Appendix A: Linear System Theory

A.1 Introduction

This appendix summarizes a number of fundamental definitions from linear system and Fourier theory. Only what is relevant for this book is discussed. More information can be found in the specific textbooks on this topic.[50]

A.2 Signals

A.2.1 Definitions and Examples

A signal represents the measurable change of some quantity with respect to one or more independent variables such as time or spatial position. Mathematically, a signal can be represented as a function. In medical imaging, the signals are multidimensional. Modern acquisition systems acquire three- (3D) and even four-dimensional (4D) data. The signal can then be written as

$$s = f(\vec{r}, t) = f(x, y, z, t) \qquad \forall\, x, y, z, t \in \mathbb{R} \quad \text{and} \quad s \in \mathbb{C}. \tag{A.1}$$

The value of the function is usually real, but it can be complex.

Signals have some particular properties. The most important for this book are defined as follows.

A signal is even if

$$s(-x) = s(x) \qquad \forall\, x \in \mathbb{R}. \tag{A.2}$$

A signal is odd if

$$s(-x) = -s(x) \qquad \forall\, x \in \mathbb{R}. \tag{A.3}$$

50 R. N. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill, New York, second edition, 1986.
E. Oran Brigham. *The Fast Fourier Transform and Its Applications*. Prentice-Hall International, Englewood Cliffs, New Jersey, first edition, 1988.
A. Oppenheim, A. Willsky, and H. Nawab. *Signals and Systems*. Prentice-Hall International, Upper Saddle River, New Jersey, second edition, 1997.

We denote even and odd signals by $s_e(x)$ and $s_o(x)$, respectively. Obviously, the product of two even signals is even, the product of two odd signals is even, and the product of an even and an odd signal is odd. From the definition it is also clear that

$$\int_{-\infty}^{+\infty} s_e(x)\,\mathrm{d}x = 2\int_{0}^{+\infty} s_e(x)\,\mathrm{d}x \tag{A.4}$$

and

$$\int_{-\infty}^{+\infty} s_o(x)\,\mathrm{d}x = 0. \tag{A.5}$$

Any signal can be written as the sum of an even and an odd part:

$$\begin{aligned} s(x) &= \left[\frac{s(x)}{2} + \frac{s(-x)}{2}\right] + \left[\frac{s(x)}{2} - \frac{s(-x)}{2}\right] \\ &= s_e(x) + s_o(x). \end{aligned} \tag{A.6}$$

A signal is periodic if

$$s(x + X) = s(x) \qquad \forall\, x \in \mathbb{R}. \tag{A.7}$$

The smallest finite X that satisfies this equation is called the *period*. If no such X exists, the function is aperiodic.

A complex function can be written in Cartesian representation. For 2D signals, we have

$$s(x, y) = u(x, y) + \mathrm{i}v(x, y), \tag{A.8}$$

where $u(x, y)$ and $v(x, y)$ are the real part and imaginary part, respectively. A complex function can also be written in polar representation as

$$s(x, y) = |s(x, y)|\, \mathrm{e}^{i\phi(x,y)}, \tag{A.9}$$

where

$$|s(x, y)| = \sqrt{u^2(x, y) + v^2(x, y)} \tag{A.10}$$

is the modulus or the amplitude and

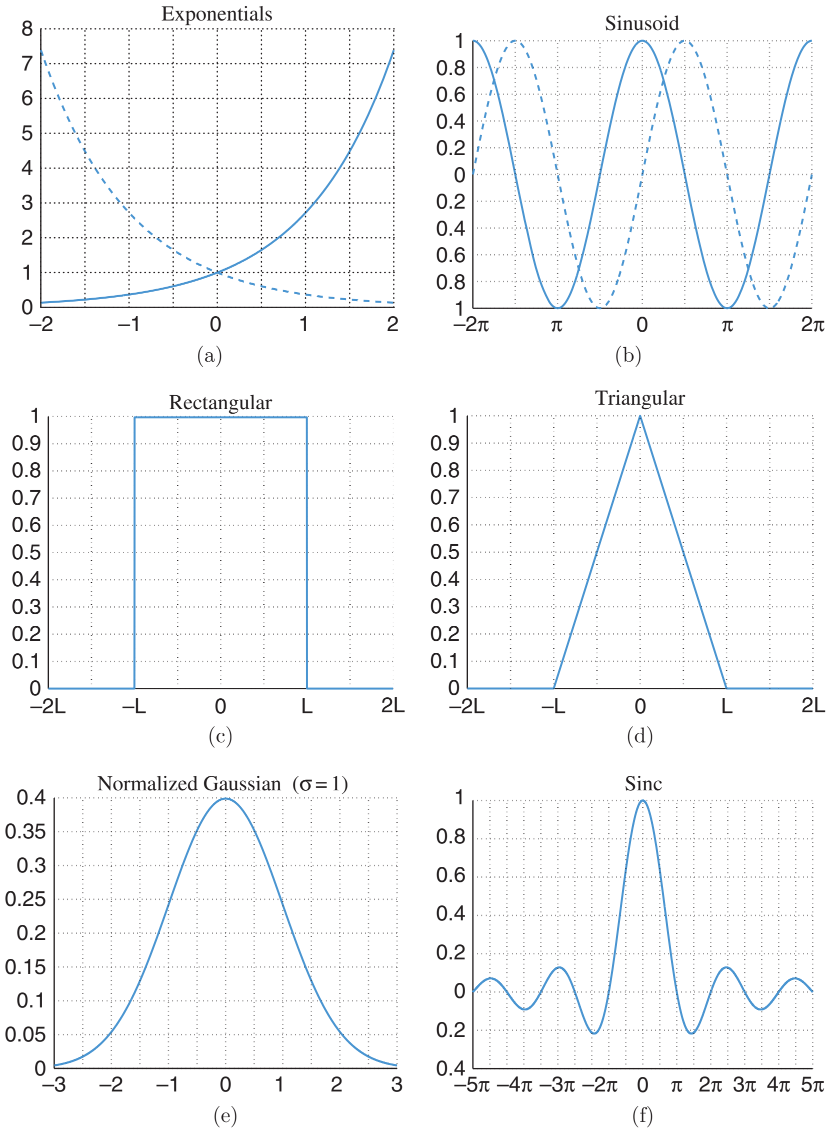

Figure A.1 Some of the most important signals in linear system theory. **(a)** The exponentials e^x (solid) and e^{-x} (dashed). **(b)** $\sin(x)$ (dashed) and $\cos(x)$ (solid). **(c)** Rectangular pulse. **(d)** Triangular pulse. **(e)** Normalized Gaussian with $\mu = 0$. **(f)** $\mathrm{sinc}(x)$.

$$\phi(x, y) = \arctan\left(\frac{v(x, y)}{u(x, y)}\right) \tag{A.11}$$

is the argument or the phase of s.

A number of signals are used extensively in system theory and are important enough to have a unique name. Here are some of them (see also Figure A.1).

- ***Exponential*** (Figure A.1(a))

$$\exp(ax) = e^{ax}. \tag{A.12}$$

When the constant $a > 0$, the exponential function increases continuously with increasing x (solid line); when $a < 0$, it decreases toward zero with increasing x (dashed line).

- ***Complex exponential or sinusoid*** (Figure A.1(b)):

$$A\,e^{i(2\pi kx+\phi)} = A\,(\cos(2\pi kx+\phi)+i\sin(2\pi kx+\phi)). \tag{A.13}$$

A sinusoid is characterized by three parameters: its modulus or amplitude A, spatial frequency k, and phase ϕ. The term i is the imaginary unit; that is $i^2 = -1$. The real and imaginary parts of a sinusoid are, respectively, a cosine (solid line) and sine function (dashed line).

– ***Unit step function (also called Heaviside's function)***

$$\begin{aligned} u(x-x_0) &= 0 \quad \text{for } x < x_0 \\ &= \frac{1}{2} \quad \text{for } x = x_0 \\ &= 1 \quad \text{for } x > x_0. \end{aligned} \tag{A.14}$$

The constant x_0 denotes the location of the step. The function is discontinuous at x_0.

– ***Rectangular function*** (Figure A.1(c))

$$\begin{aligned} \Pi\left(\frac{x}{2L}\right) &= 1 \quad \text{for } |x| < L \\ &= \frac{1}{2} \quad \text{for } |x| = L \\ &= 0 \quad \text{for } |x| > L. \end{aligned} \tag{A.15}$$

The constant $2L$ is the width of the rectangle. Because the nonzero extent of the function is finite, the function is also called a rectangular pulse.

– ***Triangular function*** (Figure A.1(d))

$$\begin{aligned} \Lambda\left(\frac{x}{2L}\right) &= 1 - \frac{|x|}{L} \quad \text{for } |x| < L \\ &= 0 \text{ for } |x| \geq L. \end{aligned} \tag{A.16}$$

Note that the base of the triangular pulse is equal to $2L$.

– ***Normalized Gaussian*** (Figure A.1(e))

$$G_n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp(-\frac{(x-\mu)^2}{2\sigma^2}). \tag{A.17}$$

The Gaussian is normalized (i.e., its integral for all x is 1). The constants μ, σ, and σ^2 are the mean, the standard deviation, and the variance, respectively.

– ***Sinc function*** (Figure A.1(f))

$$\text{sinc}(x) = \frac{\sin(x)}{x}. \tag{A.18}$$

According to L'Hôpital's rule, $\text{sinc}(0) = 1$.

Note that the rectangular, the triangular, the normalized Gaussian, and the sinc function are all even and aperiodic. The step function is neither even nor odd nor periodic. To be compatible with the theory of single-valued functions, it is common to use the mean of the value immediately left and right of the discontinuity. For example, the values at the discontinuities of the rectangular pulse equal $1/2$.

A.2.2 The Dirac Impulse

The *Dirac impulse*, also called impulse function or δ-function, is a very important function in linear system theory. It is defined as

$$\begin{aligned} &\delta(x-x_0) = 0 \quad \text{for } x \neq x_0, \\ &\int_{-\infty}^{+\infty} \delta(x-x_0)\,\mathrm{d}x = 1, \end{aligned} \tag{A.19}$$

with x_0 a constant. The value of a Dirac impulse is zero for all x except in $x = x_0$, where it is undefined. However, the area under the impulse is finite and is by definition equal to 1. A Dirac impulse can be considered as the limit of a rectangular pulse of magnitude $\frac{1}{\varepsilon}$ and spatial extent $\varepsilon > 0$ such that the area of the pulse is 1:

$$\delta(x) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Pi\left(\frac{x}{\varepsilon}\right). \tag{A.20}$$

When ε becomes smaller, the spatial extent decreases, the amplitude increases, but the area remains the same. Clearly, the Dirac impulse is not a function in the strict mathematical sense. Its rigorous definition is given by the theory of generalized functions or distributions, which is beyond the scope of this text.[51]

Using Eq. A.20, it is clear that

$$\int_{-\infty}^{+\infty}\delta(x)s(x)\,\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{-\infty}^{+\infty}\frac{1}{\varepsilon}\Pi\left(\frac{x}{\varepsilon}\right)s(x)\,\mathrm{d}x, \tag{A.21}$$

and consequently the following properties hold:

– *sifting* let $s(x)$ be continuous at $x = x_0$, then

$$\int_{-\infty}^{+\infty} s(x)\,\delta(x-x_0)\,\mathrm{d}x = s(x_0); \tag{A.22}$$

– *scaling:*

$$\int_{-\infty}^{+\infty} A\,\delta(x)\,\mathrm{d}x = A, \tag{A.23}$$

this is a special case of sifting.

The definition of the impulse function can be extended to more dimensions by replacing x by $\vec{r}$.

51 R. N. Bracewell. *The Fourier Transform and Its Applications*. McGraw-Hill, New York, second edition, 1986.
E. Oran Brigham. *The Fast Fourier Transform and Its Applications*. Prentice-Hall International, Englewood Cliffs, New Jersey, first edition, 1988.
R. F. Hoskins. *Generalised Functions*. McGraw-Hill Book Company, New York, 1979.

The properties are analogous; for example, the sifting property in 2D becomes

$$\iint_{-\infty}^{+\infty} s(\vec{r})\,\delta(\vec{r}-\vec{r}_0)\,\mathrm{d}\vec{r} = s(\vec{r}_0). \tag{A.24}$$

The impulse function is crucial for a thorough understanding of *sampling*, as discussed in Section A.5.

A.3 Systems

A.3.1 Definitions and Examples

A system transforms an input signal (also called *excitation*) into an output signal (also called *response*). Mathematically this can be written as

$$s_\mathrm{o} = \mathcal{L}\{s_\mathrm{i}\}, \tag{A.25}$$

where s_i and s_o are the input and output signals, respectively.* The term $\mathcal{L}$ is an operator and denotes the action of the system. A system can be complex and it can consist of many diverse parts. In system theory, however, it is often considered as a black box, and the detailed behavior of the different components is irrelevant. As a simple example, consider an amplifier. It consists of many electrical and electronic parts, but its essential action is to amplify any input signal by a certain amount, say A. Hence,

$$s_\mathrm{o}(t) = \mathcal{L}\{s_\mathrm{i}(t)\} = A\,s_\mathrm{i}(t). \tag{A.26}$$

The process of finding a mathematical relationship between the input and the output signal is called *modeling*. The simplest is an algebraic relationship, as in the example of the amplifier. More difficult are continuous dynamic relationships that involve (sets of) differential or integral equations, or both, and discrete dynamic relationships that involve (sets of) difference equations.

With respect to their model, systems can be linear or nonlinear. A system is linear if the *superposition principle* holds, that is,

$$L\{c_1s_1 + c_2s_2\} = c_1\mathcal{L}\{s_1\} + c_2\mathcal{L}\{s_2\} \quad \forall\, c_1, c_2 \in \mathbb{R}, \tag{A.27}$$

with s_1 and s_2 as arbitrary signals. For example, the amplifier introduced above is linear because

$$\begin{aligned}\mathcal{L}\{c_1s_1 + c_2s_2\} &= A(c_1s_1 + c_2s_2)\\ &= c_1A\,s_1 + c_2A\,s_2\\ &= c_1\mathcal{L}\{s_1\} + c_2\mathcal{L}\{s_2\}.\end{aligned} \tag{A.28}$$

A system is nonlinear if the superposition principle does not hold. For example, a system whose output is the square of the input is nonlinear because

$$\begin{aligned}\mathcal{L}\{c_1s_1 + c_2s_2\} &= (c_1s_1 + c_2s_2)^2\\ &\neq (c_1s_1)^2 + (c_2s_2)^2.\end{aligned} \tag{A.29}$$

In this text, only linear systems are dealt with.

A system is *time invariant* if its properties do not change with time. Hence, if $s_\mathrm{o}(t)$ is the response to the excitation $s_\mathrm{i}(t)$, $s_\mathrm{o}(t-T)$ will be the response to $s_\mathrm{i}(t-T)$. Analogously, a system is *shift invariant* if its properties do not change with spatial position: if $s_\mathrm{o}(x)$ is the response to the excitation $s_\mathrm{i}(x)$, $s_\mathrm{o}(x-X)$ will be the response to $s_\mathrm{i}(x-X)$. We will denote linear time-invariant systems as LTI systems and linear shift-invariant systems as LSI systems.

The response to a Dirac impulse is called the *impulse response*. From Eq. A.22 it follows that

$$s_\mathrm{i}(x) = \int_{-\infty}^{+\infty} s_\mathrm{i}(\xi)\,\delta(x-\xi)\,\mathrm{d}\xi. \tag{A.30}$$

Let $h(x)$ be the impulse response of a LSI system. Based on the superposition principle (A.27), $s_\mathrm{o}(x)$ can then be written as

$$\begin{aligned}s_\mathrm{o}(x) = \mathcal{L}\{s_\mathrm{i}\} &= \int_{-\infty}^{+\infty} s_\mathrm{i}(\xi)\mathcal{L}\{\delta(x-\xi)\}\,\mathrm{d}\xi\\ &= \int_{-\infty}^{+\infty} s_\mathrm{i}(\xi)\,h(x-\xi)\,\mathrm{d}\xi.\end{aligned} \tag{A.31}$$

A similar equation holds for a LTI system:

$$s_\mathrm{o}(t) = \int_{-\infty}^{+\infty} s_\mathrm{i}(\tau)\,h(t-\tau)\,\mathrm{d}\tau. \tag{A.32}$$

The integral in Eqs. (A.31) and (A.32) is a so-called *convolution* and is often represented by an asterisk:

$$s_\mathrm{o} = s_\mathrm{i} * h. \tag{A.33}$$

The function h is also known as the *point spread function* or PSF (see Figure 1.4). Because of its importance in this book, convolution will first be discussed in some more detail.

* We also use s_o to represent an odd signal. However, this should cause no confusion because the exact interpretation is clear from the context.

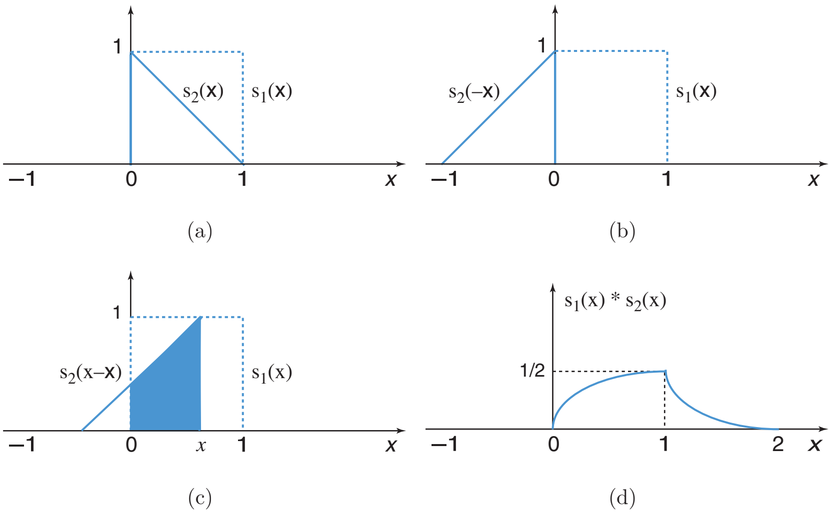

Figure A.2 Graphical interpretation of the convolution of a rectangular pulse $s_1(x)$ with a triangle $s_2(x)$. Changing the independent variable to ξ does not change the functions **(a)**. After $s_2(\xi)$ is mirrored **(b)**, it is translated over a distance x, both functions are multiplied, and the result is integrated **(c)**. The area of the overlapping part is the result for the chosen x. The convolution $s_1(x) * s_2(x)$ is shown in **(d)**.

A.3.2 Convolution

Given two signals $s_1(x)$ and $s_2(x)$, their convolution is defined as follows:

$$s_1(x) * s_2(x) = \int_{-\infty}^{+\infty} s_1(x-\xi)\, s_2(\xi)\, \mathrm{d}\xi, \tag{A.34}$$

or equivalently

$$s_2(x) * s_1(x) = \int_{-\infty}^{+\infty} s_1(\xi)\, s_2(x-\xi)\, \mathrm{d}\xi. \tag{A.35}$$

The result of both expressions is identical, as is clear when substituting ξ by $x-\xi$.

A graphical interpretation of convolution is given in Figure A.2. The following steps can be discerned:

- ***mirroring***, changing ξ to $-\xi$,
- ***translation*** over a distance equal to x,
- ***multiplication***, the product of the mirrored and shifted function $s_1(x-\xi)$ with $s_2(\xi)$ is the colored part in Figure A.2(c),
- ***integration***, the area of the colored part is the convolution value in point x.

The convolution function is found by repeating the previous steps for each value of x.

Convolution can also be defined for multidimensional signals. For 2D (two-dimensional) signals, we have

$$s_1(x,y) * s_2(x,y) = \iint_{-\infty}^{+\infty} s_1(x-\xi, y-\zeta)\, s_2(\xi,\zeta)\, \mathrm{d}\xi\, \mathrm{d}\zeta, \tag{A.36}$$

or equivalently

$$s_2(x,y) * s_1(x,y) = \iint_{-\infty}^{+\infty} s_2(x-\xi, y-\zeta)\, s_1(\xi,\zeta)\, \mathrm{d}\xi\, \mathrm{d}\zeta. \tag{A.37}$$

The graphical analysis shown above can be extended to 2D. The convolution values are then represented by *volumes* rather than by areas.

The convolution integrals (A.34)–(A.37) have many properties. The most important in the context of this book include the following.

- ***Commutativity***

$$s_1 * s_2 = s_2 * s_1. \tag{A.38}$$

- ***Associativity***

$$(s_1 * s_2) * s_3 = s_1 * (s_2 * s_3) = s_1 * s_2 * s_3. \tag{A.39}$$

- ***Distributivity***

$$s_1 * (s_2 + s_3) = s_1 * s_2 + s_1 * s_3. \tag{A.40}$$

A.3.3 Response of a LSI System

Let us first consider the response of a LSI system to a sinusoid. Using Eq. A.31 with $s_\mathrm{i}(x) = A\,\mathrm{e}^{2\pi\mathrm{i}kx}$ yields

$$\begin{aligned} s_\mathrm{o}(x) &= \int_{-\infty}^{+\infty} A\,\mathrm{e}^{2\pi\mathrm{i}k(x-\xi)}\, h(\xi)\, \mathrm{d}\xi \\ &= A\,\mathrm{e}^{2\pi\mathrm{i}kx} \int_{-\infty}^{+\infty} e^{-2\pi\mathrm{i}k\xi}\, h(\xi)\, \mathrm{d}\xi \\ &= A\,\mathrm{e}^{2\pi\mathrm{i}kx}\, H(k), \end{aligned} \tag{A.41}$$

with $H(k)$ the so-called *Fourier transform* of the PSF $h(x)$:

$$H(k) = \int_{-\infty}^{+\infty} e^{-2\pi i k\xi}\, h(\xi)\, d\xi. \tag{A.42}$$

The function $H(k)$ is also called the *transfer function*.

It can be shown that any input signal $s_i(x)$ can be written as an integral of weighted sinusoids with different spatial frequencies:

$$s_i(x) = \int_{-\infty}^{+\infty} S_i(k)\, e^{2\pi i k x}\, dk, \tag{A.43}$$

where $S_i(k)$ is the Fourier transform of $s_i(x)$. The signal $s_i(x)$ is the so-called *inverse* Fourier transform (because of the $+$ sign in the exponent instead of the $-$ sign in Eq. A.42) of $S_i(k)$.

Using Eq. A.41 and the superposition principle, the output signal s_o is then

$$s_o(x) = \int_{-\infty}^{+\infty} S_i(k)\, H(k)\, e^{2\pi i k x}\, dk. \tag{A.44}$$

Summarizing, the output function s_o of a LSI system can be calculated in two ways: either by convolving the input function s_i with the PSF, that is, $s_o = s_i * h$ (Eq. A.33), or in the k-space or frequency domain by multiplying the Fourier transform of s_i by the transfer function, that is, $S_o(k) = S_i(k)H(k)$, and calculating the inverse Fourier transform of $S_o(k)$.

In linear system theory, the transfer function $H(k)$ is often used instead of the PSF $h(x)$ because of its nice mathematical and interesting physical properties. The relationship between the PSF $h(x)$ and the transfer function $H(k)$ is given by the Fourier transform (A.42). Because of its importance in medical imaging, the Fourier transform is discussed in more detail in the next section. Note, however, that the Fourier transform is not the only possible transform. There are many others (Hilbert, Laplace, etc.), although the Fourier transform is by far the most important in the theory of medical imaging.

A.4 The Fourier Transform

A.4.1 Definitions

Let k and r be the conjugate variables in the Fourier domain and the original domain, respectively. The *forward* Fourier transform (FT) of a signal $s(r)$ is defined as

$$S(k) = \mathcal{F}\{s(r)\} = \int_{-\infty}^{+\infty} s(r)\, e^{-2\pi i r k}\, dr. \tag{A.45}$$

The operator symbol $\mathcal{F}$ (calligraphic F) is used as the notation for the transform. Uppercase letters are used for the result of the forward transform. Analogously, the *inverse* Fourier transform (IFT) is defined as

$$s(r) = \mathcal{F}^{-1}\{S(k)\} = \int_{-\infty}^{+\infty} S(k)\, e^{+2\pi i r k}\, dk. \tag{A.46}$$

It can be shown that for continuous functions s,

$$s(r) = \mathcal{F}^{-1}\{\mathcal{F}\{s(r)\}\}. \tag{A.47}$$

From the definitions A.45 and A.46, it follows that for an even function $s_e(r)$,

$$\mathcal{F}\{s_e(r)\} = \mathcal{F}^{-1}\{s_e(r)\}. \tag{A.48}$$

If r is time with dimension seconds, k is the temporal frequency with dimension hertz. Related to the temporal frequency is the *angular frequency* $\omega = 2\pi k$ with dimension radians per second. In this case, the base function of the forward FT describes a rotation in the clockwise direction with angular velocity ω. If r is spatial position with dimension mm, k is spatial frequency with dimension mm^{-1}.

In this definition, the original signal and the result of the transform are one dimensional. In medical imaging, however, the signals are often multidimensional and vectors must be used in the definitions.

$$\text{Forward: } S(\vec{k}) = \mathcal{F}\{s(\vec{r})\} = \int_{-\infty}^{+\infty} s(\vec{r})\, e^{-2\pi i \vec{k}\cdot\vec{r}}\, d\vec{r}.$$
$$\text{Inverse: } s(\vec{r}) = \mathcal{F}^{-1}\{S(\vec{k})\} = \int_{-\infty}^{+\infty} S(\vec{k})\, e^{+2\pi i \vec{k}\cdot\vec{r}}\, d\vec{k}. \tag{A.49}$$

$\vec{r}$ and $\vec{k}$ are the conjugate variables, $\vec{r}$ being spatial position and $\vec{k}$ spatial frequency. Although only one integral sign is shown, it is understood that there are as many as there are independent variables. The original signal and its transform are known as a Fourier transform pair denoted as

$$s(r) \longleftrightarrow S(k). \tag{A.50}$$

In general, the result of the forward FT of a signal is a complex function. The *amplitude spectrum* is the modulus of its FT, while the *phase spectrum* is the phase of its FT. Both spectra show how amplitude and phase vary with spatial or temporal frequencies.

Often, the phase spectrum is considered irrelevant, and only the amplitude spectrum is considered. Note, however, that a signal is completely characterized if and only if *both* the amplitude and phase spectrum are specified.

A.4.2 Examples

A.4.2.1 Example 1

The FT of a rectangular pulse (Eq. A.15), scaled with amplitude A is

$$\begin{aligned}\mathcal{F}\left\{A\,\Pi\left(\frac{x}{2L}\right)\right\} &= \int_{-\infty}^{+\infty} A\,\Pi\left(\frac{x}{2L}\right)\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x\\ &= \int_{-L}^{+L} A\,\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x \qquad \text{(A.51)}\\ &= -\frac{A}{2\pi\mathrm{i}k}(\mathrm{e}^{-2\pi\mathrm{i}kL}-\mathrm{e}^{+2\pi\mathrm{i}kL}).\end{aligned}$$

Using Eqs. A.13 and A.18, we finally obtain

$$A\,\Pi\left(\frac{x}{2L}\right) \longleftrightarrow 2AL\ \mathrm{sinc}(2\pi kL). \qquad \text{(A.52)}$$

The forward FT of a rectangular pulse is a sinc function whose maximum amplitude is equal to the area of the pulse. The first zero-crossing occurs at

$$k = \frac{1}{2L}. \qquad \text{(A.53)}$$

Thus, the broader the width of the rectangular pulse in the original domain, the closer the first zero-crossing lies near the origin of the Fourier domain or the more "peaked" the sinc function is (see Figure A.1(c) and(f)).

A.4.2.2 Example 2

The forward FT of the product of a step function (Eq. A.14) and an exponential (Eq. A.12) (we assume $a > 0$) is

$$\begin{aligned}\mathcal{F}\{u(x)\,\mathrm{e}^{-ax}\} &= \int_{-\infty}^{+\infty} u(x)\,\mathrm{e}^{-ax}\,\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x\\ &= \int_{0}^{+\infty} \mathrm{e}^{-(a+2\pi\mathrm{i}k)x}\,\mathrm{d}x\\ &= \frac{1}{a+2\pi\mathrm{i}k} \qquad \text{(A.54)}\\ &= \frac{a}{a^2+4\pi^2k^2} - \mathrm{i}\frac{2\pi k}{a^2+4\pi^2k^2}.\end{aligned}$$

The result is complex; according to Eqs. A.8 and A.9, we have the following

$$\begin{aligned}\text{real part:}&\quad \frac{a}{a^2+4\pi^2k^2},\\ \text{imaginary part:}&\quad -\frac{2\pi k}{a^2+4\pi^2k^2}. \qquad \text{(A.55)}\end{aligned}$$

$$\begin{aligned}\text{modulus:}&\quad \frac{1}{\sqrt{a^2+4\pi^2k^2}}.\\ \text{phase:}&\quad -\arctan\left(\frac{2\pi k}{a}\right).\end{aligned}$$

This transform pair is a mathematical model of the filter shown in Figure 4.23.

A.4.2.3 Example 3

The forward FT of the Dirac impulse. Direct application of the sifting property (A.22) gives

$$\begin{aligned}\mathcal{F}\{\delta(x-x_0)\} &= \int_{-\infty}^{+\infty}\delta(x-x_0)\,\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x\\ &= \mathrm{e}^{-2\pi\mathrm{i}kx_0}. \qquad \text{(A.56)}\end{aligned}$$

The FT of a Dirac impulse at x_0 is complex: in the amplitude spectrum, all spatial frequencies are present with amplitude 1. The phase varies linearly with k with slope $-2\pi x_0$.

A difficulty arises when calculating the IFT:

$$\begin{aligned}s(x) &= \int_{-\infty}^{+\infty}\mathrm{e}^{-2\pi\mathrm{i}kx_0}\,\mathrm{e}^{+2\pi\mathrm{i}kx}\,\mathrm{d}k\\ &= \int_{-\infty}^{+\infty}\cos(2\pi k(x-x_0))\,\mathrm{d}k \qquad \text{(A.57)}\\ &\quad + \mathrm{i}\int_{-\infty}^{+\infty}\sin(2\pi k(x-x_0))\,\mathrm{d}k.\end{aligned}$$

Because its integrand is odd, the second integral is zero. The first integral has no meaning, unless it is interpreted according to the distribution theory. In this case, it can be shown that

$$\begin{aligned}\int_{-\infty}^{+\infty}\cos(2\pi k(x-x_0))\,\mathrm{d}k &= \int_{-\infty}^{+\infty}\mathrm{e}^{+2\pi\mathrm{i}k(x-x_0)}\,\mathrm{d}k\\ &= \delta(x-x_0). \qquad \text{(A.58)}\end{aligned}$$

Hence,

$$\delta(x-x_0) \longleftrightarrow \mathrm{e}^{-2\pi\mathrm{i}kx_0}. \qquad \text{(A.59)}$$

Table A.1 Important Fourier transform pairs in linear system theory

Image space	Fourier space
1	$\delta(k)$
$\delta(x)$	1
$\cos(2\pi k_0 x)$	$\frac{1}{2}(\delta(k+k_0)+\delta(k-k_0))$
$\sin(2\pi k_0 x)$	$\frac{i}{2}(\delta(k+k_0)-\delta(k-k_0))$
$\Pi(\frac{x}{2L})$	$2L\ \mathrm{sinc}(2\pi Lk)$
$\Lambda(\frac{x}{2L})$	$L\ \mathrm{sinc}^2(\pi Lk)$
$G_n(x;\mu,\sigma)$	$\exp(-i2\pi k\mu)\exp(-2\pi^2k^2\sigma^2)$

A.4.2.4 Example 4

The forward FT of a cosine function is

$$\begin{aligned}
\mathcal{F}\{\cos(2\pi k_0 x)\} \\
&= \int_{-\infty}^{+\infty} \cos(2\pi k_0 x)\,\mathrm{e}^{-2\pi \mathrm{i}kx}\,\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} \left(\frac{\mathrm{e}^{+2\pi \mathrm{i}k_0 x}+\mathrm{e}^{-2\pi \mathrm{i}k_0 x}}{2}\right) \mathrm{e}^{-2\pi \mathrm{i}kx}\,\mathrm{d}x \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi \mathrm{i}(k-k_0)x}\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi \mathrm{i}(k+k_0)x}\,\mathrm{d}x \\
&= \frac{1}{2}\,\delta(k-k_0) + \frac{1}{2}\,\delta(k+k_0).
\end{aligned} \tag{A.60}$$

The spectrum of a cosine function consists of two impulses at spatial frequencies k_0 and $-k_0$. In general it can be shown that a periodic function has a discrete spectrum (i.e., not all spatial frequencies are present), whereas an aperiodic function has a continuous spectrum. Table A.1 shows a list of FT pairs used in this book.

A.4.3 Properties

- ***Linearity*** If $s_1 \longleftrightarrow S_1$ and $s_2 \longleftrightarrow S_2$, then

$$c_1 s_1 + c_2 s_2 \longleftrightarrow c_1 S_1 + c_2 S_2 \qquad \forall c_1, c_2 \in \mathbb{C}. \tag{A.61}$$

  This can easily be extended to more than two signals.

- ***Scaling*** If $s(x) \longleftrightarrow S(k)$, then

$$s(ax) \longleftrightarrow \frac{1}{|a|} S\left(\frac{k}{a}\right) \qquad a \in \mathbb{R}_0. \tag{A.62}$$

- ***Translation*** If $s(x) \longleftrightarrow S(k)$, then

$$s(x - x_0) \longleftrightarrow \mathrm{e}^{-2\pi \mathrm{i}x_0 k}\ S(k) \qquad x_0 \in \mathbb{R}. \tag{A.63}$$

  Thus, translating a signal over a distance x_0 only modifies its *phase* spectrum.

- Transfer function and impulse response (or PSF) are a FT pair. Indeed, Eq. A.42 shows that

$$h(x) \longleftrightarrow H(k). \tag{A.64}$$

In imaging, the FT of the PSF is known as the *optical transfer function* (OTF). The modulus of the OTF is the *modulation transfer function* (MTF). As mentioned in Chapter 1, the PSF and OTF characterize the resolution of the system. If the PSF is expressed in mm, the OTF is expressed in mm^{-1}. Often, line pairs per millimeter (lp/mm) is used instead of mm^{-1}. The origin of this unit can easily be understood if an image with sinusoidal intensity lines at a frequency of 1 period per millimeter or 1 lp/mm, that is, one dark and one bright line per millimeter, is observed (Figure A.3). This line pattern can be written as $\sin(2\pi x)$, x expressed in mm. The Fourier transform of this function consists of two impulses at spatial frequency 1 mm^{-1}

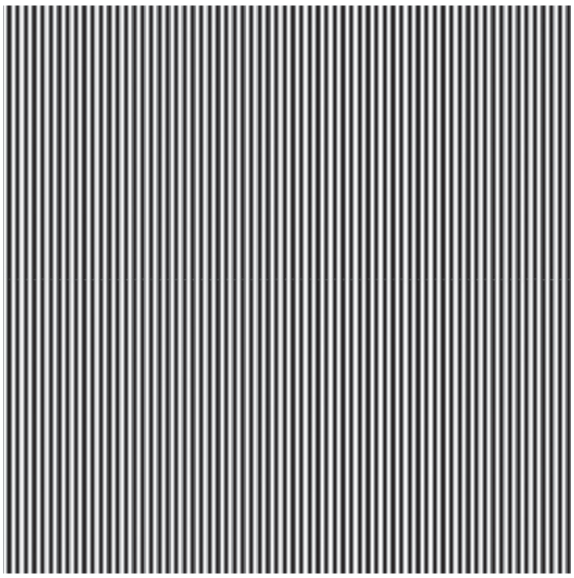

Figure A.3 Image with sinusoidal intensity lines at a frequency of 1 lp/mm.

and $-1\ \text{mm}^{-1}$. This then explains why the frequency units mm^{-1} and lp/mm can be used as synonyms.

The resolution of an imaging system is sometimes characterized by the distinguishable number of line pairs per millimeter. It is clear now that this is a limited and subjective measure, and that it is preferable to show the complete OTF curve when talking about the resolution. Nevertheless, it is common practice in the technical documents of medical imaging equipment and in the medical literature simply to list an indication of the resolution in lp/mm at a specified small amplitude (in %) of the OTF.

- ***Convolution*** In Section A.3.3 it was concluded that an output function s_o of a LSI system can be calculated in two ways: (1) $s_o = s_i * h$ in the image domain or (2) $\mathcal{F}^{-1}\{S_o(k) = S_i(k)H(k)\}$ in the Fourier domain. In general, if $s_1 \longleftrightarrow S_1$ and $s_2 \longleftrightarrow S_2$, then

$$\begin{aligned} s_1 * s_2 &\longleftrightarrow S_1 \cdot S_2 \\ s_1 \cdot s_2 &\longleftrightarrow S_1 * S_2. \end{aligned} \tag{A.65}$$

This is a very important property. The convolution of two signals can be calculated via the Fourier transform by calculating the forward–inverse FT of both signals, multiplying the FT results, and calculating the inverse–forward FT of the product.

- ***The FT of a real signal is Hermitian***

$$S(-\vec{k}) = \bar{S}(\vec{k}) \quad \text{if } s(x) \in \mathbb{R}, \tag{A.66}$$

where $\bar{S}$ denotes the complex conjugate of S (i.e., the real part is even and the imaginary part is odd). From Eqs. A.6, A.13, and A.45, we obtain

$$\begin{aligned} S(k) &= \int_{-\infty}^{+\infty} s(x)\, \mathrm{e}^{-2\pi \mathrm{i} kx}\, \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} [s_e(x) + s_o(x)] \\ &\quad \cdot [\cos(2\pi kx) - \mathrm{i}\sin(2\pi kx)]\, \mathrm{d}x \\ &= \int_{-\infty}^{+\infty} s_e(x)\cos(2\pi kx)\, \mathrm{d}x \\ &\quad - \mathrm{i}\int_{-\infty}^{+\infty} s_o(x)\sin(2\pi kx)\, \mathrm{d}x. \end{aligned} \tag{A.67}$$

The first integral is the real even part of $S(k)$, and the second is the imaginary odd part of $S(k)$. Hence, to compute the FT of a real signal, it suffices to know one half-plane. The other half-plane can then be computed using Eq. A.66.

Eq. A.67 further shows that if a function is even (odd), its FT is even (odd). Consequently, if a function is real and even, its FT is real and even, whereas if a function is real and odd, its FT is imaginary and odd.

- ***Parseval's theorem***

$$\int_{-\infty}^{+\infty} |s(x)|^2\, \mathrm{d}x = \int_{-\infty}^{+\infty} |S(k)|^2\, \mathrm{d}k. \tag{A.68}$$

- ***Separability*** In many cases, a 2D FT can be calculated as two subsequent 1D FTs. The transform is then called *separable*. For example,

$$\begin{aligned} &\mathcal{F}\{\operatorname{sinc}(x)\operatorname{sinc}(y)\} \\ &= \iint_{-\infty}^{+\infty} \frac{\sin(x)}{x}\frac{\sin(y)}{y}\, \mathrm{e}^{-2\pi\mathrm{i}(k_x x + k_y y)}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-\infty}^{+\infty} \frac{\sin(x)}{x}\mathrm{e}^{-2\pi\mathrm{i}k_x x}\, \mathrm{d}x \\ &\quad \cdot \int_{-\infty}^{+\infty} \frac{\sin(y)}{y}\mathrm{e}^{-2\pi\mathrm{i}k_y y}\, \mathrm{d}y \\ &= \mathcal{F}\{\operatorname{sinc}(x)\}\,\mathcal{F}\{\operatorname{sinc}(y)\}. \end{aligned} \tag{A.69}$$

- Another important property of a 2D FT is the projection theorem or central-slice theorem. It is discussed in Chapter 3 on X-ray computed tomography.

A.4.4 Polar Form of the Fourier Transform

Using polar coordinates

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned} \tag{A.70}$$

Eq. A.49

$$S(k_x, k_y) = \iint_{-\infty}^{+\infty} s(x, y)\, \mathrm{e}^{-2\pi\mathrm{i}(k_x x + k_y y)}\, \mathrm{d}x\, \mathrm{d}y \tag{A.71}$$

can be rewritten as

$$\begin{aligned} &S(k_x, k_y) \\ &= \int_0^{2\pi}\int_0^{+\infty} s(r,\theta)\, \mathrm{e}^{-2\pi\mathrm{i}(k_x r\cos\theta + k_y r\sin\theta)}\, r\, \mathrm{d}r\, \mathrm{d}\theta \\ &= \int_0^{\pi}\int_{-\infty}^{+\infty} s(r,\theta)\, \mathrm{e}^{-2\pi\mathrm{i}(k_x r\cos\theta + k_y r\sin\theta)}\, |r|\, \mathrm{d}r\, \mathrm{d}\theta. \end{aligned} \tag{A.72}$$

The factor r in the integrand is the Jacobian of the transformation:

$$J \triangleq \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta}. \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\,(\cos^2\theta + \sin^2\theta) = r. \quad \text{(A.73)}$$

The polar form of the inverse FT is obtained analogously. Let

$$\begin{aligned} k_x &= k\,\cos\phi \\ k_y &= k\,\sin\phi, \end{aligned} \quad \text{(A.74)}$$

then

$$s(x,y) \quad \text{(A.75)}$$

$$= \int_0^{\pi}\int_{-\infty}^{+\infty} S(k,\phi)\,\mathrm{e}^{+2\pi \mathrm{i}(xk\,\cos\phi + yk\,\sin\phi)}\,|k|\,\mathrm{d}k\,\mathrm{d}\phi.$$

A.5 Sampling

Equation (A.1) represents an analog continuous signal, which is defined for all spatial positions and can have any (real or complex) value:

$$s(x) \;\; \forall\, x \in \mathbb{R}. \quad \text{(A.76)}$$

In practice, the signal is often *sampled*, that is, only discrete values at regular intervals are measured:

$$s_s(x) = s(n\Delta x)\; n \in \mathbb{Z}. \quad \text{(A.77)}$$

The constant Δx is the sampling distance. Information may be lost by sampling. However, *under certain conditions*, a continuous signal can be completely recovered from its samples. These conditions are specified by the *sampling theorem*, which is also known as the *Nyquist criterion*. If the Fourier transform of a given signal is band limited and if the sampling frequency is *larger than twice the maximum spatial frequency* present in the signal, then the samples *uniquely* define the given signal. Hence,

$$\text{if} \quad \begin{cases} S(k) = 0 & \forall\, |k| > k_{\max} \quad \text{and} \\ \dfrac{1}{\Delta x} > 2k_{\max} \end{cases} \quad \text{(A.78)}$$

$$\text{then} \quad s_s(x) = s(n\Delta x) \quad \text{uniquely defines } s(x).$$

To prove this theorem, sampling is defined as a multiplication with an impulse train (see Figure A.4):

$$s_s(x) = s(x) \cdot \text{Ш}(x), \quad \text{(A.79)}$$

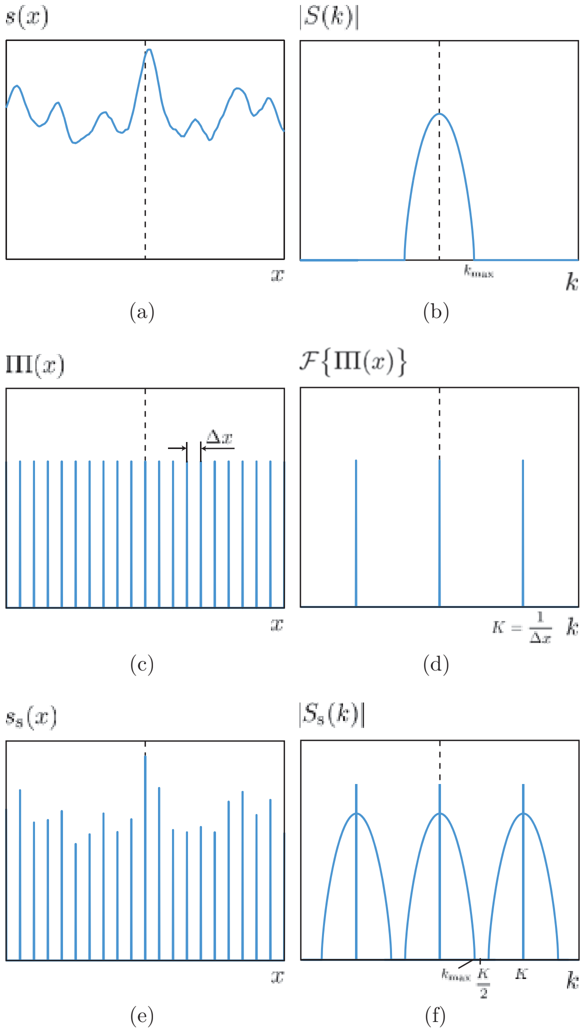

Figure A.4 A signal with an infinite spatial extent **(a)** and its band-limited Fourier transform **(b)**. The sampled signal **(e)** is obtained by multiplying **(a)** by the impulse train **(c)**. The spectrum **(f)** of the sampled signal is found by convolving the original spectrum **(b)** with the Fourier transform of the impulse train **(d)**. This results in a periodic repetition of the original spectrum.

where $\text{Ш}(x)$ is the comb function or impulse train:

$$\text{Ш}(x) = \sum_{n=-\infty}^{+\infty} \delta(x - n\Delta x). \quad \text{(A.80)}$$

The sampling distance Δx is the distance between any two consecutive Dirac impulses. Note that this formula is a *formal* notation because the product is only valid as an integrand.

Based on Eq. A.80 and using the convolution theorem, the Fourier transform $S_s(k)$ can be written as

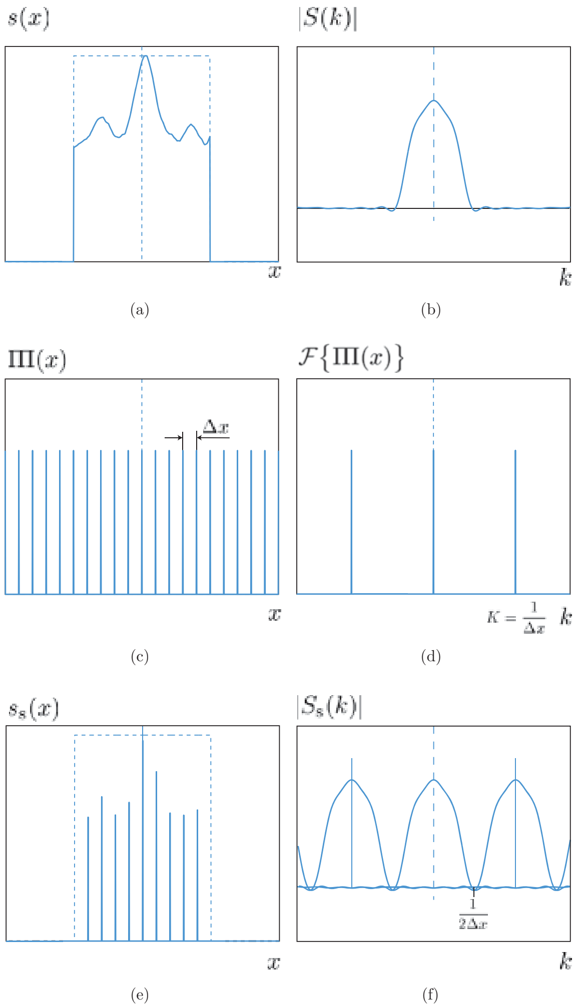

Figure A.5 A signal with a finite spatial extent **(a)** is not band limited **(b)**. The sampled signal **(e)** is obtained by multiplying **(a)** by the impulse train **(c)**. The spectrum **(f)** of the sampled signal is found by convolving the original spectrum **(b)** with the Fourier transform of the impulse train **(d)**. This results in a periodic repetition of the original spectrum. Because of the overlap, aliasing cannot be avoided.

follows:

$$S_s(k) = S(k) * \mathcal{F}\{Ш(x)\}. \tag{A.81}$$

It can be shown that

$$\mathcal{F}\{Ш(x)\} = K \sum_{l=-\infty}^{+\infty} \delta(k - lK), \tag{A.82}$$

which is again an impulse train with consecutive impulses separated by the sampling frequency

$$K = \frac{1}{\Delta x}. \tag{A.83}$$

Hence,

$$\begin{aligned} S_s(k) = K(&S(k) + S(k-K) + S(k+K) \\ &+ S(k-2K) + S(k+2K) + \cdots). \end{aligned} \tag{A.84}$$

Because

$$S(k) = 0 \;\forall\; |k| \geq \frac{K}{2}$$

it follows that

$$K\, S(k) = S_s(k)\, \Pi\left(\frac{x}{K}\right), \tag{A.85}$$

and consequently $s(x)$ can be recovered from $S_s(k)$.

If the signal $s(x)$ is not band limited or if it is band limited but $\frac{1}{\Delta x} \leq 2k_{\max}$, the shifted replicas of $S(k)$ in Eq. A.84 will overlap (see Figure A.5). In that case, the spectrum of $s(x)$ cannot be recovered by multiplication with a rectangular pulse. This phenomenon is known as *aliasing* and is unavoidable if the original signal $s(x)$ is not band limited. As an important example, note that a patient *always* has a limited spatial extent, which implies that the FT of an image of the body is never band limited and, consequently, aliasing is unavoidable. Several practical examples of aliasing are given in this textbook

Numerical methods calculate the Fourier transform for a limited number of discrete points in the frequency band $(-k_N, +k_N)$. This means that not only the signal but also its Fourier transform is sampled. Sampling the Fourier data implies that it yields shifted replicas in the signal s, which may overlap. To avoid such overlap or aliasing of the signal, the sampling distance Δk must also be chosen small enough. It can easily be shown that this condition can be satisfied if the number of samples in the Fourier domain is at least equal to the number of samples in the signal domain. In practice they are chosen equal.

Based on the preceding considerations, the *discrete Fourier transform* (DFT) for 2D signals can be written as (more details can be found in Brigham[52]):

$$S(m\Delta k_x, n\Delta k_y) = \sum_{q=0}^{N-1} \sum_{p=0}^{M-1} s(p\Delta x, q\Delta y)\, \mathrm{e}^{-2\pi \mathrm{i}\left(\frac{mp}{M} + \frac{nq}{N}\right)},$$
$$s(p\Delta x, q\Delta y) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} S(m\Delta k_x, n\Delta k_y)\, \mathrm{e}^{2\pi \mathrm{i}\left(\frac{mp}{M} + \frac{nq}{N}\right)}. \tag{A.86}$$

52 E. Oran Brigham. *The Fast Fourier Transform and Its Applications*. Prentice-Hall International, Englewood Cliffs, New Jersey, first edition, 1988.

In both cases, $m, p = 0, 1, \ldots, M - 1$ and $n, q = 0, 1, \ldots, N - 1$. Here, M and N need not be equal because both directions can be sampled differently. However, for a particular direction, the number of samples in the spatial and the Fourier domain is the same.

Direct computation of the DFT is a time-consuming process. However, when the number of samples is a power of two, a computationally very fast algorithm can be employed: the *fast Fourier transform* or FFT. The FFT algorithm has become very important in signal and image processing, and hardware versions are frequently used in today's medical equipment. The properties and applications of the FFT are the subject of Brigham's *The Fast Fourier Transform and Its Applications*.[52]