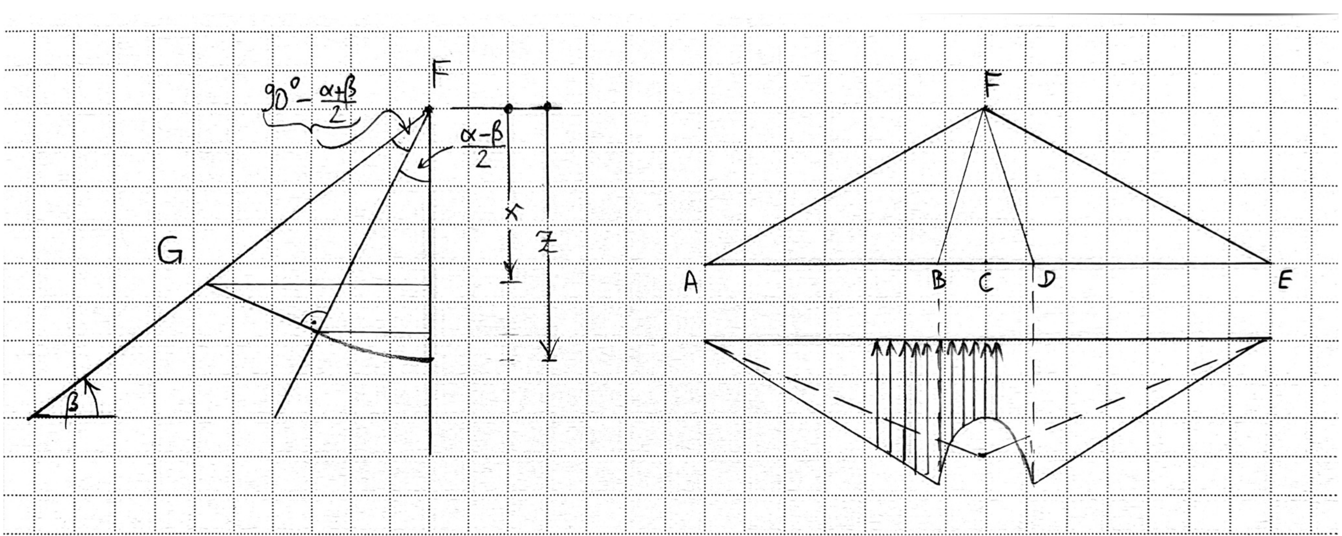

Figure 8: Principal stress trajectory in axis of symmetry and distribution of vertical stress at the basis of a sand pile. Dashed: triangular distribution.

Exercise 9: Mass conservation: $[\rho(u-v)] = 0$, i.e. $\rho_1(u-v_1) = \rho_2(u-v_2)$, hence

$$u = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 - \rho_2} = \frac{0.03 \cdot 70 - 0.16 \cdot 0}{0.03 - 0.16} = -16.15 \quad \text{km/h} \tag{22}$$

The negative sign indicates that u is oriented in the counter direction of the cars.

Exercise 10: Particles heavier than water can only float if they interact with a repulsive force. This interaction force can be seen as a sort of effective hydrostatic stress tensor $q\delta_{ij}$. Thus, the 'real' specific of the suspension equals the one of water, γ_w, but the total hydrostatic stress of the suspension at a depth z is given by $[\gamma_w + (\gamma_B - \gamma_w)]z\delta_{ij}$.

Exercise 11:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{23}$$

$$\sigma_{ii} = \lambda\varepsilon_{kk}3 + 2\mu\varepsilon_{ii} \tag{24}$$

$$\rightsquigarrow \varepsilon_{ii} = \frac{1}{2\mu}(\sigma_{ii} - 3\lambda\varepsilon_{kk}), \quad \varepsilon_{kk} \equiv \varepsilon_{ii} \ , \tag{25}$$

$$\rightsquigarrow \varepsilon_{ij} = -\frac{\lambda}{2\mu}\frac{\sigma_{kk}}{(3\lambda + 2\mu)}\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \ . \tag{26}$$

For a volume-preserving deformation, Hooke's law reads: $\sigma_{ij} = 2\mu\varepsilon_{ij}$.

Exercise 12: We denote the z-direction with 1, and the radial directions x and y with 2 and 3. We apply Equation 12.5 (right) of the book.

(a): With $\sigma_{22} = \sigma_{33} = 0$ and $\sigma_{kk} = \sigma_{11}$ we obtain

$$\varepsilon_{22} = -\frac{1}{E}\nu\sigma_{11} = -\nu\varepsilon_{11} \ . \tag{27}$$

(b):

$$\frac{d\sigma_{11}}{d\varepsilon_{11}} = E \ . \tag{28}$$

(c): We set $\varepsilon_{22} = \varepsilon_{33} = 0$ in Equation 12.5 (left) and obtain:

$$\sigma_{11} = 2G\left(\varepsilon_{11} + \frac{\nu}{1-2\nu}\,\varepsilon_{11}\right) = 2G\frac{1-\nu}{1-2\nu}\,\varepsilon_{11} = E\frac{1-\nu}{1-\nu-2\nu^2}\,\varepsilon_{11}. \tag{29}$$

The stiffness reads: $E\frac{1-\nu}{1-\nu-2\nu^2}$

(d): $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} = (1-2\nu)\varepsilon_{11} = 0 \rightsquigarrow \nu = 1/2$.

Exercise 13: (a) For oedometric compression we have $\delta = -1$ and

$$\mathbf{D}^0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \mathbf{D}^{0\star} = \frac{1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

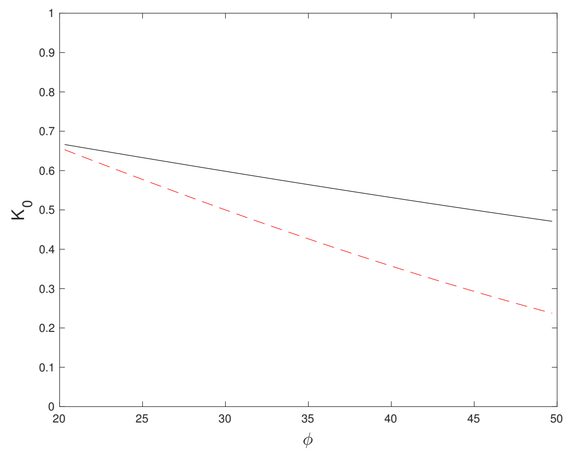

Figure 9: Relation between K_0 and φ_c. Dashed: Jaky's relation between K_0 and φ.

Thus we have

$$R_2 = -\exp\left(c_1 \cdot \frac{1}{3}\right) - c_2 \tag{30}$$

$$R_1 = -\exp\left(c_1 \cdot \frac{-2}{3}\right) - c_2 \tag{31}$$

hence

$$K_0 = \frac{R_2}{R_1} = \frac{\exp(c_1/3) + c_2}{\exp(-2c_1/3) + c_2} \tag{32}$$

With the relation 17.26 from the book,

$$c_1 = \sqrt{\frac{2}{3}} \ln\left(\frac{1 - \sin\varphi_c}{1 - \sin\varphi_c}\right), \tag{33}$$

we obtain a relation between K_0 and φ_c as plotted in Fig. 13 together with the relation $K_0 = 1 - \sin\phi$ (dashed line). Note that Jaky's relation, $K_0 = 1 - \sin\varphi$, does not refer to the *critical* friction angle φ_c.

(b) The relation between the stress $\sigma_2 = \sigma_3 = \sigma_r$ in the soil and the tensile stress σ_e in the steel ring reads: $r \cdot \sigma_r = d \cdot \sigma_e$. Hence, $\dot\sigma_2 = (r/d) \cdot E \cdot \dot\varepsilon_2 = (r/d) \cdot E \cdot D_2$. The numerical calculation proceeds in the same way as for the drained triaxial test. Instead of the equation $\dot\sigma_2 = 0$, the aforementioned equation must be numerically solved at each step to determine D_2. The ratio σ_2/σ_1 changes in the course of the deformation.

Exercise 14: The equation $\mathbf{R}(\mathbf{D}) = -\exp\left(c_1 \mathbf{D}^{0\star}\right) + c_2\delta\mathbf{1}$ is used.

(a) For oedometric deformation we have $\mathbf{D} = \mathbf{D}^0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, und $\mathbf{D}^{0\star} = \begin{pmatrix} -2/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$.

With $c_1 = -1.025, c_2 = 0.50$ and $\delta = -1$ we get: $R_1 = -\exp(-2c_1/3) - c_2 = -2,48$ und $R_2 = -\exp(c_1/3) - c_2 = -1.21 \rightsquigarrow K_0 = R_2/R_1 = 1,21/2,48 = 0,49$. **(4)**

(b) For undrained triaxial compression is $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \leadsto \mathbf{D}^{0\star} = \sqrt{\frac{2}{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \leadsto$ $R_1 = -\exp(-\sqrt{\frac{2}{3}}c_1) = -2,31$ und $R_2 = -\exp(\sqrt{\frac{2}{3}}c_1/2) = -0,66 \leadsto \sin\varphi_c = \frac{R_1 - R_2}{R_1 + R_2} = 0.556 \leadsto$ $\varphi_c = 33,8°$. **(4)**

(c) For undrained plane compression is $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leadsto \mathbf{D}^{0\star} = \sqrt{\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \leadsto$ $R_1 = -\exp(-\sqrt{\frac{1}{2}}c_1) = -2,06$ und $R_3 = -\exp(0) = -1 \leadsto K_1 = R_3/R_1 = 0,48$. **(4)**

Exercise 15: First we calculate the unit tensor $\mathbf{D}^0$ that corresponds to the dilatancy δ. We set

$$\mathbf{D}^0 = \begin{pmatrix} -\sin\alpha & 0 & 0 \\ 0 & (\cos\alpha)/\sqrt{2} & 0 \\ 0 & 0 & (\cos\alpha)/\sqrt{2} \end{pmatrix} \leadsto -\sin\alpha + \sqrt{2}\cos\alpha = \delta .$$

With $\cos^2\alpha = 1 - \sin^2\alpha$, this is a quadratic equation for $\sin\alpha$ leading to $\sin\alpha = \frac{1}{3}(\sqrt{6 - 2\delta^2} - \delta)$. For $\delta = 0.15$ we obtain $\sin\alpha = 0.7634$ and, hence,

$$\mathbf{D}^0 = \begin{pmatrix} -0.7634 & 0 & 0 \\ 0 & 0.4567 & 0 \\ 0 & 0 & 0.4567 \end{pmatrix} \leadsto \mathbf{D}^{0*} = \begin{pmatrix} -0.8134 & 0 & 0 \\ 0 & 0.4067 & 0 \\ 0 & 0 & 0.4067 \end{pmatrix}$$

In the limit state we have $\mathbf{R}^0 = \mathbf{T}^0$. Hence $\sigma_1 = \frac{R_1}{R_2}\sigma_2$. This is a relation between stress and dilatancy. With $c_1 = -1.025, c_2 = 0.50$ (for Hostun sand) we obtain:

$$R_1 = -\exp(-c_1 \cdot 0.8134) + c_2 \cdot 0.15 = -2.2269 \quad (34)$$

$$R_2 = -\exp(c_1 \cdot 0.4067) + c_2 \cdot 0.15 = -0.5841 \quad (35)$$

Hence,

$$\sigma_{1,max} = \frac{2.2269}{0.5841} 400 = 1,525.0 \text{ (kPa)} . \quad (36)$$

Exercise 16: With $\mathbf{N} := h[f\mathbf{R}^0 + g\mathbf{T}^0]$ we write the barodetic constitutive equation as $\dot{\mathbf{T}} = \mathbf{N}\,|\mathbf{D}| = (\mathbf{N} \otimes \mathbf{D}^0)\,\mathbf{D}$. Hence, $\mathbf{H} = \mathbf{N} \otimes \mathbf{D}^0$. Considering stress and strain increments and using the actual configuration as the reference one, we can write $\Delta\mathbf{T} = \mathbf{H}\,\Delta\mathbf{E}$. $\mathbf{H}$ depends on $\mathbf{T}$ and $\mathbf{D}^0 = (\Delta\mathbf{E})^0$.

Exercise 17: We use the invariants $J_1 = \text{tr}\,\mathbf{A}$, $J_2 = \text{tr}\,\mathbf{A}^2$, $J_3 = \text{tr}\,\mathbf{A}^3$ as well as the invariants I_1, I_2, I_3 as given in Section 8.5 of the book. It holds: $I_1 = J_1, I_2 = \frac{1}{2}(J_1^2 - J_2), I_3 = \frac{1}{3}(J_3 + \frac{1}{2}J_1^3 - \frac{3}{2}J_1J_2)$. We now consider the characteristic polynom

$$-\lambda^3 + I_1\lambda^2 - I_2\lambda + I_3 = 0 , \quad (37)$$

replace λ by $\mathbf{A}$ and multiply by $\mathbf{A}$ obtaining thus

$$\mathbf{A}^4 = I_1\mathbf{A}^3 - I_2\mathbf{A}^2 + I_3\mathbf{A} \leadsto \text{tr}\,\mathbf{A}^4 = I_1\text{tr}\,\mathbf{A}^3 - I_2\text{tr}\,\mathbf{A}^2 + I_3\text{tr}\,\mathbf{A} . \quad (38)$$

Exercise 18: Besides the approximation given by equation 17.25 of the book, the matrix exponential $\exp \mathbf{A}$ can in fact be strictly represented as $a\mathbf{1} + b\mathbf{A} + c\mathbf{A}^2$, and this is a consequence of the CAYLEY-HAMILTON theorem, according to which every function of $\mathbf{A}$ can be represented in this form. Let the principal values of $\mathbf{A}$ be A_1, A_2, A_3. We consider the equations

$$\exp \mathbf{A} = a\mathbf{1} + b\mathbf{A} + c\mathbf{A}^2 \tag{39}$$
$$\mathbf{A} \exp \mathbf{A} = a\mathbf{A} + b\mathbf{A}^2 + c\mathbf{A}^3 \tag{40}$$
$$\mathbf{A}^2 \exp \mathbf{A} = a\mathbf{A}^2 + b\mathbf{A}^3 + c\mathbf{A}^4 \tag{41}$$

Now we form the traces of these equations, noting that $\mathrm{tr}\mathbf{1} = 3$, $\mathrm{tr}(\mathbf{A} \exp \mathbf{A}) = A1 \exp A_1 + A_2 \exp A_2 + A_3 \exp A_3$, etc.:

$$\mathrm{tr}(\exp \mathbf{A}) = a\ \mathrm{tr}\mathbf{1} + b\ \mathrm{tr}\mathbf{A} + c\ \mathrm{tr}\mathbf{A}^2 \tag{42}$$
$$\mathrm{tr}(\mathbf{A} \exp \mathbf{A}) = a\ \mathrm{tr}\mathbf{A} + b\ \mathrm{tr}\mathbf{A}^2 + c\ \mathrm{tr}\mathbf{A}^3 \tag{43}$$
$$\mathrm{tr}(\mathbf{A}^2 \exp \mathbf{A}) = a\ \mathrm{tr}\mathbf{A}^2 + b\ \mathrm{tr}\mathbf{A}^3 + c\ \mathrm{tr}\mathbf{A}^4 \tag{44}$$

We thus have a system of three linear equations the solution of which yields a, b and c.

Exercise 19: Creep is described by the constitutive equation (b), because for $\sigma = \text{const}$ it gives $\dot{\varepsilon} \neq 0$. Relaxation is described by the constitutive equation (c), because for $\varepsilon = \text{const}$ it gives $\dot{\sigma} \neq 0$.

Exercise 20: For $\sigma = a$, this constitutive equation yields $\dot{\sigma} = 0$ for $\dot{\varepsilon} > 0$. So, a is the yield stress. The initial stiffness is $\frac{d\sigma}{d\varepsilon} = \frac{\dot{\sigma}}{\dot{\varepsilon}} = ca$.

Exercise 21: The following solution is based on the assumption that the deformation of the column is uniform and that no buckling occurs.
When the collapse load is reached, the radius of the column is r_u and the thickness of the membrane is d_u. With $r_u = 1.80 \cdot r_0$ we obtain for a volume-preserving deformation of the rubber (note that deformations are large) $d_u = d_0/1.80$. Using the known equation that relates the tensile stress in the containment with the internal pressure p in a vessel,

$$p = \frac{d}{r}\,\sigma\ ,$$

we obtain the radial stress σ_r prevailing in the sand column as

$$\sigma_r = \frac{(d_0/1.80)}{(r_0 \cdot 1.80)}\,\sigma_u = \frac{3}{30 \cdot 1.80^2} \cdot 1,200 = 37\ \text{kPa}$$

The maximum axial stress prevails at the lower edge of the column and reads

$$\frac{1+\sin\varphi}{1-\sin\varphi}\,\sigma_r = \frac{F}{\pi r_u^2} + \gamma_u h_u$$

Assuming $h_u \approx h_0$ and $\gamma_u \approx \gamma_0$ we obtain

$$F = \pi \cdot r_u^2 \cdot \left[\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \cdot \sigma_r - \gamma_0 h_0\right] = \pi \cdot (0.3 \cdot 1.80)^2 \cdot \left[\left(\frac{1+\sin 34^\circ}{1-\sin 34^\circ}\right) \cdot 37 - 18 \cdot 3\right] = 70\ \text{kN}$$

Exercise 22: Interestingly, equation 19.31 predicts the same consolidation time for a layer loaded by a small and by a large load. It is the load that controls the amount of compression in dependence of the compressibility of the grain skeleton.

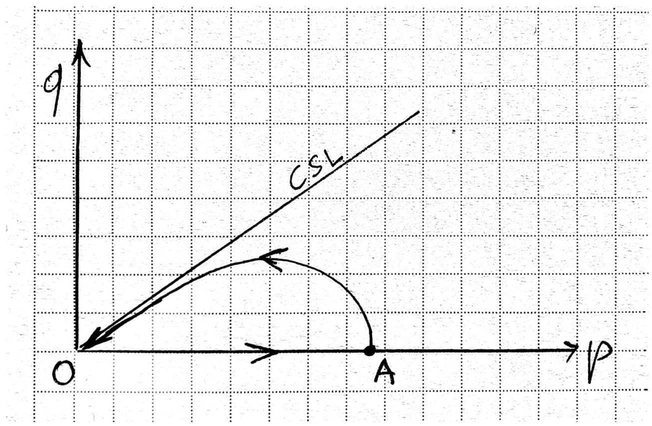

Figure 10: The curved stress path A→O leads to complete liquefaction

Exercise 23: With opening of the drainage, the sample undergoes a hydrostatic compression (path O→A in Figure 10). This path corresponds to reloading, hence the associated densification is small. In accordance with this result, cone penetration tests indicate that earthquake-induced liquefaction does not induce any significant densification.

Exercise 24: F is given by $F = W + \mu \int_0^l e(z)dz$ where W is the weight of the strip, μ is the friction coefficient and $l(t) = H - vt$. v is the pull-out velocity (assumed as constant) and H is the initial length of the embedded strip. With $F = F(l(t))$ we have $dF/dt = (dF/dl)dl/dt = -vdF/dl = -ve(l)$, hence

$$e(z) = -v\frac{dF(t = \frac{H-l}{v})}{dt} \, . \tag{45}$$

Exercise 25: The dimensional analysis of the problem refers to exactly the same variables as the ones used for the derivation of Darcy's law in Section 19.4 of the book. Here, the spacing d between two grains denotes the spacing between the glas plates.

Exercise 26: For an assumed central field $v_r = f(r), v_\phi = v_\theta = 0$, the condition div$\mathbf{v} = 0$ implies $v_r(r) = v_0 \left(\frac{r_0}{r}\right)^2$. Introducing Darcy's law $v_r = -k\,\nabla h$ yields

$$h = -\frac{v_0}{k} r_0^2 \int_{r_0}^{r} \frac{dr}{r^2} = -\frac{v_0}{k} r_0^2 \left(\frac{1}{r} - \frac{1}{r_0}\right) + H$$

Setting $h \to 0$ for $r \to \infty$ yields finally

$$v_0 = \frac{kH}{r_0} \, .$$

Exercise 27: We set $\mathbf{D} = -\mathbf{1}$ and obtain $\delta = -\sqrt{3}$. With $\mathbf{D}^{0*} = \mathbf{0}$ we obtain $\mathbf{R}^0 = -\mathbf{1}$. We furthermore have $\sigma = 735$ and

$$\mathbf{T}^0 = \frac{-1}{\sqrt{6}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \dot{\mathbf{T}} = c_4 \cdot 735^{c_5} \cdot [-(e_c - c_3 \cdot \sqrt{3})\mathbf{1} + (-e - c_3 \cdot \sqrt{3})\mathbf{T}^0] \cdot \sqrt{3} \rightsquigarrow$$

$$\dot{\mathbf{T}} = \begin{pmatrix} -177,896 & 0 & 0 \\ 0 & -211,776 & 0 \\ 0 & 0 & -211,776 \end{pmatrix}.$$

The stiffnesses are:

$$\frac{\Delta T_1}{\Delta \varepsilon_1} = \frac{\dot{T}_1}{D_1} = 177,896; \quad \frac{\Delta T_2}{\Delta \varepsilon_2} = \frac{\dot{T}_2}{D_2} = 211,776;$$

$$\Delta\sigma_1 \approx 177,896 \cdot 0.001 = 178 \text{ kPa}; \quad \Delta\sigma_2 \approx 211,776 \cdot 0.001 = 212 \text{ kPa}.$$

Exercise 28: Equating pressure and buoyancy at the bottom edge of the ice sheet gives $d \cdot \gamma_{ice} = b \cdot \gamma_w \rightsquigarrow b = (\gamma_{ice}/\gamma_w)d = 0.92$ m. A downwards displacement Δz of the ice sheet causes an increase of buoyancy $\Delta p = \gamma_w \Delta z$. Hence, the subgrade reaction modulus equals γ_w.

Exercise 29: The rate-independent part, $\mathbf{h}(\mathbf{T}, \mathbf{D}, \ldots)$, does not have any contribution to stress due to a jump of $\mathbf{D}$. So, it is not further considered here.

Differentiating the expression

$$\mathbf{G} = \text{Arsinh}\left(\frac{1}{c}\mathbf{D}\right)$$

with respect to t we obtain

$$\dot{\mathbf{G}} = \frac{d\mathbf{G}}{dt} = \frac{\dot{\mathbf{D}}}{\sqrt{c^2\mathbf{1} + \mathbf{D}^2}} = \left(\sqrt{c^2\mathbf{1} + \mathbf{D}^2}\right)^{-1}\dot{\mathbf{D}}$$

Thus, the increase of stress due to a change of $\mathbf{D}$ reads

$$\Delta\mathbf{T} = \int_{t_1}^{t_2} \dot{\mathbf{T}} dt = b \text{ Arsinh}\left(\frac{1}{c}\mathbf{D}\right)_{t_1}^{t_2} = b \ln\left(\frac{1}{c}\mathbf{D} + \sqrt{\mathbf{1} + \frac{1}{c^2}\mathbf{D}^2}\right)_{t_1}^{t_2} \tag{46}$$

With $\mathbf{D}_1 := \mathbf{D}(t = t_1)$ and $\mathbf{D}_2 := \mathbf{D}(t = t_2)$ we thus have:

$$\Delta\mathbf{T} = b \ln\left(\frac{1}{c}\mathbf{D}_2 + \sqrt{\mathbf{1} + \frac{1}{c^2}\mathbf{D}_2^2}\right) - b \ln\left(\frac{1}{c}\mathbf{D}_1 + \sqrt{\mathbf{1} + \frac{1}{c^2}\mathbf{D}_1^2}\right) \tag{47}$$

Taking that for large positive values of x ($x \gg 1$) holds:

$$\ln(x + \sqrt{1 + x^2}) \approx \ln(x + \sqrt{x^2}) = \ln(2x) \ ,$$

we can retrieve the logarithmic relation (19.24) of the book for $c \ll |\mathbf{D}|$.

Exercise 30: For groundwater percolating above the water table the downwards oriented hydraulic gradient equals 1.

1. Thus, the vertical filter velocity equals k and, hence, the run-out time on the earth equals $t_e = h/k = 0.15 \cdot 10^5$ s = 4.17 h.

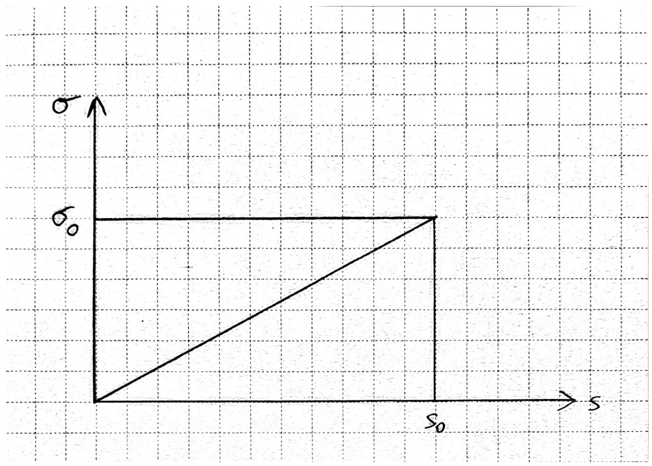

Figure 11: The works for deformation of the soil and squeezing the water correspond to the triangular areas shown. Herein, a constant compressibility of the grain skeleton is assumed.

2. With $k/k_{moon} = g/g_{moon}$ we obtain $k_{moon} = k \cdot (g_{moon}/g) = k \cdot (1.63/9.81)$. Hence, the run-out time on the moon equals $4.17 \cdot (9.81/1.63) = 25.1$ h.

Exercise 31: Assuming a constant compressibility of the soil (this is the usual assumption in the consolidation theory) leads to the partition of work shown in Fig. 11, i.e. the works spent for deformation and for water flow read both:

$$W = \frac{1}{2}\sigma_0 s_0$$

This presupposes that the load σ_0 is applied instantaneously.

Exercise 32:

(a) Tidal deformation: With **i** and **j** being the unit vectors in x and y directions, and if the moon is in the direction of **j** and elongates the earth radius by the factor λ, the deformation is given by

$$x_1 = \lambda^{-1} X_1 \tag{48}$$

$$x_2 = \lambda X_2 \tag{49}$$

Consequently, we have

$$\mathbf{F}_0 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} .$$

Now we let this deformation rotate with the angular velocity ω, i.e.

$$\mathbf{F} = \mathbf{R}\mathbf{F}_0\mathbf{R}^T \qquad \text{with} \qquad \mathbf{R} = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} , \tag{50}$$

and obtain

$$\mathbf{F} = \begin{pmatrix} \lambda^{-1}\cos^2\omega t + \lambda\sin^2\omega t & (\lambda^{-1}-\lambda)\sin\omega t\cos\omega t \\ (\lambda^{-1}-\lambda)\sin\omega t\cos\omega t & \lambda\cos^2\omega t + \lambda^{-1}\sin^2\omega t \end{pmatrix} \tag{51}$$

$$= \frac{1}{2}\begin{pmatrix} \lambda^{-1}(1+\cos 2\omega t)+\lambda(1-\cos 2\omega t) & (\lambda^{-1}-\lambda)\sin 2\omega t \\ (\lambda^{-1}-\lambda)\sin 2\omega t & \lambda(1+\cos 2\omega t)+\lambda^{-1}(1-\cos 2\omega t) \end{pmatrix} \tag{52}$$

$$= \begin{pmatrix} \Lambda_1 + \Lambda_2\cos 2\omega t & \Lambda_2 \sin 2\omega t \\ \Lambda_2\sin 2\omega t & \Lambda_1 - \Lambda_2\cos 2\omega t \end{pmatrix} \tag{53}$$

with

$$\Lambda_1 = \frac{1}{2}(\lambda^{-1}+\lambda) \qquad \text{and} \qquad \Lambda_2 = \frac{1}{2}(\lambda^{-1}-\lambda)\,. \tag{54}$$

We can easily check that det$\mathbf{F} = 1$, i.e. this deformation is volume (or, here, area) preserving.

The relation $\mathbf{x}(\mathbf{X}, t)$ reads:

$$x_1 = (\Lambda_1 + \Lambda_2\cos 2\omega t)\; X_1 + \Lambda_2\sin 2\omega t\; X_2 \tag{55}$$

$$x_2 = \Lambda_2\sin 2\omega t\; X_1 + (\Lambda_1 - \Lambda_2\cos 2\omega t)\; X_2\,. \tag{56}$$

This is an affine (linear) relation between $\mathbf{x} = (x_1\,;x_2)$ and $\mathbf{X} = (X_1\,;X_2)$. Straight lines remain straight. $\mathbf{x}(\mathbf{X}, t)$ describes the trajectory of the particle $\mathbf{X}$. In our case, the trajectories are circles (check!).

Differentiating equation 53 with respect to t yields $\mathbf{v} = \dot{\mathbf{x}} = \dot{\mathbf{F}}\mathbf{X}$. The velocity gradient $\mathbf{L}$ refers, however, to the spatial coordinates $\mathbf{x}$. To express $\mathbf{v}$ in dependence of $\mathbf{x}$, we invert (48) and (49) obtaining $\mathbf{X} = \mathbf{F}^{-1}\mathbf{x}$ and, hence, $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. To invert $\mathbf{F}$ we use the relation

$$\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det\mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\,, \tag{57}$$

hence

$$\mathbf{F}^{-1} = \begin{pmatrix} \Lambda_1 - \Lambda_2\cos 2\omega t & -\Lambda_2\sin 2\omega t \\ -\Lambda_2\sin 2\omega t & \Lambda_1 + \Lambda_2\cos 2\omega t \end{pmatrix} \tag{58}$$

and

$$\mathbf{L} = 2\omega\Lambda_2\begin{pmatrix} -\Lambda_1\sin 2\omega t & \Lambda_1\cos 2\omega t + \Lambda_2 \\ \Lambda_1\cos 2\omega t - \Lambda_2 & \Lambda_1\sin 2\omega t \end{pmatrix} \tag{59}$$

Again, div$\mathbf{v}$ = tr$\mathbf{L} = 0$ indicates constant volume (area).

(b) Uniqueness: In Section 18.2 of the book is stated that an affine deformation $\mathbf{x} = \mathbf{AX}$ can be obtained if the constitutive condition given by equation 18.8 is fulfilled *and* the imposed boundary velocity is also given by the matrix $\mathbf{A}$, i.e. $\mathbf{v} = \mathbf{Ax}$. However, a rotating stencil can only impose a velocity normal to its surface, whereas the tidal deformation implies also velocities that are *not* normal to the stencil. Consider e.g. the point $\mathbf{x} = (1\,;\;0)$ at the right apex of the stencil at the time $t = 0$. Its velocity $\mathbf{v} = \mathbf{Lx}$ reads $(0\,;\;-2\omega\Lambda_2\lambda)$ and thus has only a component tangential to the stencil. So, the deformation imposed by the rotating stencil is not necessarily affine (homogeneous).

A cyclic deformation of an elastic body requires no energy in addition to the one spent for the initial deformation. This is similar to a 1D wave that propagates within an elastic rod. Once this rod is deformed, no additional energy is needed. In the course of time, the deformation affects different particles, but the deformation itself is not altered. Of course, in reality we always have damping.

Referring to tidal deformation, it is conceivable that the outer parts of the earth, being softer, get more and more involved in tidal locking, i.e. they start rotating, and the angular velocity (being initially very small) increases with increasing distance from the center of the earth. In this way, initially straight lines through the center become spiral.

(c) Tidal locking: We have a rigid body rotation: $\mathbf{F} = \mathbf{R}$.

Exercise 33: At the peak of the triaxial compression σ_1 is the maximum stress. Es gilt $\frac{\sigma_1 - \sigma_2}{\sigma_1 + \sigma_2} = \sin\varphi \leadsto \frac{\sigma_1}{\sigma_2} = \frac{1+\sin\varphi}{1-\sin\varphi}$. With $\varphi = 30°$ follows $\sigma_1 = 3\sigma_2 = 3 \cdot 200 = 600$ kPa.
At the peak of the triaxial extension, σ_1 is the minimum principal stress. It holds $\frac{\sigma_2 - \sigma_1}{\sigma_2 + \sigma_1} = \sin\varphi \leadsto \frac{\sigma_1}{\sigma_2} = \frac{1-\sin\varphi}{1+\sin\varphi}$. With $\varphi = 30°$ follows $\sigma_1 = \frac{1}{3}\sigma_2 = \frac{1}{3} \cdot 200 \approx 67$ kPa.