Problems for Dynamics of Galaxies, Second Edition

The set of problems provided in this book is meant primarily to offer the opportunity for additional discussion of a few selected points that are judged to be interesting. In many cases the resolution is very simple, and the problem is an excuse to attract the attention to the scales associated with some important quantities. In other cases the resolution may require a non-trivial analytical discussion or a simple numerical investigation. Sometimes the problem is a way to focus on some derivation aspects that are thought to be instructive. In only very few cases are the problems plain exercises, that is, a routine application of formulas provided in the text.

Chapter 1

1. What is the ratio of the gravitational to the electrostatic force between two electrons?
2. What is the typical angular size of a galaxy at a distance of 100 Mpc?
3. What is the minimum angular resolution required to detect directly the rotation of M31 by comparing images taken one year apart? (From Fig. 20.1, refer to conditions applicable at a distance from the center of $\approx$10 kpc, where the rotation speed is $\approx$250 km s^{-1}.)
4. As will be shown in Part II (see Section 6.3 and Chapter 12), a natural dynamical frequency is set by the quantity $(G\rho)^{1/2}$, where ρ is the mass density and G is the gravitational constant. What is the related natural dynamical time scale thus associated with the solar neighborhood in our Galaxy? What is the corresponding scale associated with the cosmological density 10^{-29} g cm^{-3}? Similarly, what is the scale appropriate for dynamical processes inside the Sun? And for a neutron star?
5. What are the relevant time scales for stellar evolution? Check the value of the Kelvin-Helmholtz time scale for the Sun (based on an estimate of the available gravitational energy and the solar luminosity; the virial theorem states the way energy is divided into thermal and gravitational forms). Check the value of the nuclear burning time (based on the available energy $0.1M_\odot c^2$ (where c is the speed of light) and the efficiency 0.7 percent (appropriate for the fusion of four protons into an α-particle).
6. In Chapter 13 we will show that particles in a disk have a typical epicyclic excursion of $c/\kappa \approx c/\sqrt{2}\Omega$, where c is the velocity dispersion of the particles and Ω is the mean angular velocity around the galaxy. In the solar neighborhood, consider a system of clouds of atomic hydrogen, characterized by $c = 5$ km s^{-1}, and a population of disk stars, characterized by $c = 30$ km s^{-1}. Estimate the size of the epicyclic excursion in the two cases.
7. What is the scale of the acceleration experienced by the Sun in its orbit around the center of the Milky Way Galaxy? What is the corresponding scale for a star in a globular cluster?

(Refer to 5 km s^{-1} as a typical velocity and 10 pc as a typical radial distance from the center of the cluster.) And for a galaxy inside a cluster of galaxies? (For the corresponding velocity and radius, refer to 1,000 km s^{-1} and 500 kpc.)

8. Compare the value of the gravitational binding energy for a short-period binary star with that of a globular cluster.
9. One supernova explosion may release 10^{51} erg of energy. How much mass (in solar units, $M_\odot$) would a clump of atomic hydrogen moving at a speed of 100 km s^{-1} have to possess in order to carry a similar amount of kinetic energy?
10. Suppose that a quasar shines with an intrinsic power of 10^{47} erg s^{-1}. If the power derives from accretion onto a massive black hole, what is the scale for the accretion rate involved in the process (in units of $M_\odot$ yr^{-1})?

Chapter 2

1. Consider a 1-W lamp shining at the distance of the Moon ($\approx$400,000 km). What would be its apparent magnitude?
2. From the well-known expression of the energy levels of the hydrogen atom $E_n = -(13.6\text{ eV})/n^2$, check that the transitions to the third level (Paschen lines) and to the fourth level (Brackett lines) fall in the infrared (IR) region. Here the positive integer $n = 1, 2, 3, \ldots$ labels the sequence of energy levels.
3. Suppose that hot ionized hydrogen is confined in quasi-hydrostatic equilibrium by the gravitational field of a spherical galaxy. Suppose that the star velocity dispersion in such galaxy is $\approx$300 km s^{-1}. What is the natural temperature of the hot gas, and which observational window is expected to show evidence of it?
4. A massive elliptical galaxy (such as NGC 4472) may have an absolute optical luminosity (in the B band) of 2×10^{11} L$_\odot$ and an x-ray power of 10^{42} erg s^{-1} (in a soft x-ray band). Consider such a galaxy as being located at the distance of the Virgo cluster, and ignore losses along the path from the source to the Earth. What is an order of magnitude for the number of optical photons collected from such a source on a large telescope (8-m in diameter) in an exposure of 1 min? What is the corresponding number for the soft x-ray photons on a large x-ray telescope (effective collecting area of $\approx$1,000 cm^2)? (For simplicity, refer to x-ray photons of 1 keV.)
5. How does the resolution set by the diffraction limit for very long-baseline interferometry (based on antennas placed in different continents) in the radio compare with that for the *Hubble Space Telescope* in the visible?
6. What is the resolution that is required to resolve the Schwarzschild radius of the massive black hole at the center of the Milky Way Galaxy?
7. ALMA is planned to work in the wavelength range 0.3 to 9.6 mm, with baselines for the $\approx$70 antennas of the final configuration on scales from 15 m to 18 km. What is the expected best resolution that will be available?
8. Many space telescopes (among which *Herschel* and *Planck*) operate or are planned to operate at a location very close to the Lagrangian point L_2, at $\approx$1.5 million km from the Earth, in the direction Sun–Earth, opposite to the Sun. How much of the light from the Sun is expected there to be intercepted by the Earth?

9. If the velocity distribution of a stellar system is isotropic, show how a combined measurement of velocities along the line of sight (based on the Doppler effect) and tangential velocities (based on a study of *proper motions*, i.e., detection of position changes with time by accurate astrometry) for a set of stars can lead to a determination of the distance to the stellar system.
10. In March 2008 the gamma-ray burst GRB 080319B became briefly visible with the naked eye (it reached an apparent magnitude of between 5 and 6). Its measured redshift ($z = 0.94$) corresponds to a distance of $\approx$7.5 billion light-years. By ignoring cosmological correction factors in the conversion of apparent into absolute luminosity, what would be the associated power in the visible region if the light emission were isotropic?

Chapter 3

1. An ellipse can be seen as an $m = 2$ deformation of a circle. What are the Fourier characterizations of a "boxy" and a "disky" departure from a pure ellipse?
2. Imagine that we represent, at a given instant, a two-dimensional density distribution by Fourier analysis, for example, as $\sigma(r,\phi) = \sigma_0(r) + \sigma_1(r)\cos[-\phi + \psi_1(r)] + \sigma_2(r)\cos[-2\phi + \psi_2(r)] + \cdots$. What are the simplest conditions on the functions $\sigma_n(r)$ and $\psi_n(r)$ for the "perturbation" (i.e., the nonaxisymmetric part) to be representative of an m-armed spiral pattern, leading or trailing, open or tightly wound, of small or large amplitude, with amplitude modulation along the arms, grand design or flocculent, symmetric or lopsided, that is, representative of all the attributes commonly used to describe the morphology of spiral structure? What is the main characteristic of a bar? What is the main characteristic of a "linear" perturbation in contrast with a "nonlinear" one? (Many of these words, which are broadly used qualitatively, will find their proper quantitative definition later in the book, especially in Part III.)
3. In view of the representation introduced in the preceding problem, what is the main property of a logarithmic spiral, that is, a spiral associated with radial phase $\psi(r) \propto \ln r$?
4. If an axisymmetric thin mass density distribution is represented by $\rho(r,z) = \sigma(r)\delta(z)$, where δ is the Dirac delta function, what would be the simplest representation of a warped disk (as suggested by Fig. 3.4)? The large-scale warp of our Galaxy is asymmetric (i.e., it is stronger on one side with respect to the opposite side of the Galaxy). What would be a simple way to characterize this feature in a Fourier representation of the warp? The gas disk of our Galaxy appears to be affected by corrugations, much like those of a ballerina skirt. What would be a way to characterize such a feature in a Fourier representation of the warp?
5. Consider an axisymmetric ellipsoid (spheroid) characterized by ellipticity $\epsilon = 1 - b/a$, with a and b the length of the long and short axes, respectively. The spheroid can be either oblate (if the short axis is the symmetry axis) or prolate (if the long axis is the symmetry axis). If the symmetry axis of a spheroidal galaxy is inclined at an angle i with respect to the direction of observation, what is the observed ellipticity corresponding to a given intrinsic ellipticity?
6. Suppose that elliptical galaxies are spheroidal systems with ellipticity distribution peaked around the most probable value $\bar{\epsilon}$. What would be the distribution of observed ellipticities if the inclination angle of the symmetry axis with respect to the direction of observation

is randomly distributed? Under the hypothesis of random inclinations, can the intrinsic ellipticity distribution be derived from the observed ellipticity distribution?

7. Suppose that elliptical galaxies are spheroidal systems. Identify one physical factor that may favor the idea that they should be oblate and a different physical factor that might favor the possibility that they are prolate. (Many of these concepts will be addressed later in the book, especially in Chapters 10 and 22.)
8. For the two cases considered in the preceding problem, discuss whether the assumption of arbitrary inclination of the intrinsic symmetry axis with respect to the line of sight may be plausible or not.
9. If elliptical galaxies are rotating spheroids, what is a natural expectation for the velocity along the line of sight measured at different positions along the minor axis? Qualitatively, what would we expect instead, by making a similar measurement along the major axis?
10. Given that spiral galaxies can be modeled, in a first approximation, as zero-thickness disks in differential rotation [$\mathbf{V} = r\Omega(r)\mathbf{e}_\theta$; here $\mathbf{V}$ represents the velocity field, r and θ are standard polar coordinates, and $\mathbf{e}_\theta$ is the unit vector in the tangential direction], what would be the structure of the contours of equal velocity along the line of sight for a disk at an inclination i with respect to the observer?

Chapter 4

1. With the help of a simple numerical integration, check that half the total luminosity associated with the $R^{1/4}$ profile defined by Eq. (4.3) comes from the region inside the circle of radius R_e. Check the relation between the half-light radius R_e and the exponential scale h applicable to the exponential luminosity profile (see Subsection 4.1.1). For the two prototypical luminosity profiles, what is the value of $R_{(9/10)}/R_e$, where $R_{(9/10)}$ is the radius associated with 90 percent of the total luminosity?
2. Consider a galaxy with an exponential luminosity profile, with scale length h and a rotation curve that remains flat beyond a given radius $R_{\max} \approx 2.5h$. Suppose that inside such a radius the rotation curve can be well fitted by a maximum-disk prescription, based on a constant mass-to-light conversion of the disk luminosity profile into a disk density profile. As an order of magnitude, what is the increase in the *local* value of the mass-to-light ratio for the disk that is required for justifying a flat rotation curve out to $10h$ without invoking the presence of a spheroidal dark halo? (It may be useful to refer to properties of the singular self-similar model with disk density distribution $\propto 1/r$ that will be described in Subsection 14.3.1 and to the properties of the exponential disk described in the same subsection.)
3. For a given spiral galaxy, the value of the flat part of the rotation curve is overestimated by 20 km s^{-1} (with respect to a true value of 250 km s^{-1}). What is the implied error in distance determination if one uses the luminosity-velocity relation (4.11)?
4. Consider a dynamical determination of the mass M of a galaxy. If the distance to the galaxy is set by application of the Hubble expansion law, with constant $H_0 = 100h$ km s^{-1} Mpc^{-1}, how does the resulting estimate of the mass-to-light ratio M/L scale with h?
5. Consider a system of stars with a spherical density distribution of the form $\rho/\rho_0 = [(r/b)(1+r/b)]^{-2}$, where ρ_0 and b are scale parameters. Suppose that on the (x, y) plane all stars move on perfectly circular orbits around the center, all in the same direction about the z axis, with

angular velocity $\Omega/\Omega_0 = (b/r)(1 + r/b)^{-1/2}$. Suppose that we observe this system from the x direction by using a slit along the y axis. What is the line-of-sight velocity profile at distance R from the center for this perfectly cold system? (The mass model used here is briefly described in Subsection 22.4.4.) Compare these results, to be obtained with the help of simple numerical integrations, with those obtained for a disk observed edge-on, with density falling off exponentially [$\propto \exp(-r/h)$] and with angular velocity $\Omega/\Omega_0 = (1 + r^2/b^2)^{-1/2}$.

6. By extrapolating relation (4.17) from the case of M87 ($M_\bullet \approx 3 \times 10^9 M_\odot$, $\sigma_0 \approx 300$ km s^{-1}) down to the case of small stellar systems, what would be the expected mass range of a central black hole in globular clusters?
7. If elliptical galaxies were a class of structurally homologous stellar systems with no significant variations of the relevant mass-to-light ratio from galaxy to galaxy, what would be the exponents predicted in the fundamental plane relation by the virial theorem?
8. Consider elliptical galaxies as a class of structurally homologous stellar systems, and use the virial theorem condition to eliminate the velocity dispersion from the fundamental plane relation (4.15). By referring to the variables effective radius R_e and total luminosity L (instead of R_e and SB_e), show that given the values for α and β reported in the text, the R_e dependence basically cancels out and the fundamental plane relation is equivalent to a mass-luminosity relation (e.g., see the article by Bertin, Ciotti, and Del Principe 2002).
9. Consider an exponential disk with density $\sigma = \sigma_0 \exp(-r/h)$; let M be its integrated mass out to $r = 3h$. Then consider a Maclaurin spheroid with ellipticity $1 - a_3/a_1 = 0.7$ and equatorial radius $a_1 = 3h$ and a sphere of radius $R = 3h$, all with the same given mass M. In view of Eq. (4.7), compare the equatorial gravitational accelerations at a point at distance $3h$ from the center obtained for the three different cases (with the help of the formulas provided in Chapter 10 and Subsection 14.3.1).
10. By borrowing some formulas recorded in Chapter 10, estimate the value of the parameter λ defined in Eq. (4.9) for a Maclaurin spheroid with ellipticity $1 - a_3/a_1 = 0.7$.

Chapter 6

1. Compare the acceleration experienced by the Sun in its orbit around the Galactic Center with the acceleration induced by the closest star.
2. Compare the acceleration experienced by the Sun in its orbit around the Galactic Center with the vertical acceleration experienced by a star in the solar neighborhood just off the plane of the disk of the Galaxy.

Chapter 7

1. In a simple model in which molecules collide as hard spheres of radius R, with total (geometric) cross section $\sigma_{\text{tot}} = 4\pi R^2$, estimate the typical relaxation rate $\nu = n v_{th} \sigma_{\text{tot}}$ in air under standard conditions (here v_{th} represents the relevant thermal speed).
2. Based on approximation (7.4), estimate a typical stellar dynamical relaxation time and a typical star mean free path for various conditions applicable to stellar systems. In particular, refer to globular clusters (core or half-light radius conditions), to elliptical galaxies (core or

half-light radius conditions), to the solar neighborhood in the disk of our Galaxy, and to the case in which the scattered pointmasses are the galaxies inside a cluster of galaxies.

3. Based on Eq. (7.16), estimate the mean free path for electron-electron encounters inside a plasma under thermonuclear conditions (e.g., $n \approx 10^{15}$ cm^{-3}, $T \approx 10$ keV), and compare with the size of a few meters, applicable to the current toroidal experiments.
4. Estimate the mean free path for conditions applicable to the solar-wind plasma (e.g., $n \approx 5$ to 10 cm^{-3}, $T \approx 3$ to 10 eV), and compare with the value of 1 AU $\approx 1.5 \times 10^{13}$ cm.
5. Prove that Eq. (7.13) holds for a central force independent of the law of interaction.
6. Consider a satellite spiraling in, on quasi-circular orbits, being captured at the outskirts of a spherical primary galaxy (considered to be immobile), with star density $\rho \sim r^{-4}$ (see Chapter 22 for a discussion of this choice). If we take, for simplicity, $F(v) \ln \Lambda \approx$ constant in the dynamical friction equation, what is the anticipated behavior of the capture process in terms of the function $r = r(t)$? What happens, within the simple model of Eq. (7.19), when the satellite reaches the inner regions where $\rho \sim r^{-2}$? What happens if we now take $F(v) \ln \Lambda \propto r$?
7. For two globular clusters, NGC 5139 and NGC 6341, the estimated mass and radius of the sphere that contains half the mass of the system (M, r_M) are, respectively, 3×10^6 $M_\odot$ and 8 pc and 3×10^5 $M_\odot$ and 2 pc. Which cluster is expected to evaporate faster?
8. For a massive cluster of galaxies ($M \approx 10^{15}$ $M_\odot$; characteristic radius $R \approx 0.5$ Mpc), most of the mass ($\approx$85 percent) is in the form of dark matter. What is the expected range for the dynamical friction (in terms of the time scale T_{fr}) suffered by a galaxy of mass $M_g = 10^{11}$ $M_\odot$ belonging to the cluster?
9. A particle moves along the z axis in the potential $\Phi(z) = \alpha z^4$, with α a given positive constant. What is the dependence of the period as a function of energy $\tau = \tau(E)$? Is there any difference to be noted with respect to the simple example illustrated in Fig. 7.3? Compare with the motion of an exact pendulum, for which the specific potential is $\Phi(\theta) = -\omega_p^2 \cos\theta$, where ω_p is the frequency of small oscillations.
10. A nonrotating homogeneous sphere of radius R and initial virial ratio $2K/|W| = \epsilon \ll 1$ undergoes violent relaxation, reaching a spherical quasi-equilibrium state characterized by density profile $\rho/\rho_0 = [(r/b)(1+r/b)]^{-2}$, where ρ_0 and b are scale parameters. Suppose that during the process the total mass and total energy $E_{\rm tot} = K + W$ (the sum of total kinetic energy K and gravitational energy W) are conserved. What is the value of the final half-mass radius b in units of R?

Chapter 8

1. Transform the equations of a zero-thickness barotropic fluid disk, recorded in Subsection 8.3.2, by moving to a rotating frame of reference, defined by coordinates $(R,\phi) = (r, \theta - \Omega_p t)$ and velocities $(U,V) = (u, v - \Omega_p r)$. Identify the terms associated with the Coriolis and centrifugal forces.
2. For a Maxwell-Boltzmann distribution function $f = A \exp[-mv^2/(2\mathrm{k}T)]$, where A is a normalization constant, m is the mass of the gas particles, k is the Boltzmann constant, and T the gas temperature, prove that pressure is indeed a scalar (i.e., that $p_{ij} = p\delta_{ij}$) and that the equation of state is that of a perfect gas (i.e., $p = n\mathrm{k}T$, where n is the number density of gas particles).

3. For a stellar system described by an assigned distribution function f, show that the total kinetic energy K_{tot} can be divided in two contributions, one associated with the fluid motions $\mathbf{u}$ and the other (the internal energy) associated with the random motions related to the pressure tensor p_{ij}. For a stellar system characterized by scalar pressure $p = p(\mathbf{x})$, what is the expression of the internal energy in terms of p?
4. Prove that for any system characterized by distribution function of the form $f = f(E, J^2)$ (where E is the specific energy and J is the specific angular momentum), the system does not possess fluid motions (i.e., $\mathbf{u} = 0$), and the pressure tensor in spherical coordinates is diagonal, with $p_{\theta\theta} = p_{\phi\phi}$.
5. Prove that for any axisymmetric system characterized by distribution function of the form $f = f(E, J_z)$ (where E is the specific energy and J_z is the specific angular momentum component along the symmetry axis), the pressure tensor in cylindrical coordinates (R, z, ϕ) is diagonal, with $p_{RR} = p_{zz}$.
6. Extend the discussion provided in Subsection 8.3.1 to a 3D axisymmetric stellar system, and prove that for a system characterized by distribution function $f = f(E, J_z)$ (where E is the specific energy and J_z is the specific angular momentum component along the symmetry axis), the following hydrostatic equilibrium conditions hold (often called the *Jeans equations*):

$$\frac{1}{\rho}\frac{\partial}{\partial z}(\rho\langle v_R^2\rangle) + \frac{\partial \Phi}{\partial z} = 0,$$
$$\frac{1}{\rho}\frac{\partial}{\partial R}(\rho\langle v_R^2\rangle) + \frac{\langle v_R^2\rangle - \langle v_\phi^2\rangle}{R} + \frac{\partial \Phi}{\partial R} = 0.$$

   Here (R, z, ϕ) are standard cylindrical coordinates, and $\langle\rangle$ represents average in velocity space.
7. Write the general form of the collisionless Boltzmann equation in spherical coordinates $(r, \theta, \phi, v_r, v_\theta, v_\phi)$ by applying the procedure described in Subsection 8.3.1, that is, by starting from the equation in the associated canonical coordinates $(r, \theta, \phi, p_r, p_\theta, p_\phi)$. (This step is required to set up the stability analysis of spherical stellar systems, as will be presented in Section 23.1.)
8. From the collisionless Boltzmann equation and the Poisson equation, as given in Section 8.2, derive the expression of the scalar virial theorem, and state the conditions under which it can be written in the standard form $W + 2K_{\mathrm{tot}} = 0$.
9. For a spherical self-gravitating system, show that radial oscillations around virial equilibrium are bound to occur at a frequency comparable with $(4\pi G\rho)^{1/2}$, where ρ is the average density. Provide conditions for this statement to be correct.
10. Set up an asymptotic analysis for the boundary-layer problem described in Subsection 8.5.1, and compare the results obtained with the exact analytical calculation that is available in this simple case.

Chapter 9

1. Complete the derivation of the eigenvalues that demonstrate the secular instability of the $\theta = 0$ equilibrium point for the fast-rotating bowl described in Subsection 9.2.1. {*Hint*:

Expand Eq. (9.13) around the solution ω_0 of the problem without friction $[D_0(\omega_0) = 0]$ that becomes neutral ($\omega_0 = 0$) at the bifurcation point ($\xi = 1$).}

2. Consider a slab model of a collisionless stellar system, that is, a model in which the mean potential Φ depends on only the vertical coordinate z. Construct two different equilibrium distribution functions with assigned velocity dispersions c_x, c_y, c_z and with assigned streaming velocity u_y in the y direction. [For the purpose of this problem, it is not necessary to determine $\Phi(z)$ from the self-consistency requirement.]
3. For a collisionless spherical galaxy model described by the distribution function $f = A\exp(-aE - cJ^2)$, where $E = v^2/2 + \Phi(r)$ is the specific energy, J is the magnitude of the specific angular momentum, and A, a, and c are positive constants, what is the behavior of the pressure anisotropy with radius?
4. Consider a spherical system characterized by distribution function $f = A(-E)^s$, for $E < 0$ (and $f = 0$ otherwise), where $E = v^2/2 + \Phi(r)$ is the specific energy and s is a positive constant; the condition $\Phi = 0$ defines the outer surface of the system. Show that the resulting density distribution is of the form $\rho(r) = B[-\Phi(r)]^{s+3/2}$ and that the equation of state is that of a polytrope, $p = K\rho^{(n+1)/n}$, where $n = s + 3/2$.
5. Suppose that a stellar system of very small total mass is characterized by a Maxwell-Boltzmann distribution function $f = A\exp(-aE)$, where a and A are positive constants and E is the single-star specific energy, and is embedded in the approximately homogeneous core of a dominant, spherical dark halo. What is the resulting stellar density distribution?
6. Consider a stellar system characterized by an isotropic distribution function $f = f(E)$ that would be spherical if isolated. Place such a system on a circular orbit at angular velocity Ω around the center of a spherical hosting galaxy so that the star energy is no longer an integral of the motion, and argue, by virtue of the Jeans theorem, that the system is described by the function $f = f(H)$, where $H = E - p_\phi\Omega$ is the relevant Jacobi integral. Prove that in general the stellar system is characterized by a triaxial density distribution and isotropic pressure tensor. (This is the starting point for the construction of some interesting models that will be presented in Chapter 22.)
7. Consider a dark halo characterized by an isotropic distribution function $f = f(E)$ that would be spherical if isolated. Embed in it an axisymmetric disk with a given mass distribution, for simplicity an infinitesimally thin exponential disk characterized by surface density $\sigma = \sigma_0 \exp(-r/h)$. What can we say about the expected density distribution of the dark halo? (This is the starting point for the construction of some interesting models that will be presented in Chapter 14.)
8. Discuss the possibility of an equilibrium configuration based on the Jeans theorem generalizing the solution given in Subsection 9.1.1 to the case in which ions and electrons have different temperatures.

Chapter 10

1. From the expression of the potential inside a homogeneous ellipsoid (10.2), prove that identity (10.5) holds.
2. Provide approximate expressions for the quantities A_i for quasi-spherical oblate and prolate spheroids. Provide approximate expressions for the same quantities in the disk and needle limit, that is, $e \to 1$.

3. Expand the dimensionless energy $g(e,\eta)$ [see Eq. (10.17)] for small values of e, up to $O(e^4)$ included, for oblate and prolate spheroids. Is one of the two types of deviations from spherical symmetry favored from the gravitational energy point of view at fixed values of mass M and volume V?
4. Expand the expression that relates dimensionless angular velocity and eccentricity for a quasi-spherical Maclaurin spheroid. By modeling the Earth in such a way (mean density ≈ 5.5 g cm^{-3}), what would be its expected flattening? What about the flattening of the Sun and Jupiter?
5. If we model a millisecond pulsar as a classical ellipsoid, what is the minimum value of the average mass density implied, and how does this value compare with that of the density of nuclear matter?
6. Consider a particle moving inside a Maclaurin spheroid without direct interaction with the matter of the spheroid except for the action of its mean gravitational field. What is the condition on the eccentricity e in order for all orbits to be closed (see Chapter 21 for a detailed discussion of related issues)? What is the ratio of horizontal to vertical frequency at the Jacobi bifurcation point?
7. Show that indeed the Maclaurin and Jacobi sequences can be recovered by extremizing E_{mec}, as described in Subsection 10.2.3.
8. For a homogeneous ellipsoid with principal axes coinciding with the Cartesian axes, from definition (10.35) and expression (10.2), prove that in the notation of the second-order virial theorem $W_{ij} = -2\pi G\rho A_i I_{ij}$.
9. For a Maclaurin spheroid rotating around the third axis, from the second-order virial theorem calculate the ratio $(K_{11}+\Pi)/\Pi$, and provide an approximate expression for small values of the polar eccentricity.
10. Consider the case of Achernar, for which the apparent axial ratio is ≈ 1.56 and the apparent equatorial velocity is $V \sin i \approx 220$ km s^{-1}. Discuss whether and under which conditions these values could be reconciled with the properties of a simple homogeneous Maclaurin spheroid viewed at a suitable inclination angle i.
11. Calculate and plot the rotation parameter $t = t(e,\eta(e))$ as a function of the polar eccentricity for the triaxial Jacobi sequence.

Chapter 11

1. The dispersion relation for some electromagnetic waves, with a wave number parallel to a given homogeneous magnetic field, can be written as $c^2k^2 = \omega^2 - \omega_{pe}^2/(1 \pm \omega_{ce}/\omega)$. Here c is the speed of light in vacuum, ω_{pe} is the plasma frequency, and ω_{ce} is the electron cyclotron frequency; the $\pm$ sign corresponds to the two circular polarizations available. Discuss the dispersion and propagation properties of these waves in the plane (ω, c_{ph}), where c_{ph} is the relevant phase velocity. Can propagation occur for $\omega > \omega_{ce}$? (Waves belonging to the low-frequency propagating branch, at $\omega < \omega_{ce}/2$, are called *whistlers*, from the characteristic descending tone recognized in some ionospheric emissions.)
2. Discuss the properties of the circular water-wave signals produced by a point source of negligible duration. What is a simple explanation for the formation of expanding rings from a dropped stone, and what is the expected velocity of such water rings in a pond?

Chapter 12

1. Derive the Jeans instability condition for a homogeneous nonrotating system made of two fluids, one characterized by density ρ_0 and effective thermal speed c_s and the other by density $\alpha\rho_0$ and effective thermal speed $\beta^{1/2}c_s$, where α and β are given constants. Assume that the two coexisting fluids interact with each other only by means of gravitational forces. (The notation follows that of a similar problem studied in Chapter 16, in which the two fluids represent stars and gas in the context of density waves on a disk.)
2. From the equations of the fluid model given in Chapter 8, prove that the frequency of waves in a differentially rotating axisymmetric disk should enter the dispersion relation in terms of the so-called Doppler-shifted value $\omega - m\Omega$, where m is the integer number of the azimuthal Fourier analysis and Ω is the unperturbed angular velocity of the disk.
3. Derive the asymptotic behavior of the plasma dispersion function $Z(\zeta)$ for small and large arguments. Note that for large arguments the asymptotic expansion is subtle and has to be qualified in relation to the region of the complex plane considered; the expression given in the text (in Subsection 12.2.1) describes the regular limit, which is the one to be used in the derivation of the fluid limit of approximation (12.39).
4. In a uniform cold plasma in which the ions are at rest and two electron beams of equal density are streaming through each other with relative velocity $2\mathbf{u}$, in the absence of magnetic fields, we consider electrostatic waves with wave number $\mathbf{k}$ parallel to $\mathbf{u}$. The resulting dispersion relation is $1 = (\omega_{pe}^2/2)[1/(\omega - ku)^2 + 1/(\omega + ku)^2]$, where ω_{pe} is the plasma frequency. Prove that some perturbations subject to this dispersion relation may be unstable. This is the *two-stream instability*; it corresponds to the Kelvin-Helmholtz instability in fluid dynamics. A gravitational analogue of this plasma instability will be addressed in Chapter 19.
5. Two homogeneous fluids of different densities, initially at rest, are separated by a horizontal boundary in the presence of the gravitational acceleration g. The lower fluid is characterized by density $\rho_{\rm down}$ and the fluid above the horizontal boundary by density $\rho_{\rm up}$. In the inviscid case, the dispersion relation for waves characterized by wave number k and frequency ω is $\omega^2 = gk[(\rho_{\rm down} - \rho_{\rm up})/(\rho_{\rm down} + \rho_{up}) + (k^2T/g)/(\rho_{\rm down} + \rho_{\rm up})]$, where T represents the surface tension as in Section 11.3. Prove that the waves are stable if the denser fluid is below, that is, for $\rho_{\rm up} < \rho_{\rm down}$, whereas instability can occur for $\rho_{\rm up} > \rho_{\rm down}$. This is called the *Rayleigh-Taylor instability*.
6. In the standard cosmological model, elementary linear density perturbations with wave number k are governed by a simple differential equation (see the monograph by Peebles 1993): $\ddot{\rho}_1 + 2H\dot{\rho}_1 = (4\pi G\rho - k^2c_s^2/a^2)\rho_1$. Here $H = \dot{a}/a$ is the Hubble expansion rate, and a is the cosmological expansion parameter. The standard Jeans dispersion relation is recovered by setting $H = 0$ and $a = 1$. Consider the cold Einstein–de Sitter limit ($c_s \to 0$; $\Omega_R = 0$, $\Omega_\Lambda = 0$, so that $H^2 = 8\pi G\rho/3$ and $a \propto t^{2/3}$; see also Section 4.4). Find the two power-law solutions that describe linear perturbations in this case. (The solution growing in time describes the mode of Jeans gravitational collapse.) Why is the standard $\exp(-i\omega t)$ Fourier analysis not useful in this context?

Chapter 13

1. For a disk with a perfectly flat rotation curve $V =$ constant, let corotation be at $r = r_{co}$. Recall that the inner Lindblad resonance and outer Lindblad resonance locations are defined

by the conditions $m[\Omega(r_{co}) - \Omega]/\kappa = -1$ and $+1$. Where are the inner and outer Lindblad resonances located for $m = 2$? What happens for $m = 1$ and for $m = 3$? Compare with the case of a Keplerian rotation curve.

2. Following the epicyclic expansion procedure outlined in Section 13.2, prove that indeed the quantity A_2 has the expression recorded in Eq. (13.28).
3. Define the orbit $[r_\star(\tau), \theta_\star(\tau)]$ along the characteristics (see Chapter 11) in an axisymmetric potential, with the conditions $r_\star(\pm\tau_e) = r$, $\theta_\star(\pm\tau_e) = \pm\theta_e$. Here the quantities $2\tau_e = \tau_r$ and $2\theta_e$ are functions of E and J and denote the radial period of oscillation of the stars in the equilibrium potential and the azimuthal angle traversed in such a period. Prove that to two orders in the epicyclic expansion for the radial coordinate and to one order for the azimuthal coordinate we can write

$$r_\star = r(1 - R_1 - R_2), \qquad \theta_\star = \frac{\Omega(r_0)}{\kappa(r_0)}\left[s - 2\left(\xi + \frac{\partial R_1}{\partial s}\right)\right],$$

with $s = \kappa(r_0)\tau$, $\theta_e = \pi\Omega(r_0)/\kappa(r_0)$, $\tau_e = \pi/\kappa(r_0)$, and

$$R_1 = \eta(1 + \cos s) + \xi \sin s, \qquad R_2 = [1 - A_2(r)]R_1^2 - [3 - 2A_2(r)]\eta R_1.$$

Here $\xi = a \sin s_0$ and $\eta = a \cos s_0$ are dimensionless radial and azimuthal epicyclic velocities associated with the orbit at $\tau = \tau_e$ [see Eqs. (13.20) and (13.22)]. From the identity $r_0^2\Omega(r_0) = r_\star^2\dot{\theta}_\star$ at $\tau = \tau_e$, check also that to first order $r_0 = r(1 - \eta)$. [*Hint:* Replace the r and θ variables in Eqs. (13.22) through (13.25) by $r_\star(\tau)$ and $\theta_\star(\tau)$, integrate the equations, and then impose the desired boundary conditions. This is one key step toward the integration along the unperturbed characteristics of the stellar dynamical equations leading to the dispersion relation for tightly wound density waves; see Chapter 15 and the article by Shu 1970, cited there.]
4. Consider the problem of the stability of geostationary satellites on the equatorial plane in view of the presence of a small departure from axisymmetry of the Earth's mass distribution, and compare this problem with that of the trapping of star orbits at corotation in the presence of a rigidly rotating two-armed spiral field.
5. For the classical restricted three-body problem, with $\epsilon = m/M \ll 1$, find the approximate location of the two Lagrangian points close to m [note the singular character of the perturbation analysis involved in identification of the distance from the smaller mass, $O(\epsilon^{1/3})r_{co}$; see Section 8.5 for comments on singular perturbations].
6. For a system of two stars of equal mass orbiting around each other at a fixed distance, find the location of the associated five Lagrangian points and discuss the linear stability of orbits in their vicinity (within the framework of the restricted three-body problem).
7. Using expression (10.21), discuss the dependence on the eccentricity e of the epicyclic frequency κ for equatorial orbits just outside a Maclaurin spheroid.
8. In a pseudo-Newtonian description, the potential in the vicinity of a nonrotating black hole outside the Schwarzschild radius $r_S = 2GM_\bullet/c^2$ is often approximated by the expression $\Phi = -GM_\bullet/(r - r_S)$ (Paczyńsky, B., Wiita, P. J. 1980. *Astron. Astrophys.*, **88**, 23; see also the commentary by Abramowicz, M. A. 2009. *Astron. Astrophys.*, **500**, 213). In the Newtonian context, discuss the energy and stability of circular orbits as a function of r.
9. Calculate and plot angular rotation $\Omega(r)$, epicyclic frequency $\kappa(r)$, and rotation curve $V(r) = r\Omega(r)$ for the potential Φ_{NFW} generated by the spherical density distribution $\rho_{NFW} =$

$a/[r(r_s+r)^2]$, where a and r_s are given dimensional constants, which is believed to represent well the inner structure of dark halos found in cosmological simulations of structure formation in the universe (Navarro, J. F., Frenk, C. S., White, S. D. M. 1996. *Astrophys. J.*, **462**, 563).

10. Calculate and plot angular rotation $\Omega(r)$, epicyclic frequency $\kappa(r)$, and rotation curve $V(r) = r\Omega(r)$ for the potential Φ_J generated by the spherical density distribution $\rho_J = b/[r^2(r_M + r)^2]$, where b and r_M are given dimensional constants, which is believed to represent well the structure of the stellar distribution of bright elliptical galaxies (see Subsection 22.4.4).

Chapter 14

1. Derive Eq. (14.11), following the procedure outlined in Subsection 14.1.1 for the one-component isothermal self-gravitating slab.
2. Consider a one-dimensional motion described by the energy $E = \dot{x}^2 + U(x)$, with $U(x) = U(-x)$. Assume that orbits are bound, with period $\tau = \tau(E)$, which we take to be known. Using the Abel transform technique, infer the form of the force $F(x) = -dU/dx$. Give the explicit result for the case $\tau(E) \propto E^\alpha$ (which also can be checked by dimensional analysis).
3. (a) Consider the quasi-Maxwellian distribution function $f = f(E,J)$ introduced in Section 14.2 [see Eq. (14.18)]. Prove that [with error $O(\epsilon^2)$]

$$-\left(\omega\frac{\partial f}{\partial E} + m\frac{\partial f}{\partial J}\right) = \nu(r_0)\frac{\kappa(r_0)}{c^2(r_0)}f,$$

where $\nu(r_0) = [\omega - m\Omega(r_0)]/\kappa(r_0)$ [here, as usual, the guiding-center radius r_0 is related to the angular momentum by the identity $r_0^2\Omega(r_0) = J$].
(b) From the epicyclic expansion of Chapter 13, show that the volume element needed to integrate a given cool distribution in velocity space is [again with error $O(\epsilon^2)$]

$$\frac{dp_r dJ}{r} = \frac{r_0^4\kappa^3(r_0)}{2r^2\Omega(r_0)}d\xi\, d\eta,$$

where ξ and η are epicyclic velocities defined by $\xi^2 + \eta^2 = a^2$, $r_0\kappa(r_0)\xi = p_r$. [*Hint*: Write out the proper Jacobian, and recall the epicyclic relation $r_0 = r(1-\eta)$. As for the third problem for Chapter 13, this is a key step toward deriving the dispersion relation for tightly wound density waves in stellar dynamics; see Chapter 15 and the article by Shu 1970, cited there.]
4. Discuss the properties of the kernel $K^0(r,r')$ used in Section 14.3 in the integral relation connecting density and potential for a zero-thickness axisymmetric disk. In particular, show that it exhibits a logarithmic singularity at $r = r'$. Note that for the potential just above the plane, at $z = \delta$, the integral relation used is still applicable, provided that we take $\zeta \to \zeta_\delta = 4rr'/[(r+r')^2 + \delta^2]$; for many purposes, it may be useful to start from the case $\delta \neq 0$ and then to refer to the limit $\delta \to 0$. Show that the force field of a sharply truncated disk (i.e., a disk with discontinuous density distribution at the truncation radius) is singular at the truncation radius.
5. Consider a disk of finite but small thickness of density ρ embedded in an external density distribution ρ_{ext} representative of a centrally compact object and/or a diffuse spheroidal halo.

Prove that close to the equatorial plane of the disk, the relevant Poisson equation can be written as

$$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G\rho + A,$$

where $A = 4\pi G\rho_{\rm ext} + 2\Omega^2 - \kappa^2$ is approximately constant in z (here Ω and κ are defined as in Chapter 13).

6. Following the framework of the preceding problem, introduce a surface density variable $\sigma_z \equiv 2\int_0^z \rho(z')\,dz'$, and prove that if the disk is vertically isothermal, the equation of vertical hydrostatic equilibrium and the Poisson equation can be combined into a single equation for σ_z:

$$\frac{\partial^2 \sigma_z}{\partial z^2} \sim -\frac{1}{c^2}\frac{\partial \sigma_z}{\partial z}\,(2\pi G\sigma_z + Az)\,.$$

   From this equation, recover the limit of the isothermal slab discussed in Subsection 14.1.1 and the limit of a non-self-gravitating (Keplerian) disk around a central point mass. (This problem is discussed and solved in appendix A of the article by Bertin, G., Lodato, G. 1999. *Astron. Astrophys.*, **350**, 694.)
7. From expression (14.32), estimate to two significant orders the asymptotic behavior of the function $V_D(R)$ for $R \gg 1$, and thus check its Keplerian behavior at large radii. For $R \ll 1$, is the asymptotic behavior of the function $V_D(R)$ linear with R? (The relevant asymptotic formulas for the modified Bessel functions can be found, for example, in the book by Abramowitz and Stegun listed in the Bibliography.)
8. What is the value of the self-gravity parameter $\epsilon_0 = \pi G\sigma/(r\kappa^2)$ for a self-similar disk [see Eq. (14.35)]?
9. What would be the profile of the self-gravity parameter $\epsilon_0(R)$ for a pure self-gravitating exponential disk [in the absence of other contributions; note that such contributions play an important role in the basic state defined by Eqs. (14.45) and (14.46)]?
10. Estimate the value of the parameter $\beta = 8\pi G\sigma_0 h_\star/V_\infty^2$ for the maximum-disk decomposition of the rotation curve in the case of a pure disk galaxy characterized by an exponential disk.

Chapter 15

1. Consider the two-armed spiral pattern of a galaxy such as M81, with inclination of $\approx$12 to 15$^{\rm o}$. What would be the approximate value of the associated WKB parameter kr if we attempted a single-wave description for it? Check the numbers for the (inner) one-armed structure of NGC 4622 with a pitch angle of $\approx 4^{\rm o}$.
2. For an elementary m-armed tightly wound density wave with radial wave number k and frequency ω, what is the expected (linearized) relation between density and velocity perturbations demanded by the continuity equation? Use the Euler equations (see Subsection 8.3.2 in Part II for the relevant set of equations) to complete the description, in the fluid model, of the relative behavior of density and kinematic perturbations; the amplitude and the phase relations turn out to be pressure-independent. Note that before the self-consistency condition is imposed, these relations describe the response to a driving potential with the preceding characteristics; thus these relations also can be considered, in the linear approximation, to

describe the linear response of a gaseous disk to an imposed wave. (This problem is also of interest for a recent application described in Chapter 18.)

3. Formulate the equations needed to provide answers to the questions of the preceding problem in the context of stellar dynamics. (Note that the relations that are the subject of the last two problems have been the focus of considerable interest in the initial developments of the density-wave theory because of their direct connection with simple observational tests.)
4. Prove Eq. (15.7) by imposing the periodicity of the radial motion in the unperturbed orbits on the relevant response function f_1. [*Hint*: Equation (15.5) can be written as $Df_1/Dt = \{\Phi_1, f_0\}$, where D/Dt is the time derivative along the unperturbed orbits; see Section 12.3. By suitable manipulations, applicable to the natural variables for the axisymmetric disk, derive a relation similar to Eq. (12.44), with the orbit integral extending from t_0 to t. Impose on this relation the desired periodicity condition.]
5. In the notation of Section 15.1, let a radial wave number be defined by $\tilde{\Phi}(r) = \alpha \exp(i \int^r k(r')\,dr')$, where α is a constant [see Eq. (15.3)]. Apply the ordering of Eq. (15.8) to expand the expression $\tilde{\Phi}(r_\star(\tau))$ in the integral of Eq. (15.7) along the unperturbed orbits. What is the accuracy needed on the unperturbed epicyclic orbits for obtaining the response accurate to $O(\epsilon)$?
6. Use the epicyclic expansion discussed in the third problem for Chapter 13 and the approximate expression for a quasi-Maxwellian distribution function given in Section 14.2 (with the relations obtained in the third problem for that chapter), and thus derive the dispersion relation for tightly wound density waves in stellar dynamics given in Eq. (15.9). {*Hint*: Equate the density response $\sigma_1 = -|k|\Phi_1/2\pi G$ [see Eq. (12.21)] obtained from the Poisson equation to that obtained by integrating, over the velocity space $(d\xi\, d\eta)$, the approximate expression for f_1. Keep the integration over the τ variable last, and replace it with an integration over the phase variable $s = \kappa(r_0)\tau$.} To obtain the dispersion relation recorded in the text, the term associated with R_2 in the epicyclic expansion of the third problem for Chapter 13 can be omitted. What physical information would be obtained if the WKB investigation is pushed to the next order?
7. Based on the simple quadratic dispersion relation of the fluid model [Eq. (15.11)], give the explicit expression $\hat{k} = \hat{k}(\nu; Q)$ for the four wave branches generally available (short trailing, long trailing, short leading, and long leading), check the group propagation and wave action properties given in Section 15.3.1, and describe the (formal) behavior of the group velocity in the vicinity of resonances and turning points applicable to different conditions on the parameter Q (with respect to the reference case of marginal stability in which $Q = 1$).
8. Based on the simple quadratic dispersion relation of the fluid model [Eq. (15.11)], give the explicit expression $\hat{k} = \hat{k}(\nu; Q)$ for the short-wave branch beyond the outer Lindblad resonance (i.e., for $\nu > 1$). Is propagation allowed for $Q > 1$? (This problem is also of interest for a recent application described in Chapter 18.)
9. Calculate the function $Q = Q_{\max}(\hat{z}_0)$ for the case in which the dilution of the gravity field (associated with the finite thickness of the disk) in the quadratic dispersion relation is described by an exponential factor (dashed declining curve in Fig. 15.4) and the function $Q = Q(\hat{z}_0)$ defined by the vertical hydrostatic equilibrium condition for a non-self-gravitating thin fluid disk (lower rising curve in Fig. 15.4). The second calculation is also of interest for the study of Keplerian accretion disks (see Chapter 27).

10. Consider a wave packet characterized by group velocity c_g in an inhomogeneous disk. What is the general expression of the propagation time relative to a trip from r_1 to r_2? Discuss a procedure to estimate a typical value of such group propagation time for density waves on a given galaxy model based on the quadratic fluid dispersion relation.

Chapter 16

1. Consider a galaxy characterized by a rotation curve of the form $V = V_\infty[1-\exp(-r/h)]$, where h is the exponential scale of the disk, and suppose that the disk is dominated by a density-wave pattern with corotation at $r = 3\ h$, in the vicinity of which the cold-gas velocity dispersion is c_g. Estimate the width of the annulus around the corotation circle inside which the relative motion between the wave and the gas is subsonic (with respect to c_g). Based on the phenomenological introduction given in Part I, insert realistic values for V_∞, h, and c_g, and check the numbers for realistic cases.
2. If a 21-cm line-emission measurement determines a certain value for the gas disk density σ_{HI} in the form of atomic hydrogen, what is the factor f that should be used to convert σ_{HI} into a more realistic estimate of the gas density $\sigma_g = f\sigma_{\mathrm{HI}}$ so as to take into account the presence of a normal amount of (cosmological) helium?
3. From the two-fluid dispersion relation, prove that for small values of α, if the gas component is not too cold, the peak of the marginal stability curve is reached at $\hat{\lambda}_{\max} = 1/2 + O(\alpha^2)$, with $Q^2_{\max} \approx 1 + 4\alpha$.
4. For the two-fluid dispersion relation, show that in the two-phase region, under the ordering $\beta = O(\alpha^2) \ll 1$, the gaseous peak occurs at $\hat{\lambda}_{\max} \sim \alpha/2$, with $Q^2_{\max} \sim (\alpha^2/\beta) + 4\alpha$.
5. By considering the description provided earlier in the book (in particular, see Chapter 14 for the Oort mass discrepancy) and other data available from the literature, discuss what would be appropriate numbers for the gas-to-star density and temperature ratios α and β to be used in a two-component description of the solar neighborhood in the disk of our Galaxy.
6. What is the value of the gradient of the vorticity distribution $\kappa^2/\sigma\Omega$ governing the effects of the corotation resonance for the self-similar model described in Subsection 14.3.1?
7. To illustrate the destabilizing role of a small amount of cold gas, it is often reported (e.g., see subsection 7.1.2 in the monograph by Bertin and Lin 1996) that the effective stability parameter Q_{eff} of a system made of stars and gas (in a zero-thickness model) follows the simple approximate relation

$$\frac{1}{Q_{\mathrm{eff}}} \approx \frac{1}{Q_g} + \frac{1}{Q_\star}.$$

With the help of the analysis given in Subsection 16.1.2, discuss the limitations of this approximation (see Romeo, A. B., Wiegert, J. 2011. *Mon. Not. Roy. Astron. Soc.*, **416**, 1191).

Chapter 17

1. (See also the first problem for Chapter 13.) Consider a galaxy characterized by a rotation curve of the form $V = V_\infty[1-\exp(-r/h)]$, where h is the exponential scale of the disk, and draw the functions Ω, $\Omega-\kappa/m$, $\Omega+\kappa/m$, for various values of m. The diagram is often used

to discuss the location of the various resonances for assigned values of the pattern frequency Ω_p. What is a necessary condition for the occurrence of the inner Lindblad resonance for a one-armed pattern?

2. From the WKBJ expansion applied to the outgoing short-trailing wave beyond OLR (i.e., for $\nu > 1$), prove that the conservation of wave action in a one-component, zero-thickness fluid model requires that the perturbed density-amplitude profile follows the relation

$$\left|\frac{\sigma_1}{\sigma_0}\right|^2 \propto G(\nu, Q) r^{-1} \kappa^4 \sigma_0^{-4},$$

where σ_0 is the unperturbed density of the disk. Find the appropriate function $G(\nu, Q)$. (The proportionality factor is independent of r; this problem is also of interest for a recent application described in Chapter 18.)

3. From the cubic dispersion relation, derive the second-order ordinary differential equation for the open-wave regime in the fluid model. {*Hint*: The second-order ODE used for the description of tightly wound (normal) spiral modes is the differential counterpart to the quadratic dispersion relation of Subsection 15.1.2; in the regime in which this latter relation is applicable, we have seen that the relevant wave cycle involves short and long trailing waves. In Chapter 15 we anticipated that the cubic dispersion relation shows the possibility for wave cycles based on leading and trailing (open) waves. Expand the cubic dispersion relation of Subsection 15.1.3 in the vicinity of $k = 0$, up to and including the term $O(k^2)$. We obtain the ODE relevant for the open-wave regime by transforming this new quadratic algebraic relation back into a differential equation by using the prescription $k \to -i\partial/\partial r$ [see Chapter 11 and Eq. (15.3)]. The criterion for marginal stability with respect to leading-trailing wave interaction, corresponding to Eq. (15.33) and now replacing the $Q = 1$ condition, is obtained by studying the equation in the vicinity of corotation, that is, for $\nu^2 \ll 1$. This ODE, by analogy with Eqs. (17.28) and (17.29), can be shown to lead to open modes, which are at the basis of barred spiral structure. Note that deriving such ODE directly from the integrodifferential problem is feasible but rather tedious. See the second paper by Bertin, Lin, Lowe, and Thurstans 1989.}

Chapter 18

1. From a detailed discussion of the identification of the density of wave action in stellar dynamics (see the papers by Shu 1970; Lynden-Bell and Kalnajs 1972; Mark 1974, 1976; and Bertin 1980, cited in Chapter 15), if we refer to $D = -k_0/|k| + \mathcal{F}_\nu(x)/(1-\nu^2)$, Eq. (15.38) is to be applied with the following definition of the wave amplitude (squared): $a^2 = \pi G \sigma_1^2 / 2k_0$; here σ_1 represents the density perturbation, and the rest of the notation follows that given in Subsection 15.1.1. Estimate the angular-momentum transport associated with a global mode of small amplitude (the amplitude level is constrained by the observations) by applying the linear theory of Subsection 15.3.1 to the outgoing wave outside the corotation circle. By comparing the value of this flux with the angular momentum stored in the basic state inside the corotation circle, it has been possible to estimate a time scale for the nonlinear evolution induced by the presence of large-scale spiral structure (Bertin 1983).

Chapter 19

1. Compare the different predictions on the amplitude behavior of a warp based on the conservation of wave action for the two cases, the first of exponential decay of the disk density and the second of $1/r$ decay, which would be implied if we take a flat rotation curve fully supported by the disk.
2. Insert appropriate numbers in the relevant expression [approximation (19.7)] to estimate typical values for the growth (damping) rate associated with disk-halo interaction applicable to astrophysically interesting conditions. Provide similar estimates for the convective amplification (damping) rate described by Eq. (19.10).

Chapter 20

1. Imagine a galaxy disk with exponential luminosity profile (with exponential scale length h) to be characterized by an observed rotation curve well fitted by Eq. (14.45). With the help of the exact relations for the rotation curve associated with an exponential disk density distribution provided in Chapter 14, discuss the possibility of a maximum-disk solution (i.e., a solution with maximum and constant mass-to-light ratio for the disk able to fit the inner part of the rotation curve); assume that the influence of bulge and gas is negligible. Formulate the equation that leads to the recovery of a spherical halo density distribution required for justifying, together with such maximum disk, the entire rotation curve. Is such a density distribution ρ_h guaranteed to be monotonically decreasing with radius? What happens if the disk mass-to-light ratio of such a maximum-disk solution is halved, with the idea that a substantial part of the support to the inner rotation curve comes from the spherical halo?
2. Suppose that our Galaxy is indeed embedded in a massive spherical dark halo that supports half the rotation curve at the location of the Sun and is able to provide a flat rotation curve considerably farther outside. If we imagine the Large Magellanic Cloud to be orbiting inside such a dark halo, what would be the approximate value of the projected halo density for the dark matter between us and a star in the Large Magellanic Cloud?

Chapter 21

1. A harmonic oscillator with frequency ω and initial energy (at $t \to -\infty$) E_i is driven by a time-dependent force of the form $F(t) = f\exp(t/\tau)$, for $t < 0$, and $F(t) = -f\exp(-t/\tau)$, for $t > 0$, where f and τ are positive constants. Calculate the energy absorbed by the oscillator in such a process. Solve the problem in the limit of small perturbations $f \to 0$, and in the general case. Is it always true that the final energy (at $t \to +\infty$) of the oscillator E_f exceeds the initial energy? [*Note*: With a more realistic choice of $F(t)$, this may be a first crude zero-dimensional description of the way the motion of a star inside a stellar system is modified by an external tidal interaction.]
2. Prove that the Lenz vector $\mathbf{L} = \mathbf{v} \times \mathbf{J} + \Phi(r)\mathbf{x}$ is indeed an integral of the motion if $\Phi(r)$ is the Keplerian potential.
3. Consider the 2D isotropic harmonic oscillator. Show that the three integrals $\mathcal{J}_1 = (\dot{x}^2 - \dot{y}^2) + \omega^2(x^2 - y^2)$, $\mathcal{J}_2 = x\dot{y} - y\dot{x}$, and $\mathcal{J}_3 = \omega^2 xy + \dot{x}\dot{y}$ can be normalized so as to satisfy the $SU(2)$ algebra $\{\mathcal{J}_i, \mathcal{J}_j\} = \epsilon_{ijk}\mathcal{J}_k$ in terms of the commutation rules defined by the standard Poisson

brackets. Show also that the Casimir integral $\mathcal{C} = \mathcal{J}_1^2 + \mathcal{J}_2^2 + \mathcal{J}_3^2$ is proportional to the square of the Hamiltonian.

4. By referring to the $J = 0$ radial orbits from the condition on the period $\tau_r \propto (-E)^{-3/2}$ [see Eq. (21.15)], by means of the Abel inversion technique, show that the associated regular potential is indeed the isochrone potential given in Eq. (21.12). [*Hint*: It may be convenient to set to $-1/2$ the value of the minimum of the potential that we look for so as to match from the beginning the conditions applicable to the dimensionless form of Eq. (21.12). Note the (little) subtlety involved in comparing the $J \to 0$ limit of orbits in a spherical potential with a one-dimensional orbit, such as the one considered, for example, in the second problem for Chapter 14.]
5. What is the most general form at large radii of an axisymmetric density distribution compatible with a Stäckel potential of the form $\Phi(r,\theta) \sim -GM/r + \eta(\theta)/r^2$? At large radii, write an equation for $\eta(\theta)$ for a perfect axisymmetric ellipsoid of eccentricity e, and provide its approximate solution for the quasi-spherical case. Compare isodensity with equipotential surfaces.

Chapter 22

1. Without addressing the issues involved in the solution of the self-consistent problem, consider simple analytical forms of distribution functions $f(E,J^2)$ able to display either a tangentially biased or a radially biased pressure tensor (note that the same distribution function can have anisotropy of the two kinds at different radii, but such an example is less easy to find). For cases for which the pressure anisotropy is strong (i.e., for cases for which either in the radial or in the tangential directions the system can be considered to be cold), describe the qualitative properties of radial and tangential cuts of the distribution function in velocity space (obtained by taking either $v_\theta = 0, v_\phi = 0$, or $v_r = 0, v_\phi = 0$). What is a reasonable expectation for the associated velocity profiles projected along the line of sight in the two cases if we use the preceding models to match the observed line profiles of a spherical stellar system?
2. For an assigned potential $\Phi(r)$, write the expression of a distribution function $f(E,J^2)$ for a system populated by only circular orbits. What is the corresponding function $F(r, v_r, v_\perp)$ referred to the standard coordinates of spherical geometry?
3. For an anisotropic spherically symmetric stellar system without internal streaming motions, show that the hydrostatic equilibrium condition can be written as

$$\alpha(r) + \frac{r^2\Omega^2}{\langle v_r^2\rangle} + \frac{d\ln(\rho\langle v_r^2\rangle)}{d\ln r} = 0$$

with the same notation used in the text.
4. Consider a modification of a spherical distribution function $f(E,J^2) \to [1 + g(J_z)]f(E,J^2)$ by a term that is odd in J_z, such as $g(J_z) \propto J_z/\sqrt{J_0^2 + J_z^2}$, where J_0^2 is a constant, so that the final function is positive definite. If the initial distribution corresponds to a fully self-consistent model, is this true also for the new function? Because the transformation leaves the even moments unaltered, is it true that the velocity-dispersion profiles remain unchanged?

5. Consider the integration over velocity space of an isotropic function. How can the integral be transformed into an integral in dE? Suppose now that the function to be integrated is anisotropic, but within the overall spherical symmetry (so that v_θ and v_ϕ are equivalent). How can the integral be transformed into an integral in $dEdJ$? What is the Jacobian of the transformation $d^3v\,d^3x$ into $dEdJdr$ appropriate for the case of spherical symmetry? In this last case, if the integrand does not depend on r explicitly, what is the result of a first integration in the radial coordinate?
6. Show that the quantity $p_{rr}(\Phi,r) = \int v_r^2 f\, d^3v$, based on a distribution function $f(E,J^2)$ associated with a spherical potential $\Phi(r)$, is related to density and pressure anisotropy by simple differentiation, that is,

$$\rho(\Phi,r) = -\frac{\partial p_{rr}}{\partial \Phi}, \qquad \alpha(\Phi,r) = -\frac{\partial \ln p_{rr}}{\partial \ln r}.$$

Note that in the isotropic case p_{rr} thus depends on r only implicitly through Φ. Note also that the two relations lead to the correct hydrostatic equilibrium condition (see Problem 3 for this chapter).
7. For a purely isotropic distribution $f = A(-E)^\beta \exp(-aE)$, where a and A are positive constants ($f = 0$ for $E > 0$), what is the value of β that is compatible with a density distribution declining as r^{-4} at large radii?
8. Consider a stellar system, such as a globular cluster, to be initially characterized by energy E, mass M, and truncation radius r_t. Suppose that, as a result of some evolutionary processes (internal evaporation, disk-shocking, etc.), the stellar system changes energy by a small amount ΔE, and its mass changes by a small amount ΔM while changing r_t under the usual condition that r_t is determined by tidal interaction with the host galaxy (and under the assumption that although the energy and the mass have changed, the location of the stellar system inside the host galaxy can be considered as practically unchanged). If initial and final states are well fitted by a King model, formulate and discuss a procedure to determine the parameter transformation $(a,A,C) \to (a',A',C')$ induced by the combined effect of ΔE and ΔM.
9. Using the simple analytical model of Subsection 22.4.4, check that for a spherical system conforming to the $R^{1/4}$ law we expect $r_M \approx 1.3R_e$ between volume and projected half-mass radii.
10. A one-component f_∞ model with $\Psi = 12$ and mass-to-light ratio $M/L_B = 6$, fitting a galaxy characterized by absolute blue magnitude $B = -20.7$, effective radius $R_e = 5$ kpc, and observed central velocity dispersion 250 km s^{-1}, is taken to be strictly correct and applicable all the way down to the center. What would be the expected core radius? What would be an estimate for the central relaxation time in this system? [*Hint*: The solution requires an investigation of the properties of the f_∞ models for high Ψ at $r \to 0$, with the scales set by the available data. One way to set the scales is to recall that $R_e \approx \sqrt{a/c}$ (to be more precise, a numerical study gives, for high-Ψ models, $\sqrt{a/c} \approx 0.85 r_M$, and it is known that for a model close to the $R^{1/4}$ law, the relation $r_M \approx 1.3R_e$ holds), that the one-dimensional velocity dispersion at the center is $\approx \sqrt{1/a}$, and that for high-Ψ models $\gamma \approx 18$. Once the scales are set, a discussion of the density behavior in the vicinity of $r = 0$, as carried out in Subsection 22.3.1 for the King models, quickly leads to the desired answers.]

11. Consider a cool homogeneous sphere characterized by total energy $E_{\rm tot}$, mass M, and initial virial ratio $(2K/|W|)_{\rm in} = u \ll 1$. As a result of collisionless collapse, suppose that a quasi-equilibrium is reached, well represented by an f_∞ model (with the same values of energy and mass). In this final configuration, the maximum phase-space density should not exceed its maximum initial value. Find a relation between initial virial ratio u and final value Ψ of the dimensionless central potential. [*Hint*: Make use of dimensional analysis and the fact that the global properties of the high-Ψ models are basically Ψ independent; more precisely, we may recall that a numerical study gives $\gamma \approx 18$, $q = |W|r_M/GM^2 \approx 1/2$, $a|W|/M \approx 3/4$, and $\sqrt{a/c} \approx 0.85 r_M$. A comparison between initial and final virialized states gives the relation $r_M/R \approx 5/12$ between final half-mass radius and radius R of the initial homogeneous sphere. The constraint on maximum phase-space density gives the maximum depth of the central potential well that can be formed by collisionless collapse as $\Psi_{\max}^{3/2}\exp(\Psi_{\max}) \approx 2\pi[u(1-u/2)]^{-3/2}$. For a related discussion, see Londrillo, Messina, and Stiavelli 1991, cited in Chapter 23.]
12. What is the relation between the central dynamical time scale $[G\rho(0)]^{-1/2}$ and the global crossing time $t_{cr} = GM^{5/2}/|2E_{\rm tot}|^{3/2}$ for high-Ψ f_∞ models?

Chapter 23

1. For a spherical stellar system described by a distribution function $f(E, J^2)$, prove the identity (see also Subsection 23.1.5)

$$\int v_r^2 \frac{\partial f}{\partial E}\, d^3v = \int \frac{J^2}{2r^2}\left(\frac{\partial f}{\partial E} + \frac{r^2}{J}\frac{\partial f}{\partial J}\right) d^3v = -\rho.$$

Chapter 24

1. Under spherical symmetry, consider a diffuse halo coexisting with a density distribution compatible with the $R^{1/4}$ law, with quasi-isotropic underlying distribution functions, for a case for which streaming motions can be ignored; for simplicity, the luminous density distribution can be taken to be of the form (22.59). Compare the condition of hydrostatic equilibrium at $r \to 0$, where the halo is assumed to be dynamically unimportant and the luminous matter may be considered to be close to an isothermal sphere, with that at $r = r_M$, where the halo may provide a significant contribution to the gravitational support. Derive a relation for the $(M_D/M_L)_{r_M}$ mass ratio (of the dark to luminous mass inside a sphere of radius r_M) in terms of the existing drop (between $r = 0$ and $r = r_M$) in velocity dispersion for the luminous matter. Estimate the impact of a moderate amount of pressure anisotropy and a realistic gradient in the velocity-dispersion profile present at $r = r_M$. [*Note*: This analysis brings us close to Eq. (24.1), but a justification of the latter relation would require further discussion because it involves velocity dispersions projected along the line of sight.]