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Problems for Chapter 16 of *Advanced Mathematics for Applications*

Green's Functions: Partial Differential Equations

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1 Elliptic equations

1. Solve
$$\nabla^2 u = -4\pi\, \delta^{(3)}(\mathbf{x} - \mathbf{x}_s),$$
subject to $u \to 0$ at infinity, $\mathbf{n} \cdot \nabla u = 0$ on a sphere of radius a placed at a distance $b > a$ from $\mathbf{x}_s$. Try to sum the resulting series in closed form by using the generating function for the Legendre polynomials.

2. Consider the equation
$$\nabla^2 u \;=\; -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}_0)$$
inside the sphere of radius a. The point $\mathbf{x}_0$ is internal to the sphere. The solution is regular at the center of the sphere and, on the sphere surface, it satisfies
$$\alpha u + \beta \mathbf{n} \cdot \boldsymbol{\nabla} u \;=\; 0$$
with α and β given constants. Calculate the average value of u over the surface of the sphere. Find the exact solution of the problem and check your answer to the previous question. Does a solution exist if $\alpha = 0$? If not, why?

3. A spherical source of waves placed at $\mathbf{x}_s$, is at a distance d from a "soft" sphere of radius a. The source size is very small compared with d, a and the wavelength, and it is therefore reasonable to model it as point-like. Find the effect of the sphere on the wave emitted by the source by solving
$$\nabla^2 u + k^2 u = -4\pi\, \delta^{(3)}(\mathbf{x} - \mathbf{x}_s),$$
$$u \to \frac{1}{|\mathbf{x} - \mathbf{x}_s|}\; \exp(ik|\mathbf{x} - \mathbf{x}_s|) \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_s,$$
and $u = 0$ on the sphere.

4. (*a*) Solve the two-dimensional Helmholtz equation in free space for the case of a unit point source radiating waves outward.
(*b*) With the aid of the previous result find the Green's function for the Helmholtz equation outside a circle of radius a with Dirichlet boundary conditions and the radiation condition at infinity.

5. Find a Green's function for the two-dimensional Helmholtz equation inside the unit circle for Neumann boundary conditions.