Calculus: Solutions Manual

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Contents

1 | Real Numbers and Functions

1.1 Field and Order Properties

Task 1.1.5. To verify that a given number y is the additive inverse of another number x, we have to check whether $x + y$ equals zero. In our case, we have $y = (-a) + (-b)$ and $x = a + b$. Therefore, we compute as follows:

$$\begin{aligned}(a + b) + ((-a) + (-b)) &= (b + a) + ((-a) + (-b)) = ((b + a) + (-a)) + (-b) \\ &= (b + (a + (-a))) + (-b) = (b + 0) + (-b) = b + (-b) \\ &= 0.\end{aligned}$$

Similarly, to verify that one number is the multiplicative inverse of another, we need to check whether their product is one.

$$\begin{aligned}(ab)(a^{-1}b^{-1}) &= (ba)(a^{-1}b^{-1}) = ((ba)a^{-1})b^{-1} \\ &= (b(a \cdot a^{-1}))b^{-1} = (b \cdot 1)b^{-1} = b \cdot b^{-1} = 1.\end{aligned}$$

Task 1.1.6. This is a special case of $(-a)(-b) = ab$, which has already been established:

$$(-x)^2 = (-x)(-x) = x \cdot x = x^2.$$

Task 1.1.7.

(a) Apply the same principles as in Task 1.1.5:

$$\frac{a}{b} + \frac{-a}{b} = ab^{-1} + (-a)b^{-1} = (a + (-a))b^{-1} = 0 \cdot b^{-1} = 0 \implies -\frac{a}{b} = \frac{-a}{b}.$$

Next, note that $(-b)^{-1} = ((-1)b)^{-1} = (-1)^{-1}b^{-1} = (-1)b^{-1} = -(b^{-1})$. Hence,

$$\frac{a}{b} + \frac{a}{-b} = ab^{-1} + a(-b)^{-1} = a(b^{-1} + (-b)^{-1}) = a \cdot 0 = 0 \implies -\frac{a}{b} = \frac{a}{-b}.$$

Alternately, make repeated use of $-x = (-1)x$. For example,

$$-\frac{a}{b} = -(ab^{-1}) = (-1)(ab^{-1}) = ((-1)a)b^{-1} = (-a)b^{-1} = \frac{-a}{b}.$$

(b) We present a terse solution below, in which the use of the commutative and associative properties is hidden.

$$\begin{aligned}\frac{a}{b}+\frac{c}{d} &= ab^{-1}+cd^{-1} = add^{-1}b^{-1}+cbb^{-1}d^{-1}\\ &= (ad)(bd)^{-1}+(cb)(bd)^{-1} = (ad+bc)(bd)^{-1}\\ &= \frac{ad+bc}{bd}.\end{aligned}$$

Task 1.1.11. We shall prove that for every element $x \in A$, we can find a $y \in A$ with $y > x$. Hence A has no greatest element.

Let us consider any $x \in A$. Then we have $x < 1$. Hence $y = \dfrac{x+1}{2}$ satisfies $x < y < 1$. Therefore, $y \in A$ but $y > x$.

Task 1.1.13. We note that x equals either $|x|$ or $-|x|$. Now $|x| \leq a$ gives $-|x| \geq -a$. Therefore,

$$|x| \leq a \implies -a \leq -|x| \leq |x| \leq a \implies -a \leq x \leq a.$$

For the converse, we similarly use the fact that $|x|$ equals either x or $-x$. Then $-a \leq x \leq a$ gives $a \geq -x \geq -a$. Therefore,

$$-a \leq x \leq a \implies -a \leq \pm x \leq a \implies |x| \leq a.$$

Alternately, we could carry out a case-by-case proof based on the sign of x.

Task 1.1.15. $\dfrac{a}{b} = \dfrac{c}{d} \iff ab^{-1} = cd^{-1} \iff ab^{-1}bd = cd^{-1}bd \iff ad = bc$.

Task 1.1.17. We have already proved that $(x^{-1})^n = (x^n)^{-1}$ for every $n \in \mathbb{N}$. It is also true when $n = 0$, since both sides become 1. Now consider a negative integer n. Then $n = -m$, with $m \in \mathbb{N}$, and

$$\begin{aligned}(x^{-1})^n &= (x^{-1})^{-m} = ((x^{-1})^m)^{-1} \quad \text{(by definition of } a^{-m})\\ &= ((x^m)^{-1})^{-1} = (x^{-m})^{-1} = (x^n)^{-1}.\end{aligned}$$

Task 1.1.18. Let $P(n)$ be the statement that $\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$.

Then $P(1)$ is the statement $|x_1| \leq |x_1|$, which is certainly true. Further, $P(2)$ is true, as it is the triangle inequality.

Now assume that $P(n)$ is true. We use this assumption to establish $P(n+1)$ as follows:

$$\begin{aligned}\left|\sum_{i=1}^{n+1} x_i\right| &= \left|\sum_{i=1}^{n} x_i + x_{n+1}\right|\\ &\leq \left|\sum_{i=1}^{n} x_i\right| + |x_{n+1}| \quad \text{(by } P(2))\\ &\leq \sum_{i=1}^{n} |x_i| + |x_{n+1}| \quad \text{(by } P(n))\end{aligned}$$

$$= \sum_{i=1}^{n+1} |x_i|.$$

Exercises for §1.1

2. We just subtract numbers and see if the result is positive or negative. Let's first compare the positive numbers with each other: 3, $\frac{14}{10}$, $\frac{17}{12}$, 2.

$$3 - 2 = 1 > 0 \implies 3 > 2$$
$$2 - \frac{17}{12} = \frac{24-17}{12} = \frac{7}{12} > 0 \implies 2 > \frac{17}{12}$$
$$\frac{17}{12} - \frac{14}{10} = \frac{85-84}{60} = \frac{1}{60} > 0 \implies \frac{17}{12} > \frac{14}{10}$$

Therefore,

$$\frac{14}{10} < \frac{17}{12} < 2 < 3.$$

We can similarly compare the two negative numbers. The final arrangement is:

$$-2 < -\frac{3}{2} < \frac{14}{10} < \frac{17}{12} < 2 < 3.$$

4. The conditions of Exercise 3 yield the following sets:

(a) $A = \{ x \in \mathbb{R} \mid x > 1 \text{ or } x < 0 \}$

(b) $B = \{ x \in \mathbb{R} \mid x \geq 1/3 \text{ or } x \leq -1 \}$

(c) $C = \{ x \in \mathbb{R} \mid 2 < x < 3 \}$

We need to find $A \cap B \cap C$. Since C has the simplest structure, we investigate its intersections with the other sets. We observe that every element of C meets the requirements for being in A as well as B. Therefore,

$$A \cap B \cap C = C = \{ x \in \mathbb{R} \mid 2 < x < 3 \}.$$

6.

(a) Consider $n = 1$. The only possibility for $k \in \mathbb{N}$ with $k \leq 1$ is $k = 1$. And we are given that $1 \in A$. Therefore $1 \in S$.

Next let $n \in S$. Consider any $k \in \mathbb{N}$ with $k \leq n + 1$. If $k \leq n$ then $n \in S \implies k \in A$. Therefore, $1, \ldots, n \in A$ and so $n + 1 \in A$. This gives $n + 1 \in S$.

(b) By the Principle of Mathematical Induction, we obtain $S = \mathbb{N}$. By considering $k = n$ in the definition of S, we see that $S \subseteq A$. Therefore $A = \mathbb{N}$.

8. Let $S \subseteq \mathbb{N}$ such that $1 \in S$ and $n \in S \implies n + 1 \in S$.

Suppose that $S \neq \mathbb{N}$. Then A, defined to be the complement of S in $\mathbb{N}$, is non-empty. By the Well Ordering Principle, A has a least element N. Now, $1 \in S \implies 1 \notin A \implies N \neq 1 \implies N - 1 \in \mathbb{N}$. Further, $N - 1 \notin A$, since N is the laest element of A. But then $N - 1 \in S$ and $N \notin S$, a contradiction.

10. We shall apply mathematical induction.

(a) Let $A = \{ n \in \mathbb{N} \mid 1^n = n \}$. Then $1^1 = 1 \cdot 1^0 = 1 \cdot 1 = 1 \implies 1 \in A$. Now assume $n \in A$. Then

$$1^{n+1} = 1 \cdot 1^n = 1 \cdot 1 = 1 \implies n+1 \in A.$$

By mathematical induction, $A = \mathbb{N}$.

(b) Let $A = \{ n \in \mathbb{N} \mid a^n < b^n \}$. Then $a^1 = a < b = b^1 \implies 1 \in A$. Now assume $n \in A$. Then

$$a^{n+1} = a \cdot a^n < b \cdot a^n < b \cdot b^n = b^{n+1} \implies n+1 \in A.$$

By mathematical induction, $A = \mathbb{N}$.

12.

(a) (This part needs $k < n$) We have a direct calculation:

$$\begin{aligned} \binom{n}{k} + \binom{n}{k+1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} \\ &= \frac{k+1}{n+1}\binom{n+1}{k+1} + \frac{n-k}{n+1}\binom{n+1}{k+1} \\ &= \binom{n+1}{k+1} \end{aligned}$$

(b) Apply mathematical induction. Let

$$A = \left\{ n \in \mathbb{N} \mid \binom{n}{k} \in \mathbb{N} \text{ for every } k = 0, \ldots, n \right\}.$$

Now,

$$\binom{1}{0} = \binom{1}{1} = 1 \in \mathbb{N} \implies 1 \in A.$$

Suppose $n \in \mathbb{N}$. Then for $k = 1, \ldots, n$ we have

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \in \mathbb{N}.$$

Further,

$$\binom{n+1}{0} = \binom{n+1}{n+1} = 1 \in \mathbb{N}.$$

Therefore, $n+1 \in A$.

14. We will mimic the proof that there is no rational number whose square is 2. For that, we will need to establish that if 3 divides the square of a natural number m, then it divides m as well. An equivalent statement is that if 3 does not divide m then it does not divide m^2. Now, if 3 does not divide m, then m has one of the forms $3k+1$ or $3k+2$, for a whole number k.

$$\begin{aligned} m = 3k+1 &\implies m^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1 = 3n + 1 \\ m = 3k+2 &\implies m^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1 = 3n + 1 \end{aligned}$$

In either case, 3 does not divide m^2.

Now we can begin our proof. Suppose $(p/q)^2 = 3$ where $p, q \in \mathbb{Z}$. We may assume that $p, q \in \mathbb{N}$ and they have no common factor except 1. Then,

$$p^2 = 3q^2 \implies 3 \text{ divides } p^2 \implies 3 \text{ divides } p.$$

Hence $p = 3k$ for some $k \in \mathbb{N}$. Now,

$$(3k)^2 = 3q^2 \implies 3k^2 = q^2 \text{ divides } q^2 \implies 3 \text{ divides } q.$$

So 3 is a factor of both p and q, contradicting our assumption that their only common factor was 1.

1.2 Completeness Axiom and Archimedean Property

Task 1.2.2. Take any real number x and ask whether it can serve as an upper bound for the empty set $\varnothing$. The only way x could *fail* to be an upper bound is if there is $y \in \varnothing$ such that $y > x$. Clearly, $\varnothing$ has no such y, so we must accept x as an upper bound. Thus, every real number is an upper bound of $\varnothing$.

Similarly, every real number is a lower bound of $\varnothing$.

Students often object, stating that x can't be an upper bound since $\varnothing$ has no smaller element. One can ask them whether they accept $\varnothing$ as a subset of every set – which they generally find less counterintuitive – and to compare the reasoning in both situations.

Task 1.2.4. An upper bound of $\mathbb{Z}$ would also be an upper bound of $\mathbb{N}$. Hence, by the Archimedean property, $\mathbb{Z}$ has no upper bound.

Now, suppose $x \in \mathbb{R}$ is a lower bound of $\mathbb{Z}$. Then $x \leq -m$ for every $m \in \mathbb{N}$, hence $-x \geq m$ for every $m \in \mathbb{N}$. That is, $-x$ is an upper bound of $\mathbb{N}$. Since this is impossible, $\mathbb{Z}$ has no lower bound.

Task 1.2.10. Consider distinct real numbers x, y with $x < y$. Suppose that $(x, y) \cap \mathbb{Q}$ is finite. Then there is $n \in \mathbb{N}$ such that $(x, y) \cap \mathbb{Q} = \{q_1, \ldots, q_n\}$ with $x < q_1 < \cdots < q_n < y$. By denseness, there is a rational $q_{n+1} \in (q_n, y)$, a contradiction to our description of $(x, y) \cap \mathbb{Q}$.

Task 1.2.11. We use proof by contradiction and the field properties of $\mathbb{Q}$.

(a) If $-t \in \mathbb{Q}$ then $t = -(-t) \in \mathbb{Q}$. Similarly, if $1/t \in \mathbb{Q}$ then $t = 1/(1/t) \in \mathbb{Q}$.

(b) If $r+t \in \mathbb{Q}$ then $t = (r+t) - r \in \mathbb{Q}$. If $r - t \in \mathbb{Q}$ then $t = r - (r-t) \in \mathbb{Q}$.

(c) If $rt \in \mathbb{Q}$ then $t = (rt)/r \in \mathbb{Q}$. If $r/t \in \mathbb{Q}$ then $t = r/(r/t) \in \mathbb{Q}$.

Task 1.2.15. We have $[a, b] = \{\, x \in \mathbb{R} \mid a \leq x \leq b \,\}$. If $x \in [a, b]$ then $x \leq b$. Therefore b is an upper bound of $[a, b]$. Let u be any upper bound of $[a, b]$. Then $b \in [a, b]$ givees $b \leq u$. Therefore b is the least among all the upper bounds.

Task 1.2.16. We have $(a, b) = \{\, x \in \mathbb{R} \mid a < x < b \,\}$. If $x \in (a, b)$ then $x < b$. Therefore b is an upper bound of (a, b). Let u be any upper bound of $[a, b]$. Then $a < (a + b)/2 < b$ implies $(a + b)/2 \in (a, b)$, hence $u \geq (a + b)/2 > a$. Suppose

$u < b$. Then $a < u < (u+b)/2 < b$ gives $(u+b)/2 \in (a,b)$ yet $(u+b)/2 > u$, contradicting the choice of u as an upper bound of (a,b). Therefore $u \geq b$, and so b is the least among the upper bounds of (a,b).

Task 1.2.17. This has to be checked for each type of interval. We illustrate the solutions for two types:

Suppose $I = [\alpha, \beta]$. Then $\alpha \leq a < x < b \leq \beta$ gives $\alpha < x < \beta$, hence $x \in [\alpha, \beta]$.

Suppose $I = [\alpha, \infty)$. Then $\alpha \leq a < x$ gives $\alpha < x$, hence $x \in [\alpha, \infty)$.

Exercises for §1.2

2. The rational numbers can be arranged using the field and order axioms, as we did in Exercise 2 of §1.1. This gives

$$-2 < -\frac{3}{2} < \frac{14}{10} < 3.$$

To compare the roots, we use the squares. For example,

$$\left(\frac{14}{10}\right)^2 = \frac{196}{100} < 2 < 5 < 3^2 \implies \frac{14}{10} < \sqrt{2} < \sqrt{5} < 3.$$

Similarly,

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2 \implies \frac{3}{2} > \sqrt{2} \implies -\frac{3}{2} < -\sqrt{2}.$$

So the final rankings are:

$$-2 < -\frac{3}{2} < -\sqrt{2} < \frac{14}{10} < \sqrt{2} < \sqrt{5} < 3.$$

4. Let A be a non-empty subset of $\mathbb{R}$ and let ℓ be a lower bound of A. Define $B = \{-x \mid x \in A\}$. Then $-\ell$ is an upper bound of B: $y \in B \implies -y \in A \implies \ell \leq -y \implies -\ell \geq y$.

By the LUB property, B has a least upper bound β. We shall show that $\alpha = -\beta$ is the greatest lower bound of A.

If $x \in A$ then $-x \in B$, hence $\beta \geq -x$ and $\alpha \leq x$. Thus, α is a lower bound of A.

Now let m be any lower bound of A. As we saw above, $-m$ is an upper bound of B. Therefore, $\beta \leq -m$ and $\alpha \geq m$. Thus α is the greatest among the lower bounds of A.

6. The general element of this set is $a(n) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$. Now,

$$a(n) = 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$

Let x be any real number. By the Archimedean property there is $n \in \mathbb{N}$ with $n > x$. Then $a(n^2) \geq n > x$, hence x is not an upper bound of this set.

8. Observe that $\dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big) = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big)$$
$$= 1 - \frac{1}{n+1} < 1.$$

So 1 is an upper bound for this set.

10. Note that the $c = 0$ cases are trivial, as then $cA = \{0\}$.

(a) Suppose $c > 0$ and A is bounded above. By LUB property, A has a supremum α. We need to show that $c\alpha$ is the supremum of cA.

$$y \in cA \implies y = cx \text{ for some } x \in A.$$

Then $x \leq \alpha \implies y = cx \leq c\alpha$, hence $c\alpha$ is an upper bound of cA.

Now let u be any upper bound of cA. For any $x \in A$ we have $u \geq cx$, hence $u/c \geq x$. Thus u/c is an upper bound of A. Therefore $\alpha \leq u/c$ and $c\alpha \leq u$. So $c\alpha$ is the least of the upper bounds of cA.

Next, suppose $c < 0$ and A is bounded below. By Exercise 4, A has an infimum α. In fact, the solution of Exercise 4 shows that $\alpha = -\sup((-1)A)$. Hence $c\alpha = (-c)\sup((-1)A) = \sup((-c)(-1)A) = \sup(cA)$, using $-c > 0$ and applying the previous case.

12. We have non-empty sets A, B that are bounded below. We have defined $A + B$ and AB.

(a) We have to prove that $\inf(A+B) = \inf(A)+\inf(B)$. First, for any $a \in A$ and $b \in B$ we have

$$a \geq \inf(A), b \geq \inf(B) \implies a + b \geq \inf(A) + \inf(B).$$

Therefore $\inf(A) + \inf(B)$ is a lower bound of $A + B$. To prove that it is the greatest lower bound, we shall show that for every $\varepsilon > 0$, $\inf(A) + \inf(B) + \varepsilon$ is not a lower bound of $A + B$. We note that we have $a \in A$ and $b \in B$ such that $a < \inf(A) + \varepsilon/2$ and $b < \inf(B) + \varepsilon/2$. Then $a + b \in A + B$ and $a + b < \inf(A) + \inf(B) + \varepsilon$.

(b) Assuming that all the members of A, B are non-negative, we have to show that $\inf(AB) = \inf(A)\inf(B)$. First, we note that 0 is a lower bound for both A and B, hence $\inf(A), \inf(B) \geq 0$. Now, for any $a \in A$ and $b \in B$ we have

$$a \geq \inf(A), b \geq \inf(B) \implies ab \geq \inf(A)\inf(B).$$

Therefore $\inf(A)\inf(B)$ is a lower bound of AB. Further, for any $\varepsilon > 0$ we have $a \in A$ and $b \in B$ such that $a < \inf(A)\sqrt{1+\varepsilon}$ and $b < \inf(B)\sqrt{1+\varepsilon}$. Then $ab \in A$ and $ab < \inf(A)\inf(B)(1 + \varepsilon)$.

1.3 Functions

Task 1.3.3. The domain is $X \times X$ and the codomain is X.

Task 1.3.4. Suppose f is one-one and $f(a) = f(b)$. If $a \neq b$ then $f(a) \neq f(b)$ gives a contradiction. Hence $f() = f(b)$.

Now suppose that f has the property that $f(a) = f(b)$ implies $a = b$. Let $x, y \in X$ with $x \neq y$. If $f(x) = f(y)$ then $x = y$, a contradiction. Therefore $f(x) \neq f(y)$.

Task 1.3.7.

(a) For $x \leq 0$ we have $f(x) = \frac{1}{2}(x + (-x)) = 0$. Hence $f(-1) = f(-2) = 0$ and f is not one-one. Further, for $x \geq 0$, we have $f(x) = \frac{1}{2}(x + x) = x$. Hence f is not onto, and its image is $[0, \infty)$.

(b) We have $g(-1) = g(1) = 1$, so g is not one-one. We know every $x^2 \geq 0$, so the image of g is a subset of $[0, \infty)$. On the other hand, we know every non-negative real number has a square root, so the image of g is all of $[0, \infty)$.

(c) For $x \geq 0$ we have $h(x) \geq 1$ and for $x < 0$ we have $h(x) < 1$. So $h(a) = h(b)$ is only possible if a, b have the same sign.

Suppose $a, b \geq 0$ and $h(a) = h(b)$. Then,

$$\begin{aligned} a^2 + a + 1 = b^2 + b + 1 &\implies (a^2 - b^2) + (a - b) = 0 \\ &\implies (a - b)(a + b + 1) = 0 \\ &\implies a - b = 0 \implies a = b. \end{aligned}$$

Next, suppose $a, b < 0$ and $h(a) = h(b)$. Then $a + 1 = b + 1 \implies a = b$.

So h is one-one.

Every $y \geq 1$ has pre-image $\frac{1}{2}(-1 + \sqrt{1 + 4(y - 1)}) \geq 0$. Every $y < 1$ has pre-image $y - 1$. So h is onto.

Task 1.3.9. Let $y, y' \in Y$ and $f^{-1}(y) = f(^{-1}y')$. Now, $x = f^{-1}(y)$ satisfies $f(x) = y$ while $x' = f^{-1}(y')$ satisfies $f(x') = y'$. Then $x = x'$ gives $y = f(x) = f(x') = y'$. So f^{-1} is one-one.

Let $x \in X$. Define $y = f(x)$. Then $f^{-1}(y) = x$. So f^{-1} is onto.

Finally, since f reverses f^{-1}, we have $f = (f^{-1})^{-1}$.

Task 1.3.10.

$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x))) = (h \circ g)(f(x)) = ((h \circ g) \circ f)(x)$.

Task 1.3.11.

First, suppose g is the inverse function of f. Then $g(y) = x \iff f(x) = y$.

Consider any $x \in X$ and $y \in Y$, with $y = f(x)$. Then:

$$(g \circ f)(x) = g(f(x)) = g(y) = x = 1_X(x),$$
$$(f \circ g)(y) = f(g(y)) = f(x) = y = 1_Y(y).$$

Now suppose $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then:

$$y = f(x) \implies g(y) = g(f(x)) \implies g(y) = 1_X(x) = x,$$
$$x = g(y) \implies f(x) = f(g(y)) \implies f(x) = 1_Y(y) = y.$$

Exercises for §1.3

2.

(a) $f(x) = f(y) \implies 1/x = 1/y \implies x = y$, so f is one-one. And if $y \neq 0$, then its preimage is $1/y$. So f is onto. From $1/(1/x) = x$ we deduce that $f^{-1} = f$.

(b) $g(x) = g(y) \implies \dfrac{x}{1-x} = \dfrac{y}{1-y} \implies x(1-y) = y(1-x) \implies x = y$, so g is one-one. We can find the preimage of $y \neq -1$ by solving $g(x) = y$:

$$\frac{x}{1-x} = y \iff x = y(1-x) \iff (1+y)x = y \iff x = \frac{y}{1+y}.$$

This also shows that $g^{-1}(y) = \dfrac{y}{1+y}$.

(c) First we check that h is one-one. For this, observe that $x + y \geq 1$ with equality only if $x = y = 1/2$. So,

$$\begin{aligned} h(x) = h(y) &\implies \frac{1}{x(1-x)} = \frac{1}{y(1-y)} \implies x - x^2 = y - y^2 \\ &\implies (x-y)(1-x-y) \implies x = y. \end{aligned}$$

For the onto property, we find the pre-image in $[1/2, 1)$ of any $y \geq 4$:

$$\begin{aligned} h(x) = y &\iff \frac{1}{x(1-x)} = y \iff xy(1-x) = 1 \\ &\iff yx^2 - yx + 1 = 0 \iff x = \frac{y + \sqrt{y(y-4)}}{2y} \end{aligned}$$

Therefore $h^{-1}(y) = \dfrac{y + \sqrt{y(y-4)}}{2y}$.

4.

(a) Suppose $(g \circ f)(x) = (g \circ f)(y)$. Then $g(f(x)) = g(f(y))$. Since g is one-one, we get $f(x) = f(y)$. Since f is one-one, we get $x = y$.

(b) Let $z \in Z$. Since g is onto, there is $y \in Y$ with $g(y) = z$. Since f is onto, there is $x \in X$ with $f(x) = y$. Then $(g \circ f)(x) = z$.

(c) Apply (a) and (b).

6.

(a) $f(x) = x$, $f(x) = |x|$, $f(x) = [x]$.

(b) $f(x) = -x$.

(c) $f(x) = \begin{cases} 0 & \text{if } x = 0 \text{ or } 1, \\ 1 & \text{else.} \end{cases}$

8. We will build a bijection $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ based on this diagram. The first observation is

$$f(1, n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

In general,

$$f(k,l) = f(1, k+l-1) - k + 1 = \frac{(k+l-1)(k+l)}{2} - k + 1$$

Observe that $f(1, k+l-2) < f(k,l) \leq f(1, k+l-1)$. Therefore $f(k,l) = f(k',l') \implies k+l = k'+l' \implies k = k' \implies l = l'$. So f is one-one.

Now let $m \in \mathbb{N}$. We have $n \in \mathbb{N}$ such that $\frac{(n-1)n}{2} < m \leq \frac{n(n+1)}{2}$. Let $k = \frac{n(n+1)}{2} - m + 1$ and $l = n - k + 1$. Then

$$f(k,l) = f(1, k+l-1) - k + 1 = f(1,n) - k + 1 = m.$$

1.4 Real Functions and Graphs

Task 1.4.1. In each case, let D be the domain of the function.

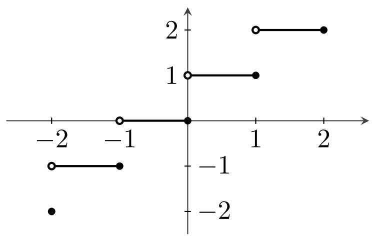
(a) $x \in D \iff 1 - x^2 \geq 0 \iff x^2 \leq 1 \iff -1 \leq x \leq 1$. So $D = [-1, 1]$.

(b) $x \in D \iff x \neq 0$. So $D = \mathbb{R}^*$.

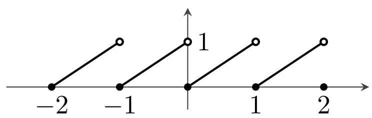
(c) $x \in D \iff (x-1)(x-2) \geq 0$. So $D = (-\infty, 1] \cup [2, \infty)$.

(d) $x \in D \iff 1 - x^2 > 0 \iff x^2 < 1 \iff -1 < x < 1$. So $D = (-1, 1)$.

Task 1.4.2.



Task 1.4.3.



Task 1.4.4. The domain of $f(x+c)$ is $\{\, x - c \mid x \in A \,\}$.

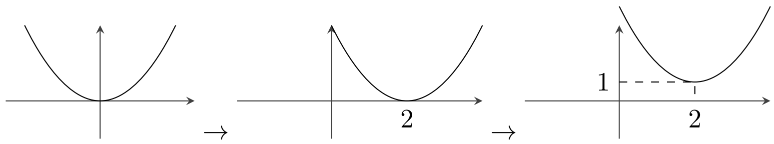
The domain of $f(cx)$ is $\{\, x/c \mid x \in A \,\}$ if $c \neq 0$. If $c = 0$, the domain is $\mathbb{R}$ if $0 \in A$ and $\varnothing$ otherwise.

Task 1.4.5. Shift the graph of f to the right by $|c|$ units.

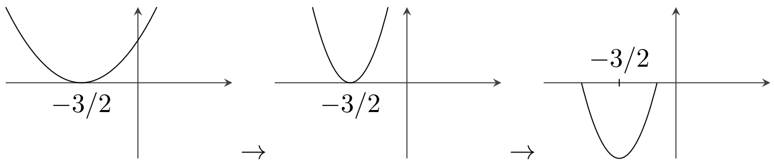
Task 1.4.6. Scale the graph of f horizontally by a factor of $1/|c|$ and then reflect in the y-axis.

Task 1.4.7.

(a) Shift the $y = x^2$ graph to the right by 2 units and then up by 1 unit.



(b) $h(x) = 4x^2 + 12x + 5 = 4(x + 3/2)^2 - 4$. So shift the graph of $y = x^2$ to the left by $3/2$ units, scale it vertically by 4, and then lower it by 4 units.



Task 1.4.8. $|x|$ is even and $\operatorname{sgn}(x)$ is odd.

$[x]$ is neither, since $[-1/2] = -1$ does not equal either $\pm[1/2] = 0$.

Task 1.4.9. If f is both, then for any point a in its domain we have $f(-a) = f(a) = -f(a) \implies 2f(a) = 0 \implies f(a) = 0$. So the only such function is the zero function.

Task 1.4.10.

(a) Increasing

(b) Increasing

(c) Not monotonic

(d) Strictly decreasing

Task 1.4.11. Proof by induction. For $n = 1$, the truth is given. Assume true for some n and consider $n + 1$:

$$f(x + (n + 1)T) = f((x + T) + nT) = f(x + T) = f(x).$$

Task 1.4.13. The domain of $f + g$, $f - g$ and fg is $A \cap B$. The domain of f/g is $\{\, x \in A \cap B \mid g(x) \neq 0 \,\}$.

Task 1.4.14.

(a) All the combinations are even. For example,

$$(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x).$$

(b) $f + g$ and $f - g$ are odd. For example,

$$(f + g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f + g)(x).$$

fg and f/g are even. For example,

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x).$$

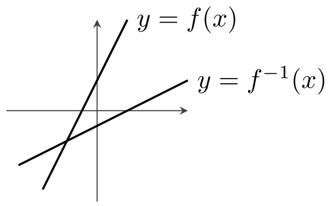
(c) $f+g$ and $f-g$ need not be either even or odd. For example, 1 is even, x is odd and $1+x$ is neither. However, fg and f/g are odd. For example,

$$(fg)(-x) = f(-x)g(-x) = -f(x)g(x) = -(fg)(x).$$

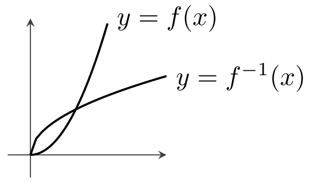
(d) As in (c).

Task 1.4.15.

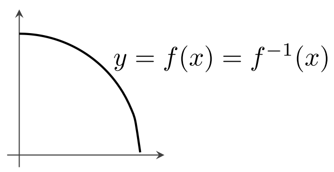
(a) To obtain f^{-1} we solve $y = 2x+1$ for x in terms of y. We get $x = \frac{1}{2}(y-1)$. Switching x and y gives the inverse function to be $f^{-1}(x) = \frac{1}{2}(x-1)$.



(b) The inverse function $f^{-1}\colon [0,\infty) \to [0,\infty)$ is $f^{-1}(x) = \sqrt{x}$.



(c) The inverse function $f^{-1}\colon [0,1] \to [0,1]$ is $f^{-1}(x) = \sqrt{1-x^2} = f(x)$.



Task 1.4.16. $(-x)^n = (-1)^n x^n = \begin{cases} x^n & \text{if n is even,} \\ -x^n & \text{if n is odd.} \end{cases}$

Therefore, monomials of even degree are even functions and monomials of odd degree are odd functions.

Task 1.4.17. Let $\deg p = m$ and $\deg q = n$. Then

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0, \quad q(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0$$

with a_m and b_n non-zero. Therefore,

$$p(x)q(x) = a_m b_n x^{m+n} + (a_m b_{n-1} + a_{m-1}b_n)x^{m+n-1} + \cdots + a_0 b_0.$$

Since $a_m b_n \neq 0$, we get $\deg(pq) = m+n = (\deg p) + (\deg q)$.

Next, let $m = max\{\deg p, \deg q\}$. Then,

$$p(x) = a_m x^m + a_{m-1}x^{m-1} + \cdots + a_0$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

with at least one of a_m and b_m being non-zero. Now,

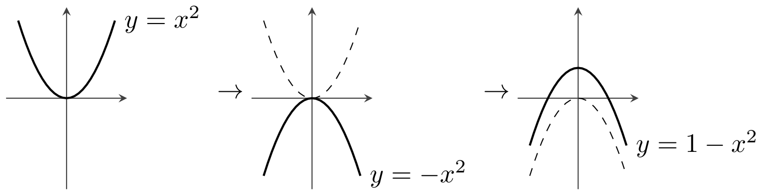
$$p(x) + q(x) = (a_m + b_m)x^m + (a_{m-1} + b_{m-1})x^{m-1} + \cdots + a_0 + b_0,$$

and so $\deg(p + q) \leq m = \max\{\deg p, \deg q\}$.

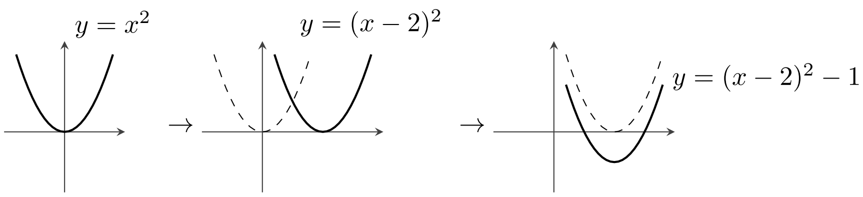
Exercises for §1.4

2.

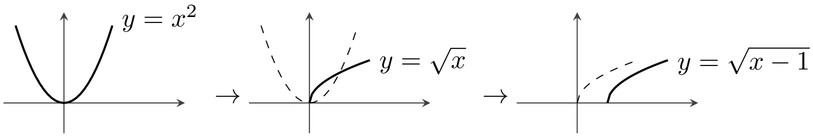
(a)



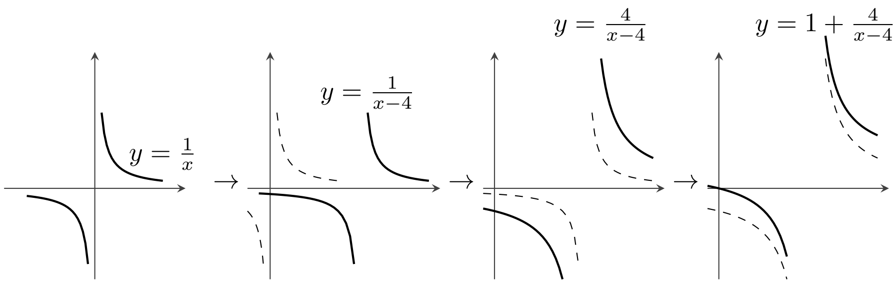
(b) $x^2 - 4x + 3 = (x - 2)^2 - 1$.



(c)

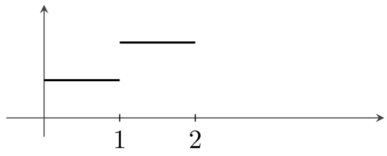


(d) $\dfrac{x}{x-4} = 1 + \dfrac{4}{x-4}$.

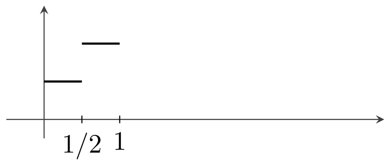


4.

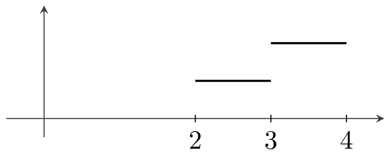
(a)



(b) The domain of g is $[0, 1]$.

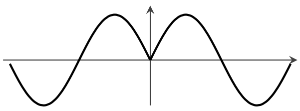


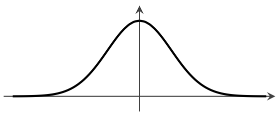
(c) The domain of h is $[2, 4]$.



(d) The domain of k is empty, so there is no graph to draw!

6. The extended graphs are given below:

(a) 

(b) 

8.

(a) Even: $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$.

(b) Odd: $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = -f(g(x)) = -(f \circ g)(x)$.

(c) Even: $(f \circ g)(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x)$.

(d) Even: $(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x)$.

10.

(a) We wish to write $f = f^+ - f^-$ where $f^\pm$ are non-negative. If f itself is non-negative, we can take $f^+ = f$ and $f^- = 0$. If f is non-positive, we can take $f^+ = 0$ and $f^- = -f$. These observations motivate the following definitions:

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{else} \end{cases}, \quad f^-(x) = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{else} \end{cases}.$$

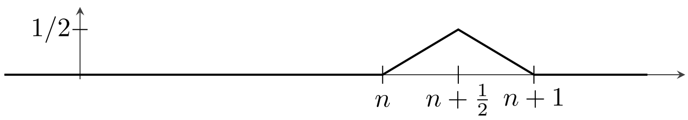
It is easy to check that $f^\pm$ are non-negative and $f = f^+ - f^-$.

(b) Define $g(x) = \frac{1}{2}(f(x) + f(-x))$ and $h(x) = \frac{1}{2}(f(x) - f(-x))$. Then it is easily seen that g is even, h is odd, and $f = g + h$.

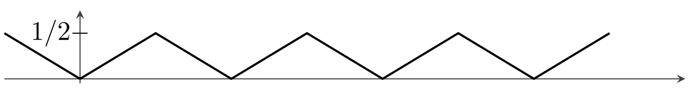
12. In Exercise 11, we defined $x^+ = \max\{0, x\}$. This function changes its behaviour at zero. Consequently, f_n will change its behaviour at n, $n+1/2$ and $n+1$. We have the following:

$$f_n(x) = \begin{cases} 0 & \text{if } x \le n \\ x - n & \text{if } n < x \le n + 1/2 \\ n - x + 1 & \text{if } n + 1/2 < x \le n + 1 \\ 0 & \text{if } n + 1 < x \end{cases}$$

(a)



(b)



14.

(a) Odd, not monotonic,not periodic.

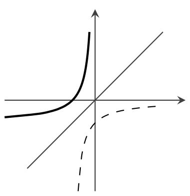
(b) Even, increasing, not periodic.

(c) Not odd or even, not monotonic, period of 1.

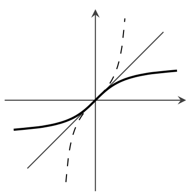
(d) Not odd or even, not monotonic, period of 1.

16. The function and its inverse are shown below:

(a)



(b)



Thematic Exercises

Curve Fitting: Interpolation and Least Squares

A1.

(a) The condition $w_i(x_j) = 0$ makes $x - x_j$ a factor of $w_i(x)$ for each $j \neq i$. Hence we have

$$w_i(x) = C \prod_{j:j\neq i} (x - x_j)$$

for some $C \in \mathbb{R}$. The condition $w_i(x_i) = 1$ further gives $1 = C \prod_{j:j\neq i}(x_i - x_j)$. Hence,

$$w_i(x) = \frac{\prod_{j:j\neq i}(x - x_j)}{\prod_{j:j\neq i}(x_i - x_j)}.$$

(b) It is easy to see that the given p works. Since each w_i has degree n, we have $\deg p \leq n$. Further,

$$p(x_k) = \sum_{i=0}^{n} y_i w_i(x_k) = y_k w_k(x_k) = y_k.$$

If $q(x)$ also satisfies the given properties, then $\deg(p-q) \leq n$ and $(p-q)(x_i) = 0$ for each $i = 0, \ldots, n$. This gives $p - q = 0$, hence $p = q$.

Remark: This $p(x)$ is called the **Lagrange interpolating polynomial** for the data $(x_0, y_0), \ldots, (x_n, y_n)$.

A2.

$$\begin{aligned} w_0(x) &= \frac{(x-0)(x-h)}{(-h-0)(-h-h)} = \frac{x(x-h)}{2h^2}, \\ w_1(x) &= \frac{(x-(-h))(x-h)}{(0-(-h))(0-h)} = -\frac{(x+h)(x-h)}{h^2}, \\ w_2(x) &= \frac{(x-(-h))(x-0)}{(h-(-h))(h-0)} = \frac{x(x+h)}{2h^2}. \end{aligned}$$

The interpolating polynomial for the given data is:

$$\begin{aligned} p(x) &= a\frac{x(x-h)}{2h^2} - b\frac{(x+h)(x-h)}{h^2} + c\frac{x(x+h)}{2h^2} \\ &= \frac{(a-2b+c)x^2}{2h^2} + \frac{(c-a)x}{2h} + b. \end{aligned}$$

An alternate approach, for those who know how to solve systems of linear equations, is to assume $p(x) = \alpha x^2 + \beta x + \gamma$ and use the data to set up the following three linear equations in the three variables α, β, γ:

$$\begin{aligned} h^2\alpha - h\beta + \gamma &= a \\ \gamma &= b \\ h^2\alpha + h\beta + \gamma &= c \end{aligned}$$

A3. We have $||\vec{x}+\vec{y}||^2 = (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x}+2\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{y} = ||\vec{x}||^2+2\vec{x}\cdot\vec{y}+||\vec{y}||^2$.

Therefore, $||\vec{x}+\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 \iff \vec{x}\cdot\vec{y} = 0 \iff \vec{x} \perp \vec{y}$.

A4. First, we note that Π will be a plane if $\vec{u}$ and $\vec{v}$ are not collinear with $\vec{0}$. If they *are* collinear, Π will be a line. Therefore, we assume this non-collinearity in our analysis below.

Next, we identify a plausible choice of $\vec{x}$. Geometry suggests that $\vec{x}$ should be the perpendicular projection of $\vec{y}$ on Π. That is, $\vec{y} - \vec{x}$ should be perpendicular to all members of Π. Let us confirm that such an $\vec{x}$ would minimize the distance. If $\vec{x}\,'$ is any other member of Π, we have

$$\begin{aligned} ||\vec{y}-\vec{x}\,'||^2 &= ||(\vec{y}-\vec{x}) + (\vec{x}-\vec{x}\,')||^2 \\ &= ||\vec{y}-\vec{x}||^2 + ||\vec{x}-\vec{x}\,'||^2 \quad \text{(by Pythagoras' theorem, since } \vec{x}-\vec{x}\,' \in \Pi) \\ &\geq ||\vec{y}-\vec{x}||^2. \end{aligned}$$

Moreover, equality can only happen if $\vec{x} = \vec{x}\,'$. This also establishes the uniqueness of $\vec{x}$.

Now, let us show the existence of $\vec{x}$ by finding a formula for it. The perpendicularity condition gives $(\vec{y} - \vec{x}) \bullet \vec{u} = (\vec{y} - \vec{x}) \bullet \vec{v} = 0$. If we set $\vec{x} = a\vec{u} + b\vec{v}$, these equations become

$$||\vec{u}||^2 a + (\vec{u} \bullet \vec{v})b = \vec{y} \bullet \vec{u}$$
$$(\vec{u} \bullet \vec{v})a + ||\vec{v}||^2 b = \vec{y} \bullet \vec{v}$$

On solving these equations for a and b we obtain:

$$a = \frac{||\vec{v}||^2(\vec{y} \bullet \vec{u}) - (\vec{u} \bullet \vec{v})(\vec{y} \bullet \vec{v})}{||\vec{u}||^2\,||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2}$$
$$b = \frac{||\vec{u}||^2(\vec{y} \bullet \vec{v}) - (\vec{u} \bullet \vec{v})(\vec{y} \bullet \vec{u})}{||\vec{u}||^2\,||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2}$$

The remaining issue is whether $||\vec{u}||^2\,||\vec{v}||^2 - (\vec{u}\bullet\vec{v})^2 \neq 0$. For this, we invoke the **Cauchy-Schwarz inequality**, which states that for numbers $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ we always have

$$\Big|\sum_{i=1}^{n} u_i v_i\Big|^2 \leq \Big(\sum_{i=1}^{n} u_i^2\Big)\Big(\sum_{i=1}^{n} v_i^2\Big),$$

with equality if and only if one of $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ is a constant times the other. The non-collinearity assumption rules out this possibility and gives the existence of $\vec{x}$.

(If $\vec{u}$ and $\vec{v}$ are collinear with $\vec{0}$, we drop $\vec{v}$ from our calculations, and we find $\vec{x} = \dfrac{\vec{y} \bullet \vec{u}}{||\vec{u}||^2}\vec{u}$.)

A5. The total squared error can be expressed as

$$E(a,b) = \sum_{i=1}^{n}(y_i - ax_i - b)^2 = ||\vec{y} - (a\vec{x} + b\vec{v})||^2,$$

which is the square of the distance of $\vec{y}$ from the member $a\vec{x}+b\vec{v}$ of Π. Therefore, minimizing the total squared error is equivalent to finding the member of Π which is closest to $\vec{y}$.

A6. Use A5 to view this as a problem of finding the closest vector from a plane, and then apply the formulas for a, b obtained in the solution of A4.

Cardinality

B1. By repeatedly dividing by 2, we can express any natural number as a power of 2 times an odd number. This shows f is a surjection.

Now suppose $f(m,n) = f(m',n')$ with $m \geq m'$. Then,

$$2^{m-1}(2n-1) = 2^{m'-1}(2n'-1) \implies 2^{m-m'}(2n-1) = 2n'-1 \implies m = m'$$
$$\implies 2n-1 = 2n'-1 \implies n = n'.$$

So f is also an injection.

B2. Any positive rational can be expressed as m/n with $m, n \in \mathbb{N}$ and having no common factors except 1. We apply the fundamental theorem of arithmetic to write $m = p_1^{\beta_1} \cdots p_k^{\beta_k}$ and $n = p_{k+1}^{\beta_{k+1}} \cdots p_\ell^{\beta_\ell}$, where the p_i are distinct primes and $\beta_i \in \mathbb{N}$. (If either m or n is 1, we express it as 2^0.) Let $f(\alpha_i) = \beta_i$ for $i = 1, \ldots, k$ and $f(\alpha_i) = -\beta_i$ for $i = k+1, \ldots, \ell$. Then we have

$$\varphi(p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}) = \frac{p_1^{\beta_1} \cdots p_k^{\beta_k}}{p_{k+1}^{\beta_{k+1}} \cdots p_\ell^{\beta_\ell}} = \frac{m}{n}.$$

So φ is surjective.

Next, suppose $\varphi(p_1^{\alpha_1} \cdots p_k^{\alpha_k}) = \varphi(q_1^{\beta_1} \cdots q_\ell^{\beta_\ell})$, where the p_i are one set of distinct primes, the q_j are another set of distinct primes, and $\alpha_i, \beta_j \in \mathbb{N}$. We get

$$p_1^{f(\alpha_1)} \cdots p_k^{f(\alpha_k)} = q_1^{f(\beta_1)} \cdots q_\ell^{f(\beta_\ell)}.$$

We can assume that there are k' and ℓ' such that

$$\begin{aligned} f(\alpha_1), \ldots, f(\alpha_{k'}) > 0, \qquad & f(\alpha_{k'+1}), \ldots, f(\alpha_k) < 0, \\ f(\beta_1), \ldots, f(\beta_{\ell'}) > 0, \qquad & f(\beta_{\ell'+1}), \ldots, f(\beta_\ell) < 0. \end{aligned}$$

Then,

$$p_1^{f(\alpha_1)} \cdots p_{k'}^{f(\alpha_{k'})} q_{\ell'+1}^{-f(\beta_{\ell'+1})} \cdots q_\ell^{-f(\beta_\ell)} = p_{k'+1}^{-f(\alpha_{k'+1})} \cdots p_k^{-f(\alpha_k)} q_1^{f(\beta_1)} \cdots q_{\ell'}^{f(\beta_{\ell'})} \in \mathbb{N},$$

with all exponents being positive. By the uniqueness of the prime factorisation, we get:

$$\{p_1, \ldots, p_{k'}\} = \{q_1, \ldots, q_{\ell'}\} \quad \text{and} \quad \{p_{k'+1}, \ldots, p_k\} = \{q_{\ell'+1}, \ldots, q_\ell\}.$$

In particular, $k = \ell$ and $k' = \ell'$. Further, matching exponents gives $f(\alpha_i) = f(\beta_i)$ for every i, and so $\alpha_i = \beta_i$ for every i.

B3. Consider the map φ defined in B2. Use it to define $\psi \colon \mathbb{N} \to \mathbb{Q}$:

$$\psi(n) = \begin{cases} 0 & \text{if } n = 1, \\ \varphi(n/2) & \text{if } n \text{ is even}, \\ -\varphi((n-1)/2) & \text{else}. \end{cases}$$

B4. Let $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$. A and B are clearly non-empty. The nesting of the intervals gives the following arrangement:

$$a_1 \leq a_2 \leq a_3 \leq \cdots \leq b_3 \leq b_2 \leq b_1.$$

Thus, $a_i \leq b_j$ for every i, j. The completeness axiom now gives a real number c such that $a_i \leq c \leq b_i$ for every i. Therefore, $c \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

B5. Set $J_1 = [f(1)+1, f(1)+2]$. Then $f(1) \notin J_1$.

Cut J_1 into three equal subintervals: $[f(1)+1, f(1)+4/3]$, $[f(1)+4/3, f(1)+5/3]$, $[f(1)+5/3, f(1)+2]$. The pieces $[f(1)+1, f(1)+4/3]$ and $[f(1)+5/3, f(1)+2]$ are disjoint, so at least one of them does not contain $f(2)$. Call that one J_2.

Repeat the process with J_2 to get J_3 with $f(3) \notin J_3$. Continue in this way to get a decreasing sequence of intervals $J_n = [a_n, b_n]$ such that $f(n) \notin J_n$.

By B4, $\bigcap_{n=1}^{\infty} J_n$ has a member c. Now, $c \neq f(n)$ for every n, which contradicts the surjectivity of f.

B6. We mimic the solution of B3. We already know there is a bijection $\psi \colon \mathbb{N} \to \mathbb{Q}$.

Suppose $\mathbb{Q}^c$ is countable, so that there is a bijection $\psi^c \colon \mathbb{N} \to \mathbb{Q}^c$. Now define $f \colon \mathbb{N} \to \mathbb{R}$ by

$$f(n) = \begin{cases} \psi(n/2) & \text{if } n \text{ is even,} \\ \psi^c((n+1)/2) & \text{else.} \end{cases}$$

We leave it to you to show that f is a bijection. This contradicts B5. Hence $\mathbb{Q}^c$ is uncountable.

2 | Integration

2.1 Integration of Step and Bounded Functions

Task 2.1.3.

(a) $P = \{-1, 0, 1\}$ (b) $P = \{0, 1, 2, 3\}$

Task 2.1.5. $\int_0^3 s(x)\,dx = 2 \cdot (1.5 - 0) + (-1) \cdot (2.5 - 1.5) + 3 \cdot (3 - 2.5) = 3.5.$

Task 2.1.6. Let Q be a refinement of P. It is enough to do the case when Q has one more point than P. If $P = \{x_0, \ldots, x_n\}$ then such a Q has the form $\{x_0, \ldots, x_{k-1}, t, x_k, \ldots, x_n\}$, with $x_{k-1} < t < x_k$. Since s is constant on (x_{k-1}, x_k), it is constant on (x_{k-1}, t) and (t, x_k). It is also constant on each (x_{i-1}, x_i) for $i \neq k$. Therefore Q is adapted to s.

Task 2.1.8. Let P be a partition that is adapted to s, and Q a partition that is adapted to t. Then $P \cup Q$ is a refinement of both P and Q. By Task 2.1.6, it is adapted to both s and t.

Task 2.1.13. The issue here is that the definition of integrability requires us to consider all lower and upper sums. However, if we find a subset L of $\mathcal{L}_f$ and a subset U of $\mathcal{U}_f$ such that there is a unique number between L and U, then the same is true of $\mathcal{L}_f$ and $\mathcal{U}_f$.

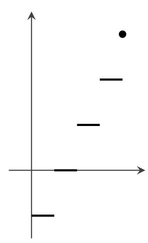
Task 2.1.15. Suppose $s\colon [a, b] \to \mathbb{R}$ is a step function. Then $I = \int_a^b s$ is both a lower and an upper sum for s. Therefore, it is the unique number between $\mathcal{L}_s$ and $\mathcal{U}_s$, hence is also the integral of s when we view s as a bounded function whose integral is to be obtained via lower and upper sums.

Exercises for §2.1

2.

(a)

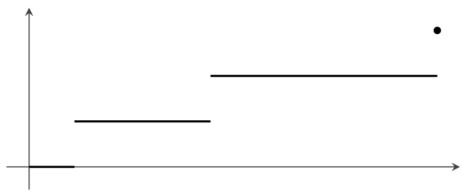
$$[2x - 1] = \begin{cases} -1 & \text{if } x \in [0, 1/2) \\ 0 & \text{if } x \in [1/2, 1) \\ 1 & \text{if } x \in [1, 3/2) \\ 2 & \text{if } x \in [3/2, 2) \\ 3 & \text{if } x = 2 \end{cases}$$



$$\int_0^2 [2x-1]\,dx = -1\cdot 0.5 + 0\cdot 0.5 + 1\cdot 0.5 + 2\cdot 0.5 = 1.$$

(b)

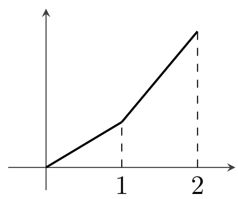
$$[\sqrt{x}] = \begin{cases} 0 & \text{if } x\in[0,1) \\ 1 & \text{if } x\in[1,4) \\ 2 & \text{if } x\in[4,9) \\ 3 & \text{if } x = 9 \end{cases}$$



$$\int_0^9 [\sqrt{x}]\,dx = 0\cdot 1 + 1\cdot 3 + 2\cdot 5 = 13.$$

4.

$$F(x) = \begin{cases} x & \text{if } x\in[0,1] \\ 1+2(x-1) & \text{if } x\in[1,2] \end{cases}$$



6. In this diagram, the shaded part represents a lower sum for $\int_0^b \sqrt{x}\,dx$. Its complement in the enclosing rectangle is an upper sum for $\int_0^{\sqrt{b}} x^2\,dx$. The area of the enclosing rectangle is $b\sqrt{b} = b^{3/2}$. Therefore,

$$\mathcal{L}_{\sqrt{x}} = \{\, b^{3/2} - u \mid u \in \mathcal{U}_{x^2} \,\}.$$

Similarly,

$$\mathcal{U}_{\sqrt{x}} = \{\, b^{3/2} - \ell \mid \ell \in \mathcal{L}_{x^2} \,\}.$$

Now let I separate $\mathcal{L}_{\sqrt{x}}$ and $\mathcal{U}_{\sqrt{x}}$. Then,

$$\begin{aligned} \ell \le I \text{ for every } \ell \in \mathcal{L}_{\sqrt{x}} &\implies b^{3/2} - u \le I \text{ for every } u \in \mathcal{U}_{x^2} \\ &\implies b^{3/2} - I \le u \text{ for every } u \in \mathcal{U}_{x^2}. \end{aligned}$$

We can similarly prove that $b^{3/2} - I \ge \ell$ for every $\ell \in \mathcal{U}_{x^2}$. Hence,

$$b^{3/2} - I = \int_0^{\sqrt{b}} x^2\,dx.$$

Therefore, I is unique, and

$$\int_0^b \sqrt{x}\,dx = I = b^{3/2} - \int_0^{\sqrt{b}} x^2\,dx = b^{3/2} - \frac{b^{3/2}}{3} = \frac{2}{3}b^{3/2}.$$

8. Let $\varepsilon > 0$. Since f is integrable on $[a,b]$, we have step functions $s,t\colon [a,b] \to \mathbb{R}$ such that $s \le f \le t$ and $\int_a^b t - \int_a^b s < \varepsilon$. Then $\int_c^d s$ is a lower sum for f on $[c,d]$, while $\int_c^d t$ is an upper sum for f on $[c,d]$. Further, $\int_c^d t - \int_c^d s \le \int_a^b t - \int_a^b s < \varepsilon$.

10. First, suppose the given conditions hold. Let $\varepsilon > 0$. From (a) and (b) we deduce that $I + \varepsilon/2$ is an upper sum and $I - \varepsilon/2$ is a lower sum. Then Theorem 2.1.17 gives $I = \int_a^b f(x)\,dx$.

Next, suppose that $I = \int_a^b f(x)\,dx$. Consider any $u > I$. By Theorem 2.1.17 we have a step function t such that $f \le t$ and $I \le u' = \int_a^b t < u$. Now consider the step function $t' = t + (u - u')/(b - a)$. Then $f \le t'$ and $\int_a^b t' = u' + (u - u') = u$. We have proved (a). There is a similar proof of (b).

12. Let $|f(x)| \le M$ for every $x \in [a, b]$, and consider any $\varepsilon > 0$.

For every $n \in \mathbb{N}$ with $1/n < b - a$, f is integrable on $[a + 1/n, b]$. So there are step functions $s', t' \colon [a + 1/n, b] \to \mathbb{R}$ such that $\int_{a+1/n}^b t' - \int_{a+1/n}^b s' < \varepsilon/2$. Define step functions $s, t \colon [a, b] \to \mathbb{R}$ by

$$s(x) = \begin{cases} -M & \text{if } x \in [a, a + 1/n) \\ s'(x) & \text{if } x \in [a + 1/n, b] \end{cases}, \quad t(x) = \begin{cases} M & \text{if } x \in [a, a + 1/n) \\ t'(x) & \text{if } x \in [a + 1/n, b] \end{cases}.$$

Then we have $s \le f \le t$ on $[a, b]$, and

$$\int_a^b t - \int_a^b s < \frac{\varepsilon}{2} + \frac{2M}{n}.$$

If we use n such that $1/n < \min\{b - a, \varepsilon/4M\}$ then we get $\int_a^b t - \int_a^b s < \varepsilon$.

2.2 Properties of Integration

Task 2.2.3.

(a) The partition $P = \{0,\ 1,\ 1.5,\ 2.5,\ 3\}$ is adapted to both s and t, hence to $s + t$. We have

$$s(x) + t(x) = \begin{cases} -1 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \le 1.5 \\ 3 & \text{if } 1.5 < x \le 2.5 \\ 4 & \text{if } 2.5 < x \le 3 \end{cases}$$

(b)
$$\int_0^3 s = -1 \cdot 1 + 1 \cdot 2 = 1$$
$$\int_0^3 t = 0 \cdot 1.5 + 2 \cdot 1 + 3 \times 0.5 = 3.5$$
$$\int_0^3 (s + t) = -1 \cdot 1 + 1 \cdot 0.5 + 3 \cdot 1 + 4 \cdot 0.5 = 4.5$$

Task 2.2.7. We have $\int_{-1}^1 |t|\,dt = \int_{-1}^0 (-t)\,dt + \int_0^1 t\,dt$. We have already found $\int_0^1 t\,dt = 1/2$. Using the partitions $P = \{-1, -1 + 1/n, -1 + 2/n, \ldots, 0\}$ we similarly find $\int_{-1}^0 t\,dt = -1/2$. Therefore,

$$\int_{-1}^1 |t|\,dt = -\int_{-1}^0 t\,dt + \int_0^1 t\,dt = 1.$$

Task 2.2.10. The domain of $f(x/k)$ will be $[kb, ka]$, and the integrals will be related by

$$\int_{kb}^{ka} f(x/k)\,dx = -k\int_a^b f(x)\,dx.$$

Task 2.2.12. In the proof of Theorem 2.2.11, we have seen that the leftmost term is a lower sum, while the rightmost term is an upper sum, for f on $[a, b]$.

Exercises for §2.2

2. We have used the shift properties of integration to simplify the calculations. In (b), we have also used Task 2.2.10 with $k = -1$.

(a) $\displaystyle\int_1^2 (x-1)(x-2)\,dx = \int_0^1 x(x-1)\,dx = \int_0^1 x^2\,dx - \int_0^1 x\,dx = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$.

(b) $$\begin{aligned}\int_1^2 (x-1)(x-2)(x-3)\,dx &= \int_{-1}^0 (x+1)x(x-1)\,dx = \int_{-1}^0 (x^3 - x)\,dx\\ &= \int_0^1 ((-x)^3 - (-x))\,dx = -\int_0^1 (x^3 - x)\,dx\\ &= -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.\end{aligned}$$

4. The conditions $P(0) = P(1) = 0$ give $P(x) = cx(x-1)$ for some $c \in \mathbb{R}$. Now,

$$\int_0^1 cx(x-1)\,dx = c\Big(\int_0^1 x^2\,dx - \int_0^1 x\,dx\Big) = -\frac{c}{6} \implies c = -6.$$

Therefore, $P(x) = -6x(x-1)$.

6.

(a) $$\begin{aligned}\int_{-a}^a f(x)\,dx &= \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx\\ &= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad \text{(Task 2.2.10)}\\ &= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad (f \text{ is even})\\ &= 2\int_0^a f(x)\,dx\end{aligned}$$

(b) $$\begin{aligned}\int_{-a}^a f(x)\,dx &= \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx\\ &= -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx \quad (f \text{ is odd})\\ &= 0\end{aligned}$$

8. We shall use the results of the preceding exercise to give two proofs of the integrability of $f \vee g$. The integrability of $f \wedge g$ can be established along the same lines.

First solution:

$$(f\vee g)(x)=\max\{f(x),g(x)\}=f(x)+\max\{0,g(x)-f(x)\}=f(x)+(g(x)-f(x))^{+}$$

The integrability of f and g gives the integrability of $g-f$ and hence of $(g-f)^{+}$. Therefore, $f\vee g=f+(g-f)^{+}$ is integrable.

Second solution:

$$(f\vee g)(x)=\max\{f(x),g(x)\}=\frac{1}{2}(f(x)+g(x))+\frac{1}{2}|g(x)-f(x)|.$$

Again, $g\pm f$ are integrable, hence $|g-f|$ is integrable. Therefore, $f\vee g=\frac{1}{2}(f+g)+\frac{1}{2}|g-f|$ is integrable.

10.

(a) $$\int_a^b t^2-\int_a^b s^2=\int_a^b (t+s)(t-s)\le 2M\int_a^b (t-s)=2M\Big(\int_a^b t-\int_a^b s\Big).$$

(b) Let $\varepsilon>0$. If $0\le f\le M$, there are step functions s,t such that $0\le s\le f\le t\le M$ and $\int_a^b t-\int_a^b s<\varepsilon/(2M)$. Then, s^2,t^2 are step functions such that $s^2\le f^2\le t^2$ and

$$\int_a^b t^2-\int_a^b s^2\le 2M\Big(\int_a^b t-\int_a^b s\Big)=\varepsilon.$$

Therefore f^2 is integrable. For a general integrable f with $|f|\le M$, we have $f+M\ge 0$, hence $(f+M)^2$ is integrable. It follows that $f^2=(f+M)^2-2Mf-M^2$ is integrable.

(c) We have $fg=\frac{1}{2}((f+g)^2-f^2-g^2)$ and every term on the right hand side is integrable.

12.

(a) $$F(-x)=\int_0^{-x} f(t)\,dt=-\int_0^{x} f(-t)\,dt=-\int_0^{x} f(t)\,dt=-F(x).$$

(b) $$F(-x)=\int_0^{-x} f(t)\,dt=-\int_0^{x} f(-t)\,dt=\int_0^{x} f(t)\,dt=F(x).$$

14. Recall that Theorem 2.2.14 was proved for the case when f is decreasing and positive.

(a) Suppose f is an increasing and positive function. Then $g(x)=f(-x)$ is decreasing and positive. Therefore $G(x)=\int_{-a}^{x} g(t)\,dt$ has the intermediate value property. We have

$$F(x)=\int_a^x f(t)\,dt=\int_{-x}^{-a} f(-t)\,dt=-\int_{-a}^{-x} g(t)\,dt=-G(-x).$$

Now it is easily seen that F has the intermediate value property.

(b) The hint in the text is not helpful. Instead, proceed as follows. At this stage, we have established Theorem 2.2.14 when f is monotone and positive. We easily get the result for a monotone and negative f by considering $-f$. Observe that our proof also covers the cases when f is zero only at an endpoint of the interval.

Now, suppose f is a general increasing function on an interval I. Then we have $\alpha, \beta \in I$ such that $\alpha \le \beta$, $f < 0$ for $x < \alpha$ and $f > 0$ for $x > \beta$. Then F has the intermediate value property on the intervals $I \cap (-\infty, \alpha]$, $[\alpha, \beta]$, and $I \cap [\beta, \infty)$. Hence it has the intermediate value property on I.

We get the result for a decreasing f by considering $-f$.

16. Mimic the solution of Exercise 6 of §2.1.

2.3 Logarithm and Exponential Functions

Task 2.3.5. We already know this is true for positive reals, and of course for zero. If a is negative, we take $b = -(-a)^{1/n}$ to prove existence. For uniqueness, we first note that $b^n = c^n = a$ implies b, c have the same sign. We then apply $b^n - c^n = (b-a)(b^{n-1} + \cdots + c^{n-1})$.

Task 2.3.6. $((a^m)^{1/n})^{nq} = (a^m)^q = a^{mq}, \quad ((a^p)^{1/q})^{nq} = (a^p)^n = a^{pn} = a^{mq}$.

Task 2.3.7. $((a^{1/n})^m)^n = (a^{1/n})^{mn} = a^m$.

Task 2.3.10.

(a) $$\log(y+1) - \log(y-1) = 2\log x \implies \log\left(\frac{y+1}{y-1}\right) = \log x^2$$
$$\implies \frac{y+1}{y-1} = x^2 \implies y(x^2-1) = x^2+1 \implies y = \frac{x^2+1}{x^2-1}.$$

(b) $$x = \frac{2^y + 2^{-y}}{2^y - 2^{-y}} = \frac{4^y+1}{4^y-1} \implies 4^y(x-1) = 1+x \implies 4^y = \frac{x+1}{x-1}$$
$$\implies y = \log_4(x+1) - \log_4(x-1).$$

Task 2.3.13. $b^{(\log_a x)(\log_b a)} = a^{\log_a x} = x \implies \log_b x = (\log_a x)(\log_b a)$.

Task 2.3.14.

(a) $$\cosh(-x) = \frac{1}{2}(e^{-x} + e^{-(-x)}) = \frac{1}{2}(e^x + e^{-x}) = \cosh(x),$$
$$\sinh(-x) = \frac{1}{2}(e^{-x} - e^{-(-x)}) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh(x).$$

(b) We have $y = \frac{1}{2}(e^x + e^{-x}) \iff (e^x)^2 - 2ye^x + 1 = 0$. This quadratic in e^x has a solution if and only if $4y^2 - 4 \ge 0$, i.e., $|y| \ge 1$. The solutions are $e^x = y \pm \sqrt{y^2-1}$. The requirement $e^x > 0$ rules out $y < -1$. This shows that the image of cosh is exactly $[1, \infty)$.

The corresponding calculations for sinh lead to the equation $e^x = y \pm \sqrt{y^2+1}$. The requirement $e^x > 0$ gives $x = \log(y + \sqrt{y^2+1})$ as the unique pre-image for every y.

(c) Use $(a+b)^2 - (a-b)^2 = 4ab$.

Task 2.3.15. Apply the discussion for the graph of cosh. In this case, sinh is odd, passes through the origin, approaches $e^x/2$ as x increases in magnitude on the positive side, and approaches $-e^{-x}/2$ as x increases in magnitude on the negative side.

Exercises for §2.3

2.

(a) $\log 80 = \log(2^4 \cdot 5) = 4\log 2 + \log 5 = 4.37$.

(b) $\log 120 = \log(2^3 \cdot 3 \cdot 5) = 3\log 2 + \log 3 + \log 5 = 4.78$.

(c) $\log 2.1 = \log 21 - \log 10 = \log 3 + \log 7 - \log 2 - \log 5 = 0.75$.

(d) $\log(3/35) = \log 3 - \log 5 - \log 7 = -2.46$.

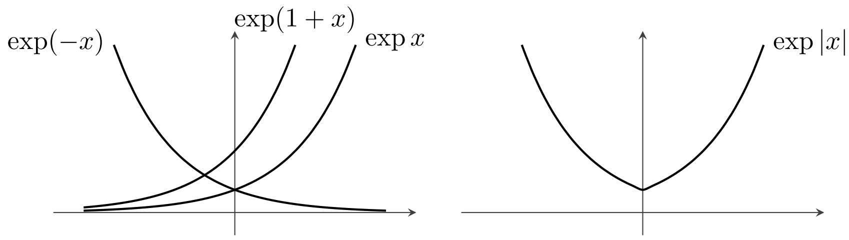
4. $2^{3/2} = 2\sqrt{2} < 3 \implies \log_2 3 > 3/2, \quad 3^{3/2} = 3\sqrt{3} > 5 \implies \log_3 5 < 3/2$.

6. For $x = 1$, all expressions are zero.

For $x > 1$, $\displaystyle\int_1^x \frac{1}{x}\,dt \le \int_1^x \frac{1}{t}\,dt \le \int_1^x 1\,dt \implies \frac{x-1}{x} \le \log x \le x - 1$.

For $x < 1$, we have $1/x > 1$, hence $1 - \dfrac{1}{1/x} \le \log(1/x) \le \dfrac{1}{x} - 1$.

8.



10. $a^x = \exp(x \log a)$ is a composition of the strictly increasing exp function with $x \log a$. And $x \log a$ is strictly increasing if $a > 1$, strictly decreasing if $0 < a < 1$.

12.

(a) $\frac{1}{4}(e^x + e^{-x})^2 + \frac{1}{4}(e^x - e^{-x})^2 = \frac{1}{2}(e^{2x} + e^{-2x}) = \cosh 2x$.

(b) $2 \cdot \frac{1}{2}(e^x + e^{-x}) \cdot \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \sinh 2x$.

(c)
$$\begin{aligned}
&\frac{1}{4}(e^x + e^{-x})(e^y + e^{-y}) + \frac{1}{4}(e^x - e^{-x})(e^y - e^{-y}) \\
&= \frac{1}{4}\Big((e^{x+y} + e^{x-y} + e^{y-x} + e^{-(x+y)}) + (e^{x+y} - e^{x-y} - e^{y-x} + e^{-(x+y)})\Big) \\
&= \frac{1}{4}(2e^{x+y} + 2e^{-(x+y)}) = \cosh(x+y).
\end{aligned}$$

(d) Similar to (c).

2.4 Integration and Area

Exercises for §2.4

2. Yes. Cut the polygon into triangles that only meet at their vertices. The contributions from a side that is shared by two triangles cancel, and only the contributions from the outer edges remain. If the vertices of the polygon are $(x_1, y_1), \ldots, (x_n, y_n)$, in counterclockwise order, then the area is

$$A = \frac{1}{2}\Big((x_1y_2 - x_2y_1) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\Big).$$

4.

(a) Find the meeting point of $y = \log(3-x)$ and $y = \log(1+x)$:

$$\log(3-x) = \log(1+x) \implies 3 - x = 1 + x \implies x = 1.$$

Find the x-intercepts of $y = x^2 - 2x$:

$$x^2 - 2x = 0 \implies x(x-2) = 0 \implies x = 0, 2.$$

Observe that $y = \log(3-x)$ and $y = \log(1+x)$ intercept the x-axis at 2 and 0 respectively: $\log(3-2) = \log 1 = 0$, $\log(1+0) = \log 1 = 0$.

Let $f(x)$ equal $\log(1+x)$ on $[0,1]$ and $\log(3-x)$ on $[1,2]$. Then the required area is given by

$$\int_0^2 (f(x) - (x^2-2x))\,dx = \int_0^1 \log(1+x)\,dx + \int_1^2 \log(3-x)\,dx - \int_0^2 (x^2-2x)\,dx.$$

(b) The arc and the slanted line segment meet at $x = -1/2$. The corresponding y-coordinate is $\sqrt{1-(-1/2)^2} = \sqrt{3}/2$. The equation of the slanted line segment is $y = -\sqrt{3}x$. Therefore the area is given by

$$\int_{-1/2}^1 \sqrt{1-x^2}\,dx + \int_{-1/2}^0 \sqrt{3}x\,dx.$$

6. Apply Theorem 2.2.9:

$$2\int_{-a}^a b\sqrt{1-x^2/a^2}\,dx = 2ab\int_{-1}^1 \sqrt{1-x^2}\,dx = \pi ab.$$

Thematic Exercises

Darboux Integral

A1. Suppose $m \le f(x) \le M$ for every $x \in [a,b]$. We have $m \le m_i \le M_i \le M$ for every i. Hence,

$$L(f,P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \le \sum_{i=1}^n M(x_i - x_{i-1}) = M\sum_{i=1}^n (x_i - x_{i-1}) = M(b-a)$$
$$U(f,P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \ge \sum_{i=1}^n m(x_i - x_{i-1}) = m\sum_{i=1}^n (x_i - x_{i-1}) = m(b-a)$$

A2. Take any partition $P = \{x_0, \ldots, x_n\}$ of $[0,1]$. By the density of rationals and irrationals, each $m_i = 0$ and $M_i = 1$. Therefore $L(f,P) = 0$ and $U(f,P) =$

1 for every P. Hence the lower Darboux integral is 0 and the upper Darboux integral is 1.

A3. We first show that if P is a partition of $[a, b]$ and P' is a refinement of P then

$$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P).$$

It is enough to prove this when P' has one more point than P. Let $P = \{x_0, \ldots, x_n\}$ and $P' = P \cup \{t\}$ with $x_{k-1} < t < x_k$. As usual, let $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}$. Further, let $m' = \inf\{f(x) : x \in [x_{k-1}, t]\}$ and $m'' = \inf\{f(x) : x \in [t, x_k]\}$. Then $m_k \leq m', m''$. Now,

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m_k(x_k - x_{k-1}) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + m'(t - x_{k-1}) + m''(x_k - t) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1}) \\ &= L(f, P') \end{aligned}$$

We can similarly prove that $U(f, P') \leq U(f, P)$.

Now, if P, Q are any two partitions of $[a, b]$, then $P \cup Q$ is a common refinement, and we get:

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q).$$

Define $A = \{\, L(f, P) \mid P \text{ is a partition of } [a, b] \,\}$
$B = \{\, U(f, P) \mid P \text{ is a partition of } [a, b] \,\}$

We have just shown that every member of B is an upper bound of A. Therefore $\sup(A)$ is a lower bound of B. Therefore $\sup(A) \leq \inf(B)$, as desired.

A4. Let $A = \{\, L(f, P) \mid P \text{ is a partition of } [a, b] \,\}$
$B = \{\, U(f, P) \mid P \text{ is a partition of } [a, b] \,\}$

Note that $A \subset \mathcal{L}_f$ and $B \subset \mathcal{U}_f$.

Now suppose that f is Darboux integrable. Let I separate $\mathcal{L}_f$ and $\mathcal{U}_f$. Then I is an upper bound of A and a lower bound of B. Hence,

$$\underline{\int_a^b} f(x)\, dx = \sup(A) \leq I \leq \inf(B) = \overline{\int_a^b} f(x)\, dx.$$

The equality of the upper and lower Darboux integrals shows that I equals them, hence I is unique and f is integrable.

For the converse, let $I = \int_a^b f$. By definition, I is an upper bound of A and a lower bound of B. We claim that $I = \sup(A) = \inf(B)$. Consider $I - \varepsilon$ with $\varepsilon > 0$. By the Riemann condition, there is a step function s such that $s \leq f$ and $\int_a^b s > I - \varepsilon$. Let P be a partition adapted to s. Then $L(f, P) \geq \int_a^b s > I - \varepsilon$, so $I - \varepsilon$ is not an upper bound of A. This shows that $I = \sup(A)$. We can similarly show that $I = \inf(B)$.

3 | Limits and Continuity

3.1 Limits

Task 3.1.4. Given an $\varepsilon > 0$ take any positive δ. For example, take $\delta = 1$. Then for any x, and in particular for those satisfying $0 < |x - p| < \delta$, we have $|f(x) - c| = |c - c| = 0 < \varepsilon$.

Task 3.1.7. We have $\lim_{x\to p} f(ax + b) = L \iff \lim_{h\to 0} f(ap + b + ah) = L$ and $\lim_{y\to ap+b} f(y) = L \iff \lim_{k\to 0} f(ap + b + k) = L$.

The definitions of the two equivalent limits are:

$\lim_{h\to 0} f(ap + b + ah) = L$: For each $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(ap + b + ah) - L| < \varepsilon$.

$\lim_{k\to 0} f(ap + b + k) = L$: For each $\varepsilon > 0$ there is a $\delta' > 0$ such that $0 < |k| < \delta'$ implies $|f(ap + b + k) - L| < \varepsilon$.

Observe that δ works in the first definition if and only if $\delta' = |a|\delta$ works in the second one.

Task 3.1.19. Apply the algebra of limits.

(a) $\lim_{x\to 2} x = 2 \implies \lim_{x\to 2} x^2 = 2^2 = 4 \implies \lim_{x\to 2} \frac{1}{x^2} = \frac{1}{4}$.

(b) $\frac{x^2 - 6x + 9}{x^2 - 9} = \frac{(x-3)^2}{(x+3)(x-3)} = \frac{x-3}{x+3}$ for $x \neq 3$. Hence,

$$\lim_{x\to 3} \frac{x^2 - 6x + 9}{x^2 - 9} = \lim_{x\to 3} \frac{x-3}{x+3} = \frac{0}{6} = 0.$$

(c) Observe that for $x \neq 0$, $\frac{|x|}{x} = \operatorname{sgn}(x)$. Apply Example 3.1.10.

Task 3.1.20. We have to rule out $\lim_{x\to a} f(x) < m$ as well as $\lim_{x\to a} f(x) > M$. Let us do the first. Suppose $\lim_{x\to a} f(x) = L < m$. Set $\varepsilon = m - L$. There is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < m - L$. Let $p = a + \delta/2$. Then $|f(p) - L| < m - L$ implies $f(p) < L + (m - L) = m$, a contradiction.

Task 3.1.24.

(a) $\lim_{x\to p+} C = C$: For any $\varepsilon > 0$, take $\delta = 1$.

(b) As in (a).

(c) $\lim_{x\to 1+} [x] = 1$: For any $\varepsilon > 0$, take $\delta = 1$.

(d) $\lim_{x\to 1-} [x] = 0$: For any $\varepsilon > 0$, take $\delta = 1$.

(e) $\lim_{x\to 0+} \frac{|x|}{x} = 1$: For any $\varepsilon > 0$, take $\delta = 1$.

(f) $\lim_{x\to 0+} \sqrt{x} = 0$: For any $\varepsilon > 0$, take $\delta = \varepsilon^2$.

Task 3.1.27. The only change in the statements is the replacement of the limits by one-sided limits. The proofs also undergo only cosmetic changes. For example, the statement of the sandwich theorem for right-hand limits is:

Suppose that $f(x) = g(x) = h(x)$ *in an interval* $(p, p+\delta')$, *with* $\delta' > 0$. *If* $\lim_{x\to p+} f(x) = \lim_{x\to p+} g(x) = L$, *then* $\lim_{x\to p+} g(x) = L$.

To prove this, consider any $\varepsilon > 0$.

There exists $\delta_f > 0$ such that $0 < x - p < \delta_f$ implies $L - \varepsilon < f(x) < L + \varepsilon$.

There exists $\delta_h > 0$ such that $0 < x - p < \delta_h$ implies $L - \varepsilon < h(x) < L + \varepsilon$.

Let $\delta = \min\{\delta_f, \delta_h, \delta'\}$. Now, if $0 < x - p < \delta$, then

- $\delta \le \delta_f \implies L - \varepsilon < f(x) < L + \varepsilon$,
- $\delta \le \delta_h \implies L - \varepsilon < h(x) < L + \varepsilon$,
- $\delta \le \delta' \implies f(x) \le g(x) \le h(x)$.

Combining these gives $L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$. Hence $L - \varepsilon < g(x) < L + \varepsilon$. Therefore $\lim_{x\to p+} g(x) = L$.

Exercises for §3.1

2. The values of δ are:

(a) 1,

(b) $0.1^{1/3} \approx 0.46$,

(c) $1.1^{1/3} - 1 \approx 0.03$,

(d) $8.1^{1/3} - 2 \approx 0.008$.

4. These are applications of the algebra of limits. The results are:

(a) 1/4,

(b) 4,

(c) $2t$,

(d) 3/5,

(e) 2,

(f) 3,

(g) 1,

(h) 6.

6. $\lim_{x\to 2}(f(x) - 5) = \lim_{x\to 2} \frac{f(x)-5}{x-2}(x-2) = 3 \cdot 0 = 0$ implies $\lim_{x\to 2} f(x) = 5$.

8.

(a) $|x^{1/n} - a^{1/n}| = \frac{|x-a|}{\sum_{k=0}^{n-1} x^{k/n} a^{1-k/n}} \le \frac{|x-a|}{a^{1/n}}$. Apply the Sandwich theorem.

(b) Given $\varepsilon > 0$, let $\delta = \varepsilon^n$.

10. Apply $\lim_{x\to 0-} f(x) = \lim_{t\to 0+} f(-t)$.

3.2 Continuity

Task 3.2.11. $x^r = \exp(r \log x)$ is a composition of continuous functions.

Task 3.2.13. $F(x) = \int_a^x f(t)\,dt = \int_a^b f(t)\,dt + \int_b^x f(t)\,dt = \int_a^b f(t)\,dt + G(x)$.

Exercises for §3.2

2.

(a) The function $\sqrt{\log x}$ is continuous. Hence, $\lim_{x\to 1+} \sqrt{\log x} = \sqrt{\log 1} = 0$.

(b) The function $\log\sqrt{x^2+1}$ is continuous. So, $\lim_{x\to 0} \log\sqrt{x^2+1} = \log\sqrt{0^2+1} = 0$.

(c) We have $\lim_{h\to 0} \log\left((1+h)^{1/h}\right) = \lim_{h\to 0} \dfrac{\log(1+h)}{h} = 1$, by Exercise 9 of §3.1. Now we invoke the continuity of the exponential function:

$$\lim_{h\to 0}(1+h)^{1/h} = \lim_{h\to 0} \exp\left(\frac{\log(1+h)}{h}\right) = \exp(1) = e.$$

(d) We have $\lim_{x\to 0+} \log\left(x^{x^2}\right) = \lim_{x\to 0+} x^2 \log x = 0$, by Exercise 7 of §3.1. Therefore,

$$\lim_{x\to 0+} x^{x^2} = \lim_{x\to 0+} \exp(x^2 \log x) = \exp(0) = 1.$$

4. Suppose f is an increasing function. Let $p \in (a,b)$. We claim that $\lim_{x\to p-} f(x) = \sup\{\, f(x) \mid x \in (a,p)\,\}$.

Let $L = \sup\{\, f(x) \mid x \in (a,p)\,\}$ and consider any $\varepsilon > 0$. There is $q \in (a,p)$ such that $f(q) > L - \varepsilon$. Let $\delta = p - q$. Then,

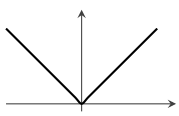
$$0 < p-x < \delta \implies q < x < p \implies L-\varepsilon < f(q) \le f(x) \le L \implies |f(x)-L| < \varepsilon.$$

We can similarly prove that $\lim_{x\to p+} f(x) = \inf\{\, f(x) \mid x \in (p,b)\,\}$.

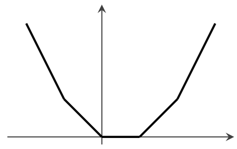
If f is a decreasing function, apply the above work to $-f$.

6. Suppose f is an increasing function. We already know f is continuous on points of $f^{-1}(a,b)$. So consider x_0 such that $f(x_0) = b$. Also consider $\varepsilon > 0$ such that $b - \varepsilon \in I$. Then there is $\delta > 0$ such that $x_0 - \delta = b - \varepsilon/2$.

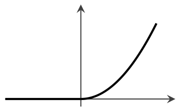
8. Let us write $F(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) $F(x) = |x|$ 

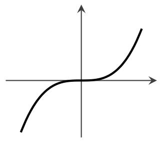
(b) $F(x) = \frac{1}{2}[x][x-1] + [x](x-[x])$



(c) $F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2/2 & \text{if } x \geq 0 \end{cases}$



(d) $F(x) = x^3/3$



3.3 Intermediate Value Theorem

Exercises for §3.3

2. This is equivalent to asking whether $x = x^3 + 1$ has a solution. Consider $f(x) = x^3 - x + 1$. Observe that f is continuous, $f(0) = 1$ and $f(-2) = -5$. By the intermediate value theorem, there is a $c \in (-2, 0)$ such that $f(c) = 0$. This c satisfies $c = c^3 + 1$.

4.

(a) A is non-empty, since $a \in A$. And A is bounded above by b. Therefore $\sup(A)$ exists, by the least upper bound property.

(b) Suppose $f(c) < 0$. By continuity, there is an interval $I = (c - \delta, c + \delta)$ such that $f(x) < 0$ for every $x \in I$. Then $f(c + \delta/2) < 0$, contradicting c being an upper bound of A.

Suppose $f(c) > 0$. By continuity, there is an interval $I = (c - \delta, c + \delta)$ such that $f(x) > 0$ for every $x \in I$. Then $f(c - \delta/2)$ is an upper bound of A, contradicting c being the *least* upper bound of A.

6. No. Suppose $f\colon \mathbb{R} \to \mathbb{Q}$ was continuous and had two distinct values a, b with $a < b$. By the intermediate value theorem, its range would contain $[a, b]$ and hence would contain some irrational numbers.

8. Convert the problem to one about zeroes of a continuous function. Define $g\colon [0,1] \to \mathbb{R}$ by $G(x) = f(x) - x$. Then g is continuous, $g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$. By the intermediate value theorem, there is a $c \in [0, 1]$ such that $g(c) = 0$. Therefore, $f(c) = c$.

10. Define $g\colon [0,1] \to \mathbb{R}$ by $g(x) = f(x) - f(x+1)$. Then g is continuous, $g(0) = f(0) - f(1)$, $g(1) = f(1) - f(2) = f(1) - f(0) = -g(0)$. By the intermediate value theorem, there is a $c \in [0, 1]$ such that $g(c) = 0$. Therefore, $f(c) = f(c+1)$.

12. Suppose that $f\colon (0,1) \to [0,1]$ is a continuous bijection. There are points $a, b \in (0, 1)$, $a < b$, such that f maps one of them to 0 and the other to 1. By the intermediate value theorem, $f([a, b]) = [0, 1]$. But then f cannot be injective on $(0, 1)$.

14. Follow the solution of Exercise 13. First, we may assume that the polynomial has the form $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n$, with n odd. Then we show that for $x \geq 1$,

$$p(x) \geq x^n \Big(1 - \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{x}\Big).$$

Hence, $x_1 = 2\sum_{i=0}^{n-1} |a_i|$ satisfies $p(x_1) > 0$. By considering $-p(-x)$ we similarly find x_2 with $p(x_2) < 0$. Now apply the intermediate value theorem.

16. We note that the only zeroes of f are at $x = \pm 1$. Let $a, b \in (-1, 1)$ with $f(a) = \sqrt{1-a^2} > 0$ and $f(b) = -\sqrt{1-b^2} < 0$. By the intermediate value theorem, there is a $c \in (-1, 1)$ with $f(c) = 0$, which is impossible.

3.4 Trigonometric Functions

Task 3.4.1. Let $\theta \in [\pi, 2\pi]$. There is an angle whose radian measure is $\theta - \pi$. Increase this angle by two right angles, to get an angle whose radian measure is θ.

Task 3.4.2. By definition, the point $(\cos t, \sin t)$ is on the unit circle, and this gives $\sin^2 t + \cos^2 t = 1$.

Task 3.4.3. Draw a figure for the angle of t radians and reflect the parts in the $y = x$ line to get the figure for $\pi/2 - t$ radians.

Task 3.4.4. From the previous Task, we have $\sin(\pi/4) = \sin(\pi/2 - \pi/4) = \cos(\pi/4)$. Hence, $\sin^2(\pi/4) + \cos^2(\pi/4) = 1$ gives $\sin^2(\pi/4) = 1/2$. Now $\sin(\pi/4) \geq 0$ gives $\sin(\pi/4) = 1/\sqrt{2}$.

Task 3.4.5. We have $\sin(\pi/6) = \sin(\pi/2 - \pi/3) = \cos(\pi/3)$.

Task 3.4.6.

$$\begin{aligned}\cos 2x &= \cos(x + x) = (\cos x)(\cos x) - (\sin x)(\sin x) = \cos^2 x - \sin^2 x,\\ \sin 2x &= \sin(x + x) = (\sin x)(\cos x) + (\cos x)(\sin x) = 2 \sin x \cos x.\end{aligned}$$

Task 3.4.7. We already know that $\sin \pi/2 = 1$ and $\sin \pi/4 = \cos \pi/4 = 1/\sqrt{2}$. Therefore, $\sin \pi/8 = \sqrt{\dfrac{1 - \cos \pi/4}{2}} = \sqrt{\dfrac{\sqrt{2} - 1}{2\sqrt{2}}}$. This gives the following table of numerical values:

x	$\sin x$	$\dfrac{\sin x}{x}$
$\pi/2$	1	0.64
$\pi/4$	0.7071	0.9003
$\pi/8$	0.3827	0.9745

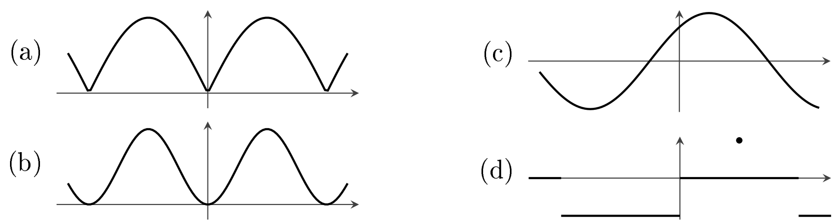
Task 3.4.10. First compute $\lim_{x\to 1} \dfrac{x^2 - 2x + 1}{x^2 - 1} = \lim_{x\to 1} \dfrac{x - 1}{x + 1} = 0$. Since sine is continuous, we get $\lim_{x\to 1} \sin\Big(\dfrac{x^2 - 2x + 1}{x^2 - 1}\Big) = \sin\Big(\lim_{x\to 1} \dfrac{x^2 - 2x + 1}{x^2 - 1}\Big) = \sin(0) = 0$.

Task 3.4.11. Consider the continuous function $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

We have, $\lim_{x\to 1}\frac{\sin(x^2-1)}{x-1}=\lim_{x\to 1}(x+1)f(x^2-1)=\lim_{x\to 1}(x+1)f(\lim_{x\to 1}x^2-1)=$ $2\cdot f(0)=2$.

Exercises for §3.4

2.



4.

(a) $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2(x/2)}{x^2}=\lim_{t\to 0}\frac{\sin^2 t}{2\,t^2}=\frac{1}{2}$.

(b) We know $\lim_{x\to 0}(1-\cos x)=0$. Use (a) to get $\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{(1-\cos x)^2}=\frac{1}{2}$.

$$\therefore \lim_{x\to 0}\frac{1-\cos(1-\cos x)}{x^4}=\lim_{x\to 0}\frac{1-\cos(1-\cos x)}{(1-\cos x)^2}\cdot\left(\frac{1-\cos x}{x^2}\right)^2=\frac{1}{8}.$$

(c) Apply the sandwich theorem. $0\le\left|x\sin\frac{1}{x}\right|\le|x|$ for $x\neq 0$ gives $\lim_{x\to 0}x\sin\frac{1}{x}=0$.

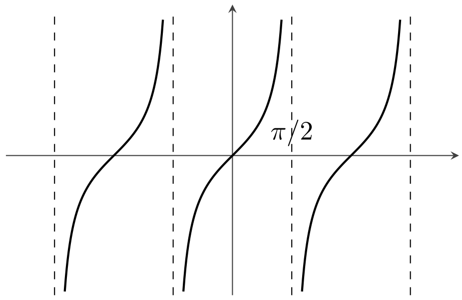
(d) We have $\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}(x+1)=2$. Note that for x close to 1, $x+1$ is positive and its square root is defined. Use the continuity of $\sin\sqrt{t}$ when $t>0$ to conclude

$$\lim_{x\to 1}\sin\sqrt{\frac{x^2-1}{x-1}}=\sin\sqrt{\lim_{x\to 1}\frac{x^2-1}{x-1}}=\sin\sqrt{2}.$$

6. The function $f(x)=x^3-\cos x$ is continuous. We have $f(0)=-1$ and $f(2)=8-\cos 2\ge 7$. By the intermediate value theorem, f has a zero in the interval $(0,2)$.

8. Use $\sin\pi=0$ and $cos\pi=-1$:

$$\tan(x+\pi)=\frac{(\sin x)(\cos\pi)+(\sin\pi)(\cos x)}{(\cos x)(\cos\pi)-(\sin x)(\sin\pi)}=\tan x.$$



10. If $A^2 + B^2 = 1$ then (A, B) is a point on the unit circle and there is an angle ϕ such that $A = \cos\phi$ and $B = \sin\phi$. Then,

$$A \sin x + B \cos x = \cos\phi \sin x + \sin\phi \cos x = \sin(x + \phi).$$

In general, let $A^2 + B^2 = R^2$. The $R = 0$ case is trivial. For $R \neq 0$:

$$A \sin x + B \cos x = R\big((A/R) \sin x + (B/R) \cos x\big) = R \sin(x + \phi).$$

12.

(a) $\sin^2(\pi/12) = \dfrac{1 - \sin(\pi/3)}{2} = \dfrac{2 - \sqrt{3}}{4}$

$\implies \sin(\pi/12) = \dfrac{(2 - \sqrt{3})^{1/2}}{2} = 0.2588,$

$\sin^2(5\pi/12) = 1 - \sin^2(\pi/12) = \dfrac{2 + \sqrt{3}}{4}$

$\implies \sin(5\pi/12) = \dfrac{(2 + \sqrt{3})^{1/2}}{2} = 0.9659.$

Now we know all the sine values in steps of $\pi/12$.

(b) $\sin^2(\pi/24) = \dfrac{1 - \sin(5\pi/12)}{2} \implies \sin(\pi/24) = 0.1305$

$\sin^2(3\pi/24) = \dfrac{1 - \sin(\pi/4)}{2} \implies \sin(3\pi/24) = 0.3827$

$\sin^2(5\pi/24) = \dfrac{1 - \sin(\pi/12)}{2} \implies \sin(5\pi/24) = 0.6088$

$\sin^2(7\pi/24) = 1 - \sin^2(5\pi/24) \implies \sin(7\pi/24) = 0.7934$

$\sin^2(9\pi/24) = 1 - \sin^2(3\pi/24) \implies \sin(9\pi/24) = 0.9239$

$\sin^2(11\pi/24) = 1 - \sin^2(\pi/24) \implies \sin(11\pi/24) = 0.9914$

(The even multiples of $\pi/24$ were covered by the $\pi/12$ calculations.)

3.5 Continuity and Variation

Task 3.5.3. On $[0, 1]$, $\mathrm{sgn}(x)$ takes the values 0 and 1. So its span is $|1-0| = 1$.

On $[0, 1]$, the range of $\sin(x)$ is $[0, 1]$. So its span is $|1 - 0| = 1$.

Task 3.5.4. The extreme value theorem gives the existence of

$$M = \max\{\, f(x) \mid x \in [a, b] \,\} \text{ and } m = \min\{\, f(x) \mid x \in [a, b] \,\}.$$

For any $x, y \in [a, b]$, we have

$$\begin{aligned} m \le f(x), f(y) \le M &\implies -M \le -f(x), -f(y) \le -m \\ &\implies m - M \le f(x) - f(y) \le M - m \\ &\implies |f(x) - f(y)| \le M - m. \end{aligned}$$

Therefore $M - m$ is at least equal to the span of f. On the other hand, from $m, M \in \{ f(x) \mid x \in [a, b] \}$ we obtain that $M - m$ does not exceed the span.

Exercises for §3.5

2.

(a) $f : (0, 1) \to \mathbb{R}$, $f(x) = 1/x$.

(b) $g : (0, \infty) \to \mathbb{R}$, $g(x) = 1 - e^{-x}$.

4.

(a) Let M' be an upper bound for f on A and M'' be an upper bound for f on B. Then $M = \max\{M', M''\}$ is an upper bound for f on $A \cup B$.

(b) A is non-empty because $a \in A$. And A is bounded above by b. Apply the least upper bound property.

(c) By continuity at a, $\alpha > a$. If $\alpha < b$ then there is $\delta' > 0$ such that $I = (\alpha - \delta', \alpha + \delta') \subset [a, b]$ and $x \in I \implies |f(x) - f(\alpha)| < 1$. Let $\alpha - \delta' < a' < \alpha$ such that $|f|$ is bounded on $[a, a']$ by M. Then $|f|$ is bounded on $[a, \alpha + \delta'/2]$ by $M + 2$. Hence, $\alpha + \delta'/2 \in A$, a contradiction.

(d) There is $\delta > 0$ such that $x \in (b - \delta, b] \implies |f(x) - f(b)| < 1$. Let $b - \delta < a' < b$ such that $|f|$ is bounded on $[a, a']$ by M. Then $|f|$ is bounded on $[a, b]$ by $M + 2$.

6.

(a) Apply the boundedness theorem.

(b) If $f(x)$ never equals M then g is continuous on $[a, b]$. Apply the boundedness theorem to g.

(c) $\dfrac{1}{M - f(x)} \le R \implies M - f(x) \ge \dfrac{1}{R} \implies f(x) \le M - 1/R.$

3.6 Continuity, Integration and Means

Task 3.6.3. $m \le f \le M \implies m(b - a) \le \int_a^b f \le M(b - a) \implies m \le \dfrac{1}{b - a} \int_a^b f \le M.$

Task 3.6.4. $\bar{f}_{[a,c]} = \dfrac{1}{c - a} \Big(\int_a^b f + \int_b^c f \Big) = \dfrac{1}{c - a} \Big((b - a)\bar{f}_{[a,b]} + (c - b)\bar{f}_{[b,c]} \Big).$

Task 3.6.5. Let $a < x < y$. Apply Task 3.6.3 to obtain $\bar{f}_{[x,y]} \le f(x) \le \bar{f}_{[a,x]}$. Now apply Task 3.6.4:

$$\bar{f}_{[a,y]} = \frac{x - a}{y - a} \bar{f}_{[a,x]} + \frac{y - x}{y - a} \bar{f}_{[x,y]} \le \frac{x - a}{y - a} \bar{f}_{[a,x]} + \frac{y - x}{y - a} \bar{f}_{[a,x]} = \bar{f}_{[a,x]}.$$

Exercises for §3.6

2. By the mean value theorem for integration, there is a $c \in (0,1)$ such that $f(c) = \frac{1}{1-0}\int_0^1 f(x)\,dx = 1$.

4. By continuity, it is enough to show that $f(c) = 0$ for every $c \in (a,b)$.

We shall show that $f(c) > 0$ gives a contradiction. Let $\varepsilon = f(c)/2$. There is a $\delta > 0$ such that $[c-\delta, c+\delta] \subseteq [a,b]$ and $x \in [c-\delta, c+\delta]$ implies $|f(x)-f(c)| < \varepsilon$. In particular, $x \in [c-\delta, c+\delta]$ implies $f(x) > f(c) - \varepsilon = f(c)/2$. Therefore,

$$\int_a^b f(x)\,dx \ge \int_{c-\delta}^{c+\delta} f(x)\,dx \ge \int_{c-\delta}^{c+\delta} (f(c)/2)\,dx \ge \delta f(c) > 0.$$

Similarly, $f(c) < 0$ also gives a contradiction.

6. Apply the mean value theorem for integration.

3.7 Limits Involving Infinity

Task 3.7.2. Take $M = -\log\varepsilon$ and $M = \log\varepsilon$ respectively.

Task 3.7.10. As we have $x \to \infty$, we can assume $x > 1$. Then we have the inequalities $0 < \log x < x - 1$. Replace x by $x^{1/2n}$ to get

$$0 < \frac{\log x}{2n} < x^{1/2n} - 1, \quad \text{hence } 0 < \frac{\log x}{x^{1/n}} < 2n(x^{-1/2n} - x^{-1/n}).$$

Now apply the sandwich theorem.

Task 3.7.12. $\lim_{x\to 0+} x\log x = \lim_{t\to\infty} \frac{1}{t}\log(1/t) = -\lim_{t\to\infty} \frac{\log t}{t} = 0,$

$$\lim_{x\to 0+} x^x = \lim_{x\to 0+} e^{x\log x} = e^{\lim_{x\to 0+} x\log x} = e^0 = 1.$$

Task 3.7.19. Since $f(x) \to \infty$, there is a real number M such that $x > M$ implies $f(x) > 1$. Working over the interval (M, ∞), we get $|g(x)| < |g(x)f(x)| \to 0$.

Exercises for §3.7

2.

(a) As $x \to \pi/2-$, $\cos x \to 0$ from the positive side. So $\lim_{x\to\pi/2-} \sec x = \infty$. Formally, for any $M > 0$, there is a $\delta > 0$ such that $\pi/2 - \delta < x < \pi/2$ implies $\cos x < 1/M$ and hence $\sec x > M$.

(b) Does not exist.

(c) For $x \ge 1$, $x - \sqrt{x} = \dfrac{x^2 - x}{x + \sqrt{x}} \ge \dfrac{x^2 - x}{2x} = \dfrac{x-1}{2}$. So, $\lim_{x\to\infty}(x - \sqrt{x}) = \infty$.

(d) $\displaystyle\lim_{x\to\infty} \frac{x+1}{\sqrt{4x^2+1}} = \lim_{x\to\infty} \frac{1 + 1/x}{\sqrt{4 + 1/x^2}} = \frac{1}{2}$.

(e) $\lim_{x\to\infty} e^{-x^2} = 0$: Given $0 < \varepsilon < 1$ consider $M = \sqrt{-\log\varepsilon}$.

(f) $\lim_{x\to 0} e^{-1/x^2} = \lim_{x\to 0+} e^{-1/x^2} = \lim_{t\to\infty} e^{-t^2} = 0.$

(g) $\lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{t\to 0+} \frac{\sin t}{t} = 1.$

(h) $\lim_{x\to\infty} \frac{x\sin x}{x^2+1} = \lim_{x\to\infty} \frac{x^2}{x^2+1}\frac{\sin x}{x} = 1\cdot 0 = 0.$

(i) $\lim_{x\to\infty} (\sqrt{x^2+x} - x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \frac{1}{2}.$

(j) $\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x}} - \sqrt{x-\sqrt{x}}\right) = \lim_{x\to\infty} \frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}} = 1.$

4. Let $p(x) = a_0 + \cdots + a_m x^m$ and $q(x) = b_0 + \cdots + b_n x^n$, with $a_m, b_n \neq 0$. Then:

- If $m = n$, the limit is a_m/b_m.
- If $m < n$, the limit is 0.
- If $m > n$, the limit is $\pm\infty$, depending on the sign of a_m/b_n.

6. There is $M > a$ such that $x > M$ implies $|f(x)| < |L| + 1$. Further, by the boundedness theorem, there is L' such that $x \in [a, M]$ implies $|f(x)| < L'$. Therefore, for every $x \in [a,\infty)$, $|f(x)| < \max\{|L| + 1, L'\}$.

8. This is obvious if $\alpha \leq 0$. For $\alpha > 0$, use $0 < e^{-x}x^{\alpha} < e^{-x}x^{[\alpha]+1}$.

Thematic Exercises

Continuity and Intervals

A1. The given hint was: "Consider cases of whether A is bounded above or below, and whether its supremum and infimum belong to it."

For example, suppose A is bounded above, but not below, and contains its supremum. Let its supremum be b. Consider any $x < b$. Since A is not bounded below, there is $a \in A$ such that $a < x < b$. So $x \in A$. Hence $A = (-\infty, b]$.

The other cases can be covered similarly.

A2.

(a) Let $c, d \in f(I)$ and $c < y < d$. We have $a, b \in I$ such that $f(a) = c$, $f(b) = d$. By the intermediate value theorem there is x between a and b such that $f(x) = y$. Hence $y \in f(I)$.

(b) We have $I = [a, b]$. By the extreme value theorem, $f(I) \subseteq [m,M]$ with $m,M \in f(I)$. Then the intermediate value theorem gives $f(I) = [m,M]$.

(c) Explore!

A3.

(a) We have shown that $f([a, b]) = [m,M]$. If $m < f(a) < M$ there is a $c \in (a, b]$ such that $f(a) = f(c)$, a contradiction. So $f(a) \in \{m,M\}$. Similarly, $f(b) \in \{m,M\}$. Hence, f maps $[a, b]$ to $[f(a), f(b)]$ or $[f(b), f(a)]$.

If f is not monotonic, there will be points $x_1, \dots, x_4$ (not necessarily distinct) such that $x_1 < x_2$, $x_3 < x_4$, $f(x_1) < f(x_2)$, and $f(x_3) > f(x_4)$. Let $a = \min\{x_1, \dots, x_4\}$ and $b = \max\{x_1, \dots, x_4\}$.

Now, suppose that $f([a,b]) = [f(a), f(b)]$. Then $a \le x_4 \le b$ gives $f(a) \le f(x_4)$, hence $f([a, x_4]) = [f(a), f(x_4)]$. But then, $a \le x_3 \le x_4$ gives $f(x_3) \le f(x_4)$, a contradiction.

The $f([a,b]) = [f(b), f(a)]$ case similarly leads to a contradiction as well.

(b) Since $f\colon I \to f(I)$ is a bijection, f^{-1} exists. As f is monotonic, so is f^{-1}. Since f is continuous, $f(I)$ is an interval. Apply Theorem 3.2.9.

(c) We know that $f(I)$ is an interval. We have to show that $f(I)$ does not have a maximum or a minimum element.

Suppose f is strictly increasing and $y \in f(I)$. Let $f(x) = y$. Since I is open, there are $a, b \in I$ such that $a < x < b$. Then $f(a) < y < f(b)$.

(d) Since $f \circ f$ is a bijection, so is f. Then the continuity of f implies that it is strictly increasing or strictly decreasing. In either case, $f \circ f$ is strictly increasing.

A4.

(a) Let $\varepsilon = 1$ and consider any $\delta > 0$. Then

$$f(x + \delta/2) - f(x) = \delta^2/4 + \delta x > \delta x = 1, \quad \text{if } x = 1/\delta.$$

(b) Let $\varepsilon = 1$ and consider any $\delta > 0$. Then $g(x/2) - g(x) = 1/x$. Let $x = \min\{1, \delta\}$.

A5. Given $\varepsilon > 0$ there is a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ such that $|f(x) - f(y)| < \varepsilon/2$ whenever x, y belong to the same $[x_{i-1}, x_i]$ subinterval. Let $\delta = \min\{\, x_i - x_{i-1} \mid i = 1, \dots, n \,\}$.

A6. Suppose f extends to a continuous function $\tilde{f}\colon [0,1] \to \mathbb{R}$. Then $\tilde{f}$ is uniformly continuous. Hence f is uniformly continuous.

Now, suppose f is uniformly continuous. For each $n \in \mathbb{N}$ there is $\delta_n > 0$ such that $|x - y| < \delta_n$ implies $|f(x) - f(y)| < 1/n$. The $n = 1$ case shows f is bounded and so each $f(0, \delta_n)$ is a bounded interval. Let $a_n = \inf f(0, \delta_n)$ and $b_n = \sup f(0, \delta_n)$. Then the nested interval property applied to the intervals $[a_n, b_n]$ gives $\cap_n [a_n, b_n] \ne \varnothing$.

Note that $b_n - a_n \le 1/n$ so $\cap_n [a_n, b_n] = \{y_0\}$. Extend f to $\tilde{f}\colon [0,1] \to \mathbb{R}$ by defining $\tilde{f}(0) = y_0$. The proof of continuity of $\tilde{f}$ at $x = 0$ is left to the reader.

A7. Let $\lim_{x \to \infty} f(x) = L$. Given $\varepsilon > 0$, there is M such that $x \ge M$ implies $|f(x) - L| < \varepsilon/4$. Now f is uniformly continuous on $[a, M]$ so there is a $\delta > 0$ such that $x, y \in [a, M]$ and $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. This is the required δ.

4 | Differentiation

4.1 Derivative of a Function

Task 4.1.1. Given any $\varepsilon > 0$, choose $\delta = 1$. Then $|x - a| < \delta$ implies $|f(x) - f(a) - m(x - a)| = 0 \leq \varepsilon|x - a|$.

Task 4.1.5. The function $f(x) = |x|$ is continuous at every point but is not differentiable at 0. We apply the characterisation of Theorem 4.1.2. If $f'(0) = L$, we have a function φ that is continuous at 0, satisfies $\varphi(0) = 0$, and $|x| - |0| - L(x - 0) = (x - 0)\varphi(x)$ for every x. Hence, for $x \neq 0$, $\varphi(x) = \dfrac{|x|}{x} - L$. But then $\lim_{x \to 0} \varphi(x)$ does not exist.

Task 4.1.7. Let $f(x) = c$. Then $\lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim_{x \to a} 0 = 0$.

Task 4.1.10. Apply Theorem 3.1.25.

Task 4.1.12. No. Consider $f(x) = x^3$.

Exercises for §4.1

2.

(a) It is not even continuous at 0.

(b) $\lim_{x \to 0+} \dfrac{\sqrt{|x|} - 0}{x - 0} = \lim_{x \to 0+} \dfrac{1}{\sqrt{x}}$ does not exist.

4.

(a) For continuity, apply the sandwich theorem. The lack of differentiability follows from the calculation below:

$$\lim_{x \to 0} \frac{x \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} \sin(1/x), \text{which does not exist.}$$

(b) $\lim_{x \to 0} \dfrac{x^2 \sin(1/x) - 0}{x - 0} = \lim_{x \to 0} x \sin(1/x) = 0$.

6.

(a) $\sin'(0) = \lim_{x \to 0} \dfrac{\sin x - 0}{x - 0} = \lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

(b) $\cos'(0) = \lim_{x \to 0} \dfrac{\cos x - 1}{x - 0} = \lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0$.

8.

(a) $f'(-a) = \lim_{x\to -a} \frac{f(x)-f(-a)}{x-(-a)} = \lim_{t\to a} \frac{f(-t)-f(-a)}{-t+a} = -\lim_{t\to a} \frac{f(t)-f(a)}{t-a} = -f'(a)$.

(b) Similar to (a).

10. f' is odd, so $f'(0) = f'(-0) = -f'(0)$.

12. (a) with (ii), (b) with (iii), (c) with (i).

4.2 Algebra of Derivatives

Task 4.2.2.

(a) The quotient rule gives the derivative to be $\frac{-1}{(x-1)^2}$, for $x \neq 1$.

(b) The derivative exists at every $x \notin \mathbb{Z}$, and is zero there.

(c) The derivative exists for $x \neq 0$, and is $-n/x^{n+1}$.

Task 4.2.4. $\sec' x = \left(\frac{1}{\cos x}\right)' = -\frac{\cos' x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} = \sec x \tan x$.

The other calculations are similar.

Task 4.2.7. Use $\log_a x = \frac{\log x}{\log a}$.

Exercises for §4.2

2.

(a) Differentiate both sides of the given formula and then multiply by x:

$$1x + 2x^2 + \cdots + nx^n = \frac{nx^n}{x-1} - \frac{x^{n+1}-x}{(x-1)^2}.$$

(b) Differentiate both sides of the formula obtained in (a) and then multiply by x:

$$1^2x + 2^2x^2 + \cdots + n^2x^n = \frac{n^2x^n}{x-1} - \frac{(2n+1)x^{n+1}-x}{(x-1)^2} + \frac{2nx^{n+1}}{(x-1)^3}.$$

4. Each differentiation lowers the degree by 1. So $p^{(n)}$ is constant and $p^{(n+1)}$ is zero.

6.

(a) The rule is $g' = f_1'f_2\cdots f_n + f_1f_2'\cdots f_n + \cdots + f_1f_2\cdots f_n'$.

(b) Apply (a).

8. First, suppose $r > 0$. Then $r = m/n$ with $m, n \in \mathbb{Z}$. Now:

$$\left(x^{m/n}\right)' = \lim_{y\to x} \frac{y^{m/n}-x^{m/n}}{y-x} = \lim_{y\to x} \frac{(y^{1/n})^m-(x^{1/n})^m}{(y^{1/n})^n-(x^{1/n})^n}$$

$$= \lim_{y \to x} \frac{y^{(m-1)/n} + y^{(m-2)/n}x^{1/n} + \cdots + x^{(m-1)/n}}{y^{(n-1)/n} + y^{(n-2)/n}x^{1/n} + \cdots + x^{(n-1)/n}}$$
$$= \frac{mx^{(m-1)/n}}{nx^{(n-1)/n}} = \frac{m}{n}x^{(m/n)-1} = rx^{r-1}$$

For $r < 0$, apply the reciprocal rule.

10. $\log x \leq x - 1 \implies \log(1+x) \leq x \implies 1 + x \leq e^x$.

4.3 Chain Rule and Applications

Task 4.3.2.

(a) $f'(x) = 20x(x^2+1)^9$, for every real x.

(b) $g'(x) = -\text{sgn}(\cos x)\sin(x)$, when x is not an odd multiple of $\pi/2$.

(c) $h'(x) = -\sin x$, for every real x.

(d) $k'(x) = \dfrac{\sin 2x \sin x^2 - 2x \sin^2 x \cos x^2}{\sin^2 x^2}$, when x^2 is not an integer multiple of π.

Task 4.3.6.

(a) $g(x)$ is the inverse function of $f(x) = x^2$ for $x \geq 0$. Therefore,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{2g(x)} = \frac{1}{2\sqrt{x}}, \text{ for } x > 0.$$

(b) Apply (a) and the chain rule:

$$h'(x) = \frac{1}{2\sqrt{x+\sqrt{x}}}\Big(1 + \frac{1}{2\sqrt{x}}\Big).$$

Task 4.3.9. $(a^x)' = (e^{x \log a})' = e^{x \log a}(x \log a)' = a^x \log a$.

Task 4.3.12. $\dfrac{d}{dx}\exp(\sqrt{x^2 + \arctan x}) = \dfrac{\exp(\sqrt{x^2+\arctan x})}{2\sqrt{x^2+\arctan x}}\Big(2x + \dfrac{1}{1+x^2}\Big)$.

Task 4.3.14.
$$\begin{aligned} f(x) = (\sin x)^{\cos x} &\implies g(x) = \log f(x) = (\cos x)\log(\sin x) \\ &\implies g'(x) = -(\sin x)\log(\sin x) + \frac{\cos^2 x}{\sin x} \\ &\implies f'(x) = f(x)g'(x) \\ &\qquad\quad = (\sin x)^{1+\cos x}(\cot^2 x - \log(\sin x)). \end{aligned}$$

Task 4.3.16.
$$\cosh' x = \Big(\frac{e^x + e^{-x}}{2}\Big)' = \frac{e^x - e^{-x}}{2} = \sinh x,$$
$$\sinh' x = \Big(\frac{e^x - e^{-x}}{2}\Big)' = \frac{e^x + e^{-x}}{2} = \cosh x.$$

Task 4.3.17. It is strictly increasing because $\sinh' x = \cosh x > 0$. To show it is a surjection we find its pre-images:

$$y = \frac{e^x - e^{-x}}{2} \iff e^{2x} - 2ye^x - 1 = 0 \iff x = \log(y + \sqrt{y^2+1}).$$

Task 4.3.18. It is strictly increasing on $(0,\infty)$ because $x>0$ gives $\cosh' x = \frac{e^x - e^{-x}}{2} > 0$. Further, $y > 1 \implies y^2 - 2y + 1 = (y-1)^2 > 0 \implies y + \frac{1}{y} > 2$. Hence $x > 0$ implies $\cosh x = \frac{e^x + e^{-x}}{2} > 1 = \cosh 0$.

To establish surjectivity, we observe that given $x \geq 0$ and $y \geq 1$ we have:

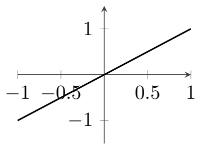
$$y = \frac{e^x + e^{-x}}{2} \iff e^{2x} - 2ye^x + 1 = 0 \iff x = \log(y + \sqrt{y^2-1}).$$

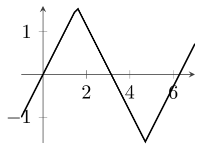
Task 4.3.19. $(\sinh^{-1} x)' = \frac{1}{\cosh(\sinh^{-1} x)} = \frac{1}{\sqrt{1+\sinh^2(\sinh^{-1} x)}} = \frac{1}{\sqrt{x^2+1}}$.

Exercises for §4.3

2. Differentiate with respect to x at x_0 to get $\frac{2x_0}{a^2} - \frac{2y_0}{b^2} y'(x_0) = 0$. Hence, $y'(x_0) = \frac{b^2}{a^2}\frac{x_0}{y_0}$, and the tangent line has equation $y - y_0 = \frac{b^2}{a^2}\frac{x_0}{y_0}(x - x_0)$. This can be rearranged to $\frac{yy_0 - y_0^2}{b^2} = \frac{xx_0 - x_0^2}{a^2}$ or $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$.

4.





6. First, $\tan^2 x = \sec^2 x - 1 \implies \tan^2(\operatorname{arcsec} x) = x^2 - 1 \implies \tan(\operatorname{arcsec} x) \in \{\pm\sqrt{x^2-1}\}$. Now, $x > 1 \implies \operatorname{arcsec} x \in (0, \pi/2) \implies \tan(\operatorname{arcsec} x) > 0$. And, $x < -1 \implies \operatorname{arcsec} x \in (\pi/2, \pi) \implies \tan(\operatorname{arcsec} x) < 0$. Hence, $\tan(\operatorname{arcsec} x) = (\operatorname{sgn} x)\sqrt{x^2-1}$. Therefore,

$$\operatorname{arcsec}' x = \frac{1}{\sec(\operatorname{arcsec} x)\tan(\operatorname{arcsec} x)} = \frac{1}{x\,(\operatorname{sgn} x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}.$$

8. Let $f(x) = x^{x^x}$ and $g(x) = \log f(x) = x^x \log x$. Example 4.3.13 gives

$$g'(x) = (x^x)' \log x + x^{x-1} = x^x(1 + \log x)\log x + x^{x-1}.$$

Therefore, $f'(x) = f(x)g'(x) = x^{x^x + x - 1}(1 + x(1+\log x)\log x)$.

10. See the solutions of Tasks 4.3.17 and 4.3.18.

4.4 The First Fundamental Theorem

Exercises for §4.4

2. (a) $F'(x) = 3x^2(1+x^6)^{-3}$, (b) $F'(x) = 2x(1+x^4)^{-3} - (1+x^2)^{-3}$.

4. $g'(x) = \sqrt{1+x^3}\sqrt{1 + (\int_0^x \sqrt{1+t^3}\,dt)^3}$.

4.5 Extreme Values and Monotonicity

Task 4.5.9. The condition $f' = f$ gives $f(x) = Ae^x$. Then $1 = f(0) = Ae^0 = A$ gives $f(x) = e^x$.

Task 4.5.10. Let $g = f^2 + (f')^2$. Then $g' = 2ff' + 2f'f'' = 2ff' - 2f'f = 0$ gives $g = C$, a constant. And $g(0) = f(0)^2 + 2f'(0)^2 = 0$ gives $C = 0$.

Task 4.5.11. Apply the previous Task to $f(x) - a\cos x - b\sin x$.

Exercises for §4.5

2.

(a) We have $f'(x) = -(2/3)x^{-1/3}$ for $x \neq 0$, while $f'(0)$ is not defined. So the only critical point is $x = 0$.

(b) The candidates for absolute extremes are $x = 0, \pm 1$. We have $f(0) = 1$, $f(\pm 1) = 0$. So the absolute maximum value is 1 (at 0) and the absolute minimum value is 0 (at ± 1).

4. We have $f(0) = -2 < 0$ and $f(\pi/2) = \pi^3/24 + \pi > 0$. By the intermediate value theorem, there is at least one zero.

Further, $f'(x) = x^2 + 2 + 2\sin x \geq x^2 > 0$ for $x \neq 0$, while $f'(0) = 2 > 0$. Therefore f is strictly increasing and has at most one zero.

6. Proceed by induction. We already know that $f' = 0$ means $f =$ constant. Now suppose $f^{(n+1)} = 0$ with $n > 0$. Then $(f')^{(n)} = 0$ and by the induction hypothesis, $f' = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1}$. Let $g = f - b_0 x - \dfrac{b_1}{2}x^2 - \cdots - \dfrac{b_{n-1}}{n}x^n$. We find that $g' = 0$, hence $g =$ constant and f is a polynomial of degree n or less.

8. We have $g'(x) = -f'(x) = -kg(x)$, so $g(x) = Ae^{-kx}$ and $f(x) = M - Ae^{-kx}$.

10. The given condition gives the equation $f'(x) = f(x)/x$. Hence we have $\left(\dfrac{f(x)}{x}\right)' = \dfrac{xf'(x) - f(x)}{x^2} = 0$ and $\dfrac{f(x)}{x} =$ constant on each interval $(-\infty, 0)$ and $(0, \infty)$. By the differentiability at $x = 0$ the two constants must be the same.

12. By Darboux's theorem f' is either positive everywhere or negative everywhere.

4.6 Derivative Tests and Curve Sketching

Task 4.6.4. If its graph is a line.

Exercises for §4.6

2.

(a) The critical points are $x = 0, \pm\pi/2$. The function is increasing on $[-\pi, -\pi/2]$ and $[0, \pi/2]$, it is decreasing on $[-\pi/2, 0]$ and $[\pi/2, \pi]$.

(b) The only critical point is $x = 0$. It is increasing on $[-3, 0]$ and decreasing on $[0, 3]$.

(c) The critical points are $x = \pm 1$. It is increasing on $[-1, 1]$, decreasing on $[-3, -1]$ and $[1, 3]$.

(d) The critical points are $x = 1/2, 1$. It is increasing on $[-1, 1/2]$ and $[1, 2]$, decreasing on $[1/2, 1]$.

4. In Example 4.6.9 we saw that the function has derivative $1/(1+x^2)$. Hence its difference with $\arctan x$ has zero derivative and is constant on each of these intervals.

6.

(a) If u is positive at a critical point, then u'' is positive at that point and the point must be a local minimum and not a local maximum. Similarly, if u is negative at a critical point, that point cannot be a local minimum.

(b) If u is not always zero, it either has a positive local maximum or a negative local minimum!

8. We already know that $e^x - 1 - x > 0$ for $x > 0$. Hence $(e^x - 1 - x - x^2/2)' = e^x - 1 - x > 0$ and $f(x) = e^x - 1 - x - x^2/2$ is strictly increasing for $x > 0$. Similarly, it is increasing for $x \geq 0$. Therefore, $x > 0$ implies $f(x) > f(x/2) \geq f(0) = 0$.

Thematic Exercises

Convex Functions and Inequalities

A1. Apply the inequality (4.1).

A2. Apply A1 to $a < b < d$ to get $\dfrac{f(b) - f(a)}{b - a} \leq \dfrac{f(d) - f(a)}{d - a}$.

We can similarly obtain $\dfrac{f(d) - f(a)}{d - a} \leq \dfrac{f(d) - f(c)}{d - c}$.

A3.

(a) Let $a \in I$. For $x \neq a$ define $g(x) = \dfrac{f(x) - f(a)}{x - a}$. Then g is an increasing function. Hence, $f'_+(a) = \lim_{x \to a+} g(x) = \inf\{\, g(x) \mid x > a \,\}$

$$f'_-(a) = \lim_{x \to a-} g(x) = \sup\{\, g(x) \mid x < a \,\}$$

By A2, we have that $m \in \{\, g(x) \mid x < a \,\}$ and $n \in \{\, g(x) \mid x > a \,\}$ implies $m \leq n$. This gives the existence of the limits as well as the desired inequality.

(b) Apply A2.

(c) The existence of the one-sided derivatives gives continuity from each side.

A4. Consider $f\colon [0, 1] \to \mathbb{R}$, $f(0) = 1$ and $f(x) = 0$ for $x > 0$.

A5.

(a) Consider a line through $(a, f(a))$, with slope m. It will be a support line if and only if $f'_+(a) \geq m \geq f'_-(a)$.

(b) The support line will be unique if and only if $f'_+(a) = f'_-(a)$.

A6. The function $(1+x)^r$ is convex and differentiable, while $y = 1 + rx$ is its tangent line through $(0, 1)$.

A7. Apply A3(b).

A8. We have seen that f' will be an increasing function. Since it also has the intermediate value property, it will be continuous.

A9.

(a) $(1-t)f(x) + tf(y)$ is the point on the secant line joining $(x, f(x))$ and $(y, f(y))$, corresponding to the input $(1-t)x + ty$.

(b) Suppose the tangent line meets the graph at a point $(b, f(b))$, with $b \neq a$. Then $f((a+b)/2) < (f(a) + f(b))/2$, violating the tangent line being a support line.

A10.

(a) If f has a local minimum at a then $f'_-(a) \leq 0 \leq f'_+(a)$, hence the line $y = f(a)$ is a support line and there is a global minimum at a.

Suppose f has local minimums at a, b, and we have $a < x < b$. Since a local minimum is also a global minimum, we have $f(a) = f(b) = L$. Convexity gives $f(x) \leq L$, hence $f(x) = L$.

(b) Apply (a). (This claim requires I to have the form $[a, b]$.)

(c) If $f'(a) = 0$ then the line $y = f(a)$ is a support line.

A11.

(a) For $n = 2$ this is the characterising property of an interval. For $n > 2$, we proceed by induction based on the following:

$$\sum_{i=1}^{n} w_i x_i = (1 - w_n) \sum_{i=1}^{n-1} \frac{w_i}{1 - w_n} x_i + w_n x_n.$$

(b) Proceed by induction, using the approach in (a).

(c) Apply (b) to $\sum_{i=1}^{n} \alpha_i x_i$ with $\alpha_i = \dfrac{w_i}{\sum_{i=1}^{n} w_i}$.

(d) $-f$ is convex.

A12. The convexity of $f(x) = x^2$, with $w_i = 1/n$ gives $\left(\sum_{i=1}^{n} \frac{x_i}{n}\right)^2 \leq \sum_{i=1}^{n} \frac{x_i^2}{n}$, that is, $RMS \geq AM$.

The convexity of $f(x) = -\log x$, with $w_i = 1/n$ gives $\log(\sum_{i=1}^{n} \frac{x_i}{n}) \geq \sum_{i=1}^{n} \frac{\log(x_i)}{n} = \log(\prod_{i=1}^{n} x_i)^{1/n}$, hence $AM \geq GM$.

Apply $AM \geq GM$ to $x_1^{-1}, \ldots, x_n^{-1}$ to get $GM \geq HM$.

A13.

(a) The convex function $f(x) = -\log x$ gives $-\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \leq -\frac{\log a^p}{p} - \frac{\log b^q}{q}$.

(b) For the special case, apply Young's inequality:

$$\sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} \left(\frac{a_i^p}{p} + \frac{b_i^q}{q}\right) = \frac{1}{p} + \frac{1}{q} = 1.$$

Reduce to the special case by considering $a_i' = a_i/(\sum_{i=1}^n a_i^p)^{1/p}$ and $b_i' = b_i/(\sum_{i=1}^n b_i^q)^{1/q}$.

(c) First,

$$\sum_{i=1}^{n} |a_i+b_i|^p \leq \sum_{i=1}^{n} (|a_i|+|b_i|)|a_i+b_i|^{p-1} = \sum_{i=1}^{n} |a_i||a_i+b_i|^{p-1} + \sum_{i=1}^{n} |b_i||a_i+b_i|^{p-1}.$$

Apply Hölder's inequality to each term on the right:

$$\begin{aligned}\sum_{i=1}^{n} |a_i + b_i|^p &\leq \left(\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}\right)\left(\sum_{i=1}^{n} |a_i + b_i|^{(p-1)q}\right)^{1/q} \\ &= \left(\left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p\right)^{1/p}\right)\left(\sum_{i=1}^{n} |a_i + b_i|^{p}\right)^{1-1/p}\end{aligned}$$

A14.

(a) Let g have minimum value m and maximum value M. Then $m, M \in I$ and $m \leq x_0 \leq M$.

(b) Let $y = F(x_0) + m(x - x_0)$ be a support line through $(x_0, F(x_0))$.

(c) Integrate both sides of the inequality in (b).

A15. Apply Jensen's inequality with $F(x) = -\log x$.

A16. Define $x_0 = \int_a^b g(x)p(x)\,dx$. Now apply the mean value theorem for weighted integration to get $x_0 \in I$.

Let $y = F(x_0) + m(x - x_0)$ be a support line for F through $(x_0, F(x_0))$. Then we have, for every $x \in [a, b]$,

$$F(x_0) + m(g(x) - x_0) \leq F(g(x)).$$

Multiply both sides by $p(x)$ and integrate to get the desired inequality.

A17. Apply A16 with $F(x) = x^2$.

A18. Apply A16 with $I = (0, \infty)$ and $F(x) = 1/x$.

A19. Denote $||f||_p = \left(\int_a^b f(x)^p\,dx\right)^{1/p}$ and $||g||_q = \left(\int_a^b g(x)^q\,dx\right)^{1/q}$. Now follow the approach for A13(b). In the special case when $||f||_p = ||g||_q = 1$, apply Young's inequality:

$$\int_a^b f(x)g(x)\,dx \le \int_a^b \left(\frac{f(x)^p}{p} + \frac{g(x)^q}{q}\right) dx = \frac{1}{p} + \frac{1}{q} = 1.$$

Reduce the general case to this special case by considering $f/||f||_p$ and $g/||g||_q$.

A20. Follow the approach of A13(c).

5 | Techniques of Integration

5.1 The Second Fundamental Theorem

Task 5.1.5. No, we do not know which constant will be in effect for $x < 0$.

Task 5.1.8. We have $(x \arctan x)' = \dfrac{x}{1+x^2} + \arctan x$, hence

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \log \sqrt{1+x^2} + C.$$

Exercises for §5.1

2.

(a) $\dfrac{2}{5}x^{5/2} + \dfrac{10}{3}x^{3/2} - 2x^{1/2} + C$.

(b) $\dfrac{1}{2}\sin 2t + C$.

(c) $\dfrac{3}{2}\log|x-1| - \dfrac{1}{2}\log|x+1| + C$.

4. $f(x) = \begin{cases} \log x + 1 & \text{if } x > 0 \\ \log(-x) + 2 & \text{if } x < 0 \end{cases}$.

6.

(a) $\dfrac{1}{2}\log\dfrac{1+\sqrt{3}/2}{1-\sqrt{3}/2}$.

(b) $1 - \dfrac{\pi}{4}$.

(c) $\dfrac{12}{\log 2}$.

8. The given hint is not needed: $\displaystyle\int_a^{a+2\pi} \cos\theta \, d\theta = \sin(2\pi + a) - \sin a = 0$.

10.

(a) The quadratic $q(x)$ has factors $x - \alpha$ and $x - \beta$. Matching the x^2 coefficient gives $q(x) = (x-\alpha)(x-\beta)$. Further, multiplying through by $(x-\alpha)(x-\beta)$ gives $Ax + B = C(x-\beta) + D(x-\alpha)$, hence $A = C + D$ and $B = -C\beta - D\alpha$. Therefore, $C = (A\alpha + B)/(\alpha - \beta)$ and $D = (A\beta + B)/(\beta - \alpha)$.

(b) Apply (a).

12.

(a) In this case, let $q(x)$ have minimum value m at $x = a$. Then $m = b^2 > 0$ and $q(x) - b^2 = (x-a)^2$.

(b) Apply (a) to get $\dfrac{p(x)}{q(x)} = \dfrac{A(x-a)}{(x-a)^2+b^2} + \dfrac{Aa+B}{(x-a)^2+b^2}$.

5.2 Integration by Substitution

Task 5.2.8. No. $1/x$ is unbounded on this domain, so this integral is not defined.

Exercises for §5.2

2.

(a) Write $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx = \int \Big(1 + \frac{2e^x}{1-e^x}\Big)\,dx = x + 2\int \frac{e^x}{1-e^x}\,dx$, and substitute $y = 1 - e^x$.

(b) Substitute $u = \sin\theta$.

(c) Substitute $u = \cos\theta$.

(d) Substitute $u = \tan\theta$.

4.
$$\begin{aligned}\int \sin^m x\cos^{2k-1} x\,dx &= \int \sin^m x(1-\sin^2 x)^{k-1}\cos x\,dx\\ &= \int u^m(1-u^2)^{k-1}\,du \qquad (u = \sin x)\\ &= \int \Big(\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}u^{m+2i}\Big)\,du\\ &= \sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{u^{m+2i+1}}{m+2i+1}\\ &= \sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}\frac{(\sin x)^{m+2i+1}}{m+2i+1}.\end{aligned}$$

6.

(a) $1/12$. (b) $3\pi/8$. (c) $16/15$. (d) $\pi/32$.

5.3 Integration by Parts

Task 5.3.5.

(a) $$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx = x^2e^x - 2xe^x + 2\int e^x\,dx\\ &= (x^2-2x+2)e^x + C.\end{aligned}$$

(b) $2x\sin x + (2-x^2)\cos x + C$.

Exercises for §5.3

2.

(a) 1.

(b) $(e^2+1)/4$.

(c) $-\log(\sqrt{2}+1)+\dfrac{\sqrt{2}-1}{2^{3/2}}\pi$.

(d) $\dfrac{\pi}{2}-1$.

4.

(a)
$$\begin{aligned}\int \cos^n x\,dx &= \int \cos^{n-1} x \cos x\,dx \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x\,dx \\ &= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

(b) $\dfrac{\sin x\cos^3 x}{4}+\dfrac{3\sin x\cos x}{8}+\dfrac{3x}{8}+C$.

(c) $\dfrac{(2n-1)!}{2^{2n}n(n-1)!^2}\pi$.

(d) $\dfrac{2^{2n}n!^2}{(2n+1)!}$.

6.

(a)
$$\begin{aligned}\int \frac{dx}{(x^2+a^2)^{n-1}} &= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{x^2\,dx}{(x^2+a^2)^n} \\ &= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\Big(\int \frac{dx}{(x^2+a^2)^{n-1}} - \int \frac{a^2\,dx}{(x^2+a^2)^n}\Big)\end{aligned}$$

(b) Similar to (a).

(c)
$$\begin{aligned}\int \sec^n x\,dx &= \int \sec^{n-2} x\,\sec^2 x\,dx \\ &= \sec^{n-2} x\,\tan x - (n-2)\int \sec^{n-2} x\,\tan^2 x\,dx \\ &= \sec^{n-2} x\,\tan x - (n-2)\int \sec^n x\,dx - (n-2)\int \sec^{n-2} x\,dx\end{aligned}$$

8.
$$\begin{aligned}\int_a^b f(x)g(x)\,dx &= \int_a^b f(x)G'(x)\,dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx \\ &= f(b)G(b) - f(a)G(a) - G(c)\int_a^b f'(x)\,dx \\ &= f(b)G(b) - (f(b)-f(a))G(c) \\ &= f(b)(G(b)-G(c)) + f(a)G(c).\end{aligned}$$

5.4 Partial Fractions

Exercises for §5.4

2. We have $p(x) = \sum_{k=0}^{n-1} A_k(x-a)^k = A_0 + A_1(x-a) + \cdots + A_{n-i}(x-a)^{n-1}$. Therefore, $p^{(i)}(x) = i!A_i + \frac{(i+1)!}{1!}A_{i+1}(x-a) + \cdots + \frac{(n-1)!}{(n-1-i)!}A_{n-1}(x-a)^{n-1-i}$.

4.

(a) $\frac{13}{5}\log(x+2) - 2\log x + \frac{2}{5}\log(x-3)$

(b) $\frac{1}{125}\log\left(\frac{x+2}{x-3}\right) - \frac{1}{25}\frac{x+12}{(x+2)(x-3)}$

(c) $\arctan x + \frac{1}{2}\log(x^2+2) - \frac{1}{\sqrt{2}}\arctan(x/\sqrt{2})$

(d) $\frac{\log(x-1)}{25} - \frac{\log(x^2+4)}{50} - \frac{2}{5}\frac{x+1}{x^2+4} + \frac{7}{25}\arctan(x/2)$

(e) $\frac{\log(x-1)}{4} - \frac{\log(x+1)}{4} - \frac{\arctan x}{2}$

(f) $\frac{1}{2^{5/2}}\log\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right) + \frac{1}{2^{3/2}}\arctan(\sqrt{2}x+1) + \frac{1}{2^{3/2}}\arctan(\sqrt{2}x-1)$

6. There is an error in the given substitution: it should be $t = \tan(x/2)$. This gives $x = 2\arctan t$, from which we get:

$$\frac{dx}{dt} = \frac{2}{1+t^2},$$

$$\sin x = \sin(2\arctan t) = 2\sin(\arctan t)\cos(\arctan t) = 2\frac{t}{\sqrt{1+t^2}}\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2},$$

$$\cos x = \cos(2\arctan t) = 1 - 2\sin^2(\arctan t) = 1 - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}.$$

(a) $$\int \frac{dx}{\sin x + \cos x} = \int \frac{2}{1+2t-t^2}\,dt = -\frac{1}{\sqrt{2}}\log\left(\frac{t-\sqrt{2}-1}{t+\sqrt{2}-1}\right)$$
$$= -\frac{1}{\sqrt{2}}\log\left(\frac{\tan(x/2)-\sqrt{2}-1}{\tan(x/2)+\sqrt{2}-1}\right).$$

(b) $x - \frac{1}{\sqrt{2}}\arctan(\sqrt{2}\tan x)$.

5.5 Improper Integrals

Task 5.5.7. The condition $\alpha < 0$ gives $\lim_{x\to 0+} x^\alpha = \infty$. Hence, for $\alpha \neq -1$,

$$\int_0^1 x^\alpha\,dx = \lim_{a\to 0+}\int_a^1 x^\alpha\,dx = \lim_{a\to 0+}\frac{1-a^{\alpha+1}}{\alpha+1} = \begin{cases} 1/(1+\alpha) & \text{if } -1<\alpha<0 \\ \infty & \text{if } \alpha<-1 \end{cases}$$

The $\alpha = 1$ case gives $\int_0^1 x^\alpha\,dx = -\lim_{a\to 0+}\log a = \infty$.

Task 5.5.10. $\Gamma(1/2) = \int_0^\infty e^{-t}t^{-1/2}\,dt$. Substitute $t = x^2$ with $x \in [0,\infty)$ to get

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2}\,dx.$$

Exercises for §5.5

2.

(a) $\pi/2$.

(b) 0. (Note: It is not enough to invoke the integrand being odd, convergence has to be established.)

(c) 1.

4.

(a) $\displaystyle\int_0^{\pi/2} \sec x\,dx = \lim_{x\to\pi/2-} \log(\sec x + \tan x) = \infty.$

(b) $\displaystyle\int_2^\infty \frac{1}{x\log x}\,dx = \lim_{x\to\infty} \log(\log x) - \log(\log 2) = \infty.$

(c) $\displaystyle\int_1^2 \frac{1}{x\log x}\,dx = \log(\log 2) - \lim_{x\to 1+} \log(\log x) = \infty.$

(d) $\displaystyle\int_0^\infty \frac{1}{\sqrt{9+x^2}}\,dx = \lim_{x\to\infty} \sinh^{-1}(x/3) = \infty.$

6. There is M such that $x \ge M$ implies $f(x)/g(x) \le L+1$, hence $0 \le f(x) \le (L+1)g(x)$. By the comparison theorem, convergence of $\int_M^\infty g(x)\,dx$ implies convergence of $\int_M^\infty f(x)\,dx$.

For the converse, use the existence of N such that $x \ge N$ implies $f(x)/g(x) \ge L/2$.

8. Statement: Let f, g be continuous on $(a,b]$ and have vertical asymptotes at a, with $0 \le f(x) \le g(x)$ for every $x \in (a,b]$. Let

$$\lim_{x\to a+} \frac{f(x)}{g(x)} = L \in \mathbb{R},\ L \ne 0.$$

Then $\int_a^b f(x)\,dx$ converges if and only if $\int_a^b g(x)\,dx$ converges.

Proof: Similar to Exercise 6.

10.

(a) No. Suppose $L > 0$. Then there is M such that $x \ge M$ implies $f(x) \ge L/2 > 0$. By the comparison theorem, $\int_M^\infty f(x)\,dx$ diverges.

(b) No. Here are two examples with $a = 0$:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{N} \\ 0 & \text{else} \end{cases},\quad g(x) = \begin{cases} 1 & \text{if } x \in [n - \frac{1}{2^n}, n + \frac{1}{2^n}] \text{ with } n \in \mathbb{N} \\ 0 & \text{else} \end{cases}$$

5.6 Ordinary Differential Equations

Task 5.6.3. $\displaystyle\int \frac{dy}{M-ky} = \int 1\,dt \implies -\frac{1}{k}\log(M-ky) = t+C \implies M-ky = Ae^{-kt}$.

Task 5.6.6. $\displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \implies \log|y| = \log|x| + C \implies y = Ax$, with $A = y(1)$.

Task 5.6.8.

$$\int (5y^4+1)\,dy = \int (3x^2+1)\,dx \implies y^5+y = x^3+x+C.$$

Task 5.6.15. In Example 5.6.13, we found the general solution: $y = A\dfrac{e^x}{x} + \dfrac{e^{2x}}{x}$. Now $y(1)=0$ gives $0 = Ae+e^2$, or $A=-e$.

Task 5.6.19. Statement: Consider an initial value problem $y' = f(y)$, $y(0) = y_0$, where $f : (a,b) \to \mathbb{R}$ is negative and continuous, and $y_0 \in (a,b)$. This initial value problem has a unique solution $y\colon (\alpha,\beta) \to (a,b)$ which is a strictly decreasing bijection.

Task 5.6.20. If $f(y_0) = 0$ then $y = y_0$ is a solution. If $f(y_0) \neq 0$ then f will not change sign in some open interval around y_0 and then Theorem 5.6.18 (and Task 5.6.19) give a solution.

Example 5.6.7 shows that a solution may not be unique.

Exercises for §5.6

2. The substitution $u = y/x$ gives $2xuu' = -u^2-1$. Separation of variables gives $u^2+1 = C/x$, hence $y^2 = Cx - x^2$.

4.

(a) $y = (x+C)e^x$.

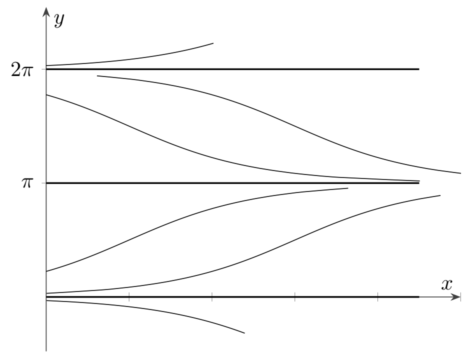
(b) $y = \dfrac{x^2}{5} + \dfrac{C}{x^3}$.

(c) $y = \dfrac{C-\cos x}{x^3}$.

(d) $y = e^{-\sin x}\Big(\displaystyle\int e^{2x+\sin x}\,dx + C\Big)$.

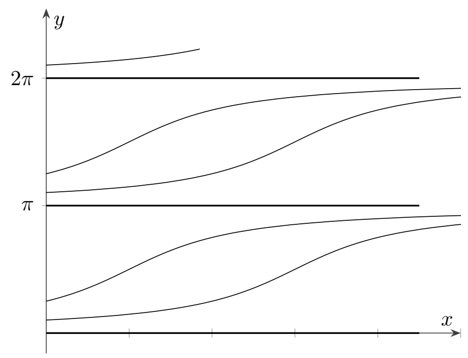
6.

(a) The equilibrium solutions are $y = n\pi$ with $n \in \mathbb{Z}$.



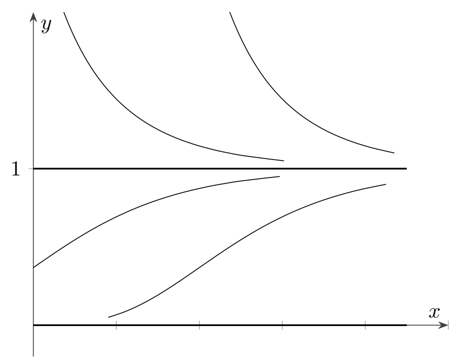
The equilibrium solutions are stable when n is odd and unstable when n is even.

(b) The equilibrium solutions are $y = n\pi$ with $n \in \mathbb{Z}$.



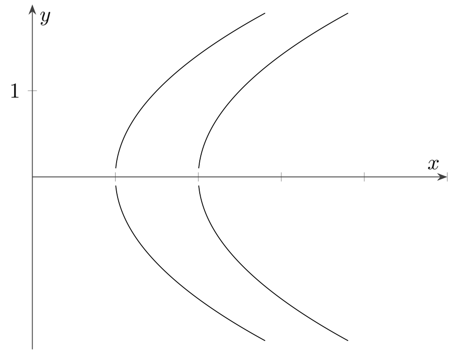
All the equilibrium solutions are semistable.

(c) The equilibrium solutions are $y = 0$ and $y = 1$.



The equilibrium solution $y = 0$ is unstable, while $y = 1$ is stable.

(d) No equilibrium solutions.



8.

(a) Let (x_0, y_0) be a point on a curve belonging to the orthogonal family. The parabola passing through it is $y = cx^2$ with $c = y_0/x_0^2$. The tangent to that parabola at (x_0, y_0) has slope $m = 2cx_0 = 2y_0/x_0$. Hence, the slope of the tangent of the curve from the orthogonal family is $-1/m = -x_0/(2y_0)$. So the orthogonal curve satisfies $y' = -x/(2y)$.

(b) $2yy' = -x \implies \int 2y\,dy = -\int x\,dx \implies 2y^2 + x^2 = C$. Clearly, $C \geq 0$ and we can write $C = k^2$.

10.

(a) From Theorem 5.6.18 we know that $y(x)$ is strictly increasing. If $y(x)$ is bounded above, then the monotone convergence theorem gives $\lim_{x\to\infty} y(x) = M$ for some $M > a$. It follows that $\lim_{x\to\infty} y'(x) = 0$ and so $0 = f(M)$.

(b) Similar to (a).

Thematic Exercises

Second Order Linear ODE

A1. We have $(f + f')' = f + f'$ and $(f - f')' = -(f - f')$. Hence, $f(x) + f'(x) = Ce^x$ and $f(x) - f'(x) = De^{-x}$. Therefore, $f(x) = (C/2)e^x + (D/2)e^{-x}$.

A2. Apply A1 to get $f(x) = Ae^x + Be^{-x}$. Note that $A + B = f(0)$ and $A - B = f'(0)$.

A3. The initial conditions should have been $f(0) = B$ and $f'(0) = A$.

(a) $k = 0 \implies f'' = 0 \implies f'(x) = A \implies f(x) = Ax + B$.

(b) Let $g(x) = f(x/w)$. Then $g''(x) = f''(x/w)/w^2 = -g(x)$, hence $g(x) = \alpha \sin x + \beta \cos x$, and $f(x) = g(wx) = \alpha \sin wx + \beta \cos wx$. Now $f(0) = B$ and $f'(0) = A$ give $\beta = B$ and $\alpha = A$.

(c) Similar to (b).

A4.

(a) Suppose $v(x) = e^{ax/2} f(x)$. Then

$$v'(x) = e^{ax/2}(f'(x) + (a/2)f(x))$$

$$v''(x) = e^{ax/2}(f''(x) + af'(x) + (a^2/4)f(x)) = (a^2/4 - b)v(x)$$

The converse has a similar justification.

(b) If $d = 0$ then $v(x) = e^{ax/2}f(x)$ is a solution of $v'' = 0$ and $\lambda = -a/2$. Hence $e^{-\lambda x}f(x) = A + Bx$.

(c) We have $\lambda_1 = (-a + \sqrt{d})/2$, $\lambda_2 = (-a - \sqrt{d})/2$ and $v'' = (d/4)v$. Therefore, by (c) of A3,

$$e^{ax/2}f(x) = Ae^{(\sqrt{d}/2)x} + Be^{-(\sqrt{d}/2)x}, \text{ so } f(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

(d) We have $r = -a/2$ and $w = \sqrt{-d}/2$. By (b) of A3, we have

$$e^{ax/2}f(x) = A\sin\left(\frac{\sqrt{-d}}{2}\right)x + B\cos\left(\frac{\sqrt{-d}}{2}\right)x,$$

$$\text{so } f(x) = e^{rx}\big(A\sin wx + B\cos wx\big).$$

A5. Consider the cases when the roots are real and distinct, real and equal, complex.

A6. Let f be any solution of $f'' + af' + bf = g$. Verify that $f - f_p$ is a solution of $f'' + af' + bf = 0$.

A7.

(a) Substitution of $f_p(x)$ in the given equation leads to:

$$(-\alpha - 3\beta + 2\alpha)\cos x + (-\beta + 3\alpha + 2\beta)\sin x = \cos x$$

This gives $\alpha - 3\beta = 1$ and $\beta + 3\alpha = 0$.

(b) Consider $f''(x) - 3f'(x) + 2f(x) = 0$. The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$, which has roots $\lambda = 1, 2$. Hence its general solution is $f_h(x) = Ae^x + Be^{2x}$.

6 | Mean Value Theorems and Applications

6.1 Mean Value Theorems

Task 6.1.5. Apply Rolle's theorem to f on $[a,b]$ and $[b,c]$ to get $a' \in (a,b)$ and $b' \in (b,c)$ such that $f'(a') = f'(b') = 0$. Now apply Rolle's theorem to f' on $[a', b']$.

Exercises for §6.1

2. Let $f(t) = \log y(t)$. Then $f'(t) \le K$, hence $f(t) \le f(0) + Kt$ for $t \ge 0$.

4. Just compute $f'(c)$ for the given f and c.

6. Apply Rolle's theorem to $f(x) - g(x)$.

8. We need to find a c such that $f'(c) = f(c)/c$. Apply Rolle's theorem to $g(x) = f(x)/x$.

10. We know that f is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Apply the mean value theorem to f on $[0,c]$ to get $t_1 \in (0,c)$ such that $f'(t_1) = \frac{f(c)-f(0)}{c-0} = \sin(1/c)$. Now pick $c_1 \in (0,t_1)$ such that $\sin(1/c_1) = \sin(1/c)$. Apply the mean value theorem to f on $[0,c_1]$ to get $t_2 \in (0,c_1)$ such that $f'(t_2) = \frac{f(c_1)-f(0)}{c_1-0} = \sin(1/c_1) = \sin(1/c)$. Repeat.

12.

(a) The required M exists and is unique because $x \ne x_1, x_2$.

(b) $L(x_1) = y_1 = f(x_1)$ implies $G(x_1) = 0$, while $L(x_2) = y_2 = f(x_2)$ implies $G(x_2) = 0$. The choice of M gives $G(x) = 0$. Apply Rolle's theorem to G to get $c_1 \in (x_1, x)$ and $c_2 \in (x, x_2)$ such that $G'(c_1) = G'(c_2) = 0$. Then apply Rolle's theorem to G' to get $\xi \in (c_1, c_2) \subset (x_1, x_2)$ such that $G''(\xi) = 0$.

(c) We have $G''(t) = f''(t) - 2M$. Hence, $0 = G''(\xi) = f''(\xi) - 2M$, and $M = f''(\xi)/2$.

6.2 L'Hôpital's Rule

Task 6.2.7. $\lim\limits_{x\to 0} \dfrac{\sin x - x + x^3/3!}{x^5} = \lim\limits_{x\to 0} \dfrac{\cos x - 1 + x^2/2}{5x^4} = -\lim\limits_{x\to 0} \dfrac{\sin x - x}{5 \cdot 4x^3} =$

$-\dfrac{1}{5!}$

Exercises for §6.2

2. A mechanical application of L'Hôpital's rule fails to simplify these expressions. For example, the expression in (a) leads to $\lim_{x\to 0+} \dfrac{\sqrt{x}\cos x}{\sqrt{\sin x}} = \lim_{x\to 0+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$. Instead, (a) can be resolved by applying the sandwich theorem, while (b) is resolved by substituting $t = 1/x$. This converts the problem to $\lim_{t\to\infty} \dfrac{e^t}{t}$.

4.

(a) Apply L'Hôpital's rule:

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{(x-a)^2} = \lim_{x\to a} \frac{f'(x) - f'(a)}{2(x-a)} = \frac{f''(a)}{2}.$$

(b) Apply L'Hôpital's rule n times.

6.

(a) We begin with $1 + x + x^2 + \cdots + x^n = \dfrac{1 - x^{n+1}}{1-x}$. Differentiating both sides gives the required identity.

(b) Computing the limit gives:

$$\begin{aligned} 1 + 2 + \cdots + n &= \lim_{x\to 1} \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} = \frac{n(n+1)}{2} \lim_{x\to 1} \frac{(x^{n-1} - x^n)}{1-x} \\ &= \frac{n(n+1)}{2} \lim_{x\to 1} x^{n-1} = \frac{n(n+1)}{2}. \end{aligned}$$

8. *Compute the following limits.*

(a) $\lim_{x\to\infty} \dfrac{x}{\log(1+e^x)}$.

(b) $\lim_{x\to -\infty} xe^x$.

(c) $\lim_{t\to 0+} \sin t \log t$.

(d) $\lim_{x\to 0+} (\cos x)^{1/x}$.

(e) $\lim_{x\to 0+} (\log(1/x))^x$.

(f) $\lim_{x\to 0+} x^{1/\log x}$.

(a) 1 (b) 0 (c) 0 (d) 1 (e) 1 (f) e

6.3 Taylor Polynomials

Task 6.3.1. *If T_n is the n^{th} Taylor polynomial of f centered at a, show that*

$$T_n^{(k)}(a) = f^{(k)}(a) \text{ for } k = 0, 1, \ldots, n.$$

Consider each term of T_n. On differentiating k times, each term whose degree is less than k will become zero. Each term with degree more than k will have a surviving $x - a$ factor, hence will become zero for $x = a$. So the only contribution will be from the $f^{(k)}(a)(x-a)^k/k!$ term, and this will give $f^{(k)}(a)$.

Task 6.3.6. *Check that the Taylor polynomials of* $\log(1 + x)$ *centred at* $a = 0$ *are* $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} \cdots + (-1)^{n+1}\dfrac{x^n}{n}$.

Let $f(x) = \log(1 + x)$. Then $f^{(k)}(x) = \dfrac{(-1)^{k-1}(k-1)!}{(1+x)^k}$ for $k \geq 1$. Hence, the k-th degree term of the Taylor polynomial is $\dfrac{f^{(k)}(0)}{k!}x^k = \dfrac{(-1)^{k-1}}{k}x^k$ if $k \geq 1$ and 0 if $k = 0$.

Exercises for §6.3

2. There is an error in the problem statement. The equality to be proved is $T_{\int f,n}(x) = \int_a^x T_{f,n-1}(t)\,dt$.

$$\begin{aligned}\int_a^x T_{f,n-1}(t)\,dt &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} \int_a^x (t-a)^k\,dt = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{(k+1)!}(x-a)^{k+1} \\ &= \sum_{k=1}^{n} \frac{f^{(k-1)}(a)}{k!}(x-a)^k = \sum_{k=1}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k\end{aligned}$$

4.

(a) $\dfrac{1}{2} + \dfrac{x+1}{2^2} + \cdots + \dfrac{(x+1)^n}{2^{n+1}}$.

(b) $e + e(x-1) + \dfrac{e}{2!}(x-1)^2 + \cdots + \dfrac{e}{n!}(x-1)^n$.

6. From $f'' = f$, we get $f^{(k)}(0) = f^{(k+2)}(0)$ for $k \geq 0$. Hence $f^{(k)}(0) = 0$ when k is even, and $f^{(k)}(0) = 1$ when k is odd. So the Maclaurin polynomials are $x + \dfrac{x^3}{3!} + \cdots + \dfrac{x^{2n+1}}{(2n+1)!}$.

8. Suppose $f'(a) > 2$. Then $f(a+2) - f(a) = 2f'(a) + 2f''(c)$ for some $c \in (a, a+2)$. Hence, $f(a+2) - f(a) > 2 \cdot 2 - 2 \cdot 1 = 2$. On the other hand, $|f(x)| \leq 1$ implies $|f(a+2) - f(a)| \leq 2$.

Similarly, $f'(a) < -2$ also leads to a contradiction.

10. We have $(p-q)^{(n)}(a) = 0$ for every $n \geq 0$. If $p - q$ is not zero, it has a degree n, and then $(p-q)^{(n)}(a) \neq 0$.

12.

(a) R can be defined in this manner because $x \neq a$.

(b) No justification needed.

(c) The definition of R gives $g(x) = 0$. The other terms are zero because $f^{(k)}(a) = T_n^{(k)}(a)$ for $k = 0, \ldots, n$.

(d) Assume $x > a$. Apply Rolle's theorem to g on $[a, x]$ to get $c_1 \in (a, x)$ such that $g'(c_1) = 0$. Then apply Rolle's theorem to g' on $[a, c_1]$ to get $c_2 \in (a, c_1)$ such that $g''(c_2) = 0$. Continue in this fashion.

(e) We have $g^{(n+1)}(t) = f^{(n+1)}(t) - R(n+1)!$. Hence $0 = f^{(n+1)}(\xi) - R(n+1)!$.

14. We get the estimate as follows:

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \implies \frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!}$$
$$\implies \int_0^1 \frac{\sin x}{x}\,dx \approx 1 - \frac{1}{18} + \frac{1}{600} = \frac{1703}{1800} = 0.94611\ldots$$

The remainder theorem gives $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{\cos(c(x))}{7!}x^7$. Hence, $\dfrac{\sin x}{x} = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{\cos c(x)}{7!}x^6$. Therefore,

$$\Big|\int_0^1 \frac{\sin x}{x}\,dx - \frac{1703}{1800}\Big| \le \frac{1}{7!}\int_0^1 x^6\,dx = \frac{1}{7!\,7} = 0.000028.$$

6.4 Riemann Sums and Mensuration

Exercises for §6.4

2. Let $f\colon [0,1] \to \mathbb{R}$ be the Dirichlet function. For each partition of $[0,1]$ choose a tag whose members are rational. Then $I = 1$ satisfies the hypotheses of Theorem 6.4.2, but the conclusion is false.

4.

(a) $\displaystyle\int_0^{\pi/4} \sqrt{1+\tan^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \log(\sqrt{2}+1)$.

(b) The function should have been $y = (x-1)^{3/2}$. Then:
$\displaystyle\int_1^4 \sqrt{1+\tfrac{9}{4}(x-1)}\,dx = \frac{31^{3/2}-8}{27}$.

(c) The arc length is $\displaystyle\int_0^1 \sqrt{1+9x^4}\,dx$. This integral cannot be evaluated by elementary means.

6. The problem statement should have $S(f,P_n)$ instead of $L(f,P_n)$. The inequalities also need slight corrections, as shown below.

We begin by observing that

$$S(f,P) = \sum_{i=1}^{n} \Big((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\Big)^{1/2} > \sum_{i=1}^{n} |y_i - y_{i-1}|.$$

(a) We have $f(\frac{1}{2k}) = \frac{(-1)^k}{2k}$ and $f(\frac{1}{2k+1}) = 0$. Hence,

$$S(f,P_n) > \sum_{i=1}^{2n-1} |y_i - y_{i-1}| = |y_0| + 2|y_2| + \cdots + 2|y_{2n-2}|$$
$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

(b) $S(f, P_n) > \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} > \displaystyle\int_2^{n+1} \frac{dx}{x} = \log(n+1) - \log 2.$

8. Let $R > r$. Consider the region bounded by the graphs of $y_+ = \sqrt{r^2 - (x-R)^2}$ and $y_- = -\sqrt{r^2 - (x-R)^2}$, with $|x - R| \le r$. A torus is obtained by rotating this around the y-axis. The shell method gives the volume as follows:

$$\begin{aligned}\int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2}\, dx &= 4\pi \int_{-r}^{r} (R+t)\sqrt{r^2 - t^2}\, dt \\ &= 4\pi R \int_{-r}^{r} \sqrt{r^2 - t^2}\, dt = 2\pi^2 R r^2.\end{aligned}$$

10. Apply the shell method: $\displaystyle\int_r^R 2\pi x \cdot 2\sqrt{R^2 - x^2}\, dx = \frac{4}{3}\pi(R^2 - r^2)^{3/2}.$

12.

(a) Surface area from curved portions (Note that $g' = -f'$ and $f + g = 2\ell$):

$$2\pi\Big(\int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx + \int_a^b g(x)\sqrt{1 + g'(x)^2}\, dx\Big) = 4\pi\ell \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

Surface area from flat caps:

$$\pi\Big(f(a)^2 - g(a)^2 + f(b)^2 - g(b)^2\Big) = 2\pi\ell((f(a) - g(a)) + (f(b) - g(b))).$$

Total surface area:

$$2\pi\ell\Big(2\int_a^b \sqrt{1 + f'(x)^2}\, dx + (f(a) - g(a)) + (f(b) - g(b))\Big) = 2\pi\ell P.$$

(b) Apply the discs method:

$$\pi \int_a^b (f(x)^2 - g(x)^2)\, dx = 2\pi\ell \int_a^b (f(x) - g(x))\, dx = 2\pi\ell A.$$

6.5 Numerical Integration

Task 6.5.5. (The last two requirements need to be corrected to $q'(a) = y'_a$ and $q'(b) = y'_b$.)

By shifting the domain we can assume $b > a = 0$. Let $q(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$. Then the supplied values of $q(0)$ and $q'(0)$ give us $\delta = y_0$ and $\gamma = y'_0$. The values of $q(b)$ and $q'(b)$ now give the following pair of linear equations for α and β:

$$\begin{aligned} b^3\alpha + b^2\beta &= y_b - by'_0 - y_0 \\ 3b^2\alpha + 2b\beta &= y'_b - y'_0 \end{aligned}$$

The coefficient matrix has determinant $-b^4 \neq 0$, so there is a unique solution.

Exercises for §6.5

2. In this example, Simpson's rule fails to outperform the Midpoint rule. The reason is that, near 1, the fourth derivative takes much larger values than the second derivative.

4. Let $x_0 < x_1 < x_2 < \cdots < x_{2n}$ be a partition of $[a, b]$ into equal subintervals. We compute the midpoint and trapeziodal rules using the partition $x_0 < x_2 < x_4 < \cdots < x_{2n}$. Let $\triangle x = (b-a)/2n$. Then,

$$\begin{aligned}\frac{2}{3}M_a^b(f) + \frac{1}{3}T_a^b(f) &= \frac{2}{3}\sum_{i=1}^{n} y_{2i-1} 2\triangle x + \frac{1}{3}\Big(\frac{y_0}{2} + \sum_{i=1}^{n-1} y_{2i} + \frac{y_{2n}}{2}\Big) 2\triangle x \\ &= \Big(y_0 + 2\sum_{i=1}^{n-1} y_{2i} + 4\sum_{i=1}^{n} y_{2i-1} + y_{2n}\Big)\frac{\triangle x}{3} = S_a^b(f).\end{aligned}$$

6. We follow the pattern for the $n = 2$ case. Let $x_1 = -h\sqrt{3/5}$, $x_2 = 0$, $x_3 = h\sqrt{3/5}$. Then, let $q(x)$ be a polynomial of degree at most 5 such that $q(x_i) = f(x_i)$ and $q'(x_i) = f'(x_i)$ for $i = 1, 2, 3$. Fix a value of x. Define

$$g(t) = f(t) - q(t) - M(t - x_1)^2(t - x_2)^2(t - x_3)^2,$$

where M is chosen so that $g(x) = 0$. Then $g(x_i) = g'(x_i) = 0$ for $i = 1, 2, 3$. The four zeroes of g give three zeroes of g', and these are distinct from the x_i. So we get 6 zeroes of g'. Repeated application of Rolle's theorem gives a zero ξ_x of $g^{(6)}$. This leads to $M = f^{(6)}(\xi_x)/6!$. Hence,

$$f(x) - q(x) = \frac{f^{(6)}(\xi_x)}{6!}(x^2 - 3h^2/5)^2 x^2.$$

This leads to:

$$\begin{aligned}\int_{-h}^{h} f(x)\,dx - G_3(f) &= \int_{-h}^{h} f(x)\,dx - G_3(q) = \int_{-h}^{h} (f(x) - q(x))\,dx \\ &= \frac{f^{(6)}(\xi)}{6!}\int_{-h}^{h} (x^2 - 3h^2/5)^2 x^2\,dx \\ &= \frac{f^{(6)}(\xi)}{6!} h^7 \int_{-1}^{1} (x^2 - 3/5)^2 x^2\,dx = \frac{8}{7!\,25} f^{(6)}(\xi) h^7.\end{aligned}$$

Thematic Exercises

Curve Fitting: Error Analysis

A1. Mimic the proof of the remainder theorem. Fix $x \neq a$ and define R by $f(x) - p(x) = R(x - x_1)\cdots(x - x_n)$. Then define $g(t) = f(t) - p(t) - R(t - x_1)\cdots(t - x_n)$.

Observe that $g(x) = g(x_1) = \cdots = g(x_n) = 0$. By repeated application of Rolle's theorem, obtain $\xi \in I$ such that $g^{(n)}(\xi) = 0$.

A2. $(L_j^2)'(x_k) = 2L_j(x_k)L_j'(x_k) = 0 = L_j^2(x_k)$ if $j \neq k$

$(L_j^2)'(x_j) = 2L_j(x_j)L_j'(x_j) = 2L_j'(x_j)$

A3. Let $H_j(x) = (a+bx)L_j^2(x)$. Then $H_j(x_k) = 0$ if $k \neq j$. And $H_j'(x) = bL_j^2(x) + 2(a+bx)L_j(x)L_j'(x)$ gives $H_j'(x_k) = 0$ for $k \neq j$. Now,

$$H_j(x_j) = 1 \implies a + bx_j = 1$$
$$H_j'(x_j) = 0 \implies b + 2L_j'(x_j) = 0$$

Next, let $K_j(x) = (a+bx)L_j^2(x)$. Again, we have $K_j(x_k) = K_j'(x_k) = 0$ if $k \neq j$.

$$K_j(x_j) = 0 \implies a + bx_j = 0$$
$$K_j'(x_j) = 1 \implies b = 1$$

A4. It is easy to see that $H(x)$ has the required properties. If H_1 and H_2 both satisfy them, then $(H_1 - H_2)(x_j) = (H_1 - H_2)'(x_j) = 0$ for every j. Thus $H_1 - H_2$ has a root of multiplicity 2 at each x_j, hence must have degree at least $2n$ if it is non-zero.

A5. Mimic the solution to A1. Fix $x \in [a,b]$ with $x \neq x_0, \ldots, x_n$. Define R by $f(x) - H(x) = R(x-x_0)^2 \cdots (x-x_n)^2$. Define

$$g(t) = f(t) - H(t) - R(t-x_0)^2 \cdots (t-x_n)^2.$$

Observe that $g(x) = g(x_0) = \cdots = g(x_n) = 0$ and $g'(x_0) = \cdots = g'(x_n) = 0$. Rolle's theorem gives $n+1$ new zeroes of g', for a total of $2n+2$. Then it gives $2n+1$ zeroes of g'' and so on, until we find a zero ξ_x of $g^{(2n+2)}$. Then:

$$0 = g^{(2n+2)}(\xi_x) = f^{(2n+2)}(\xi_x) - (2n+2)!R.$$

Riemann Integral

B1.

(a) The definition of Riemann integral gives $\delta > 0$ such that $\triangle_P < \delta$ implies $|\mathcal{R}_a^b f(x)\,dx - R(f,P^*)| < \varepsilon/2$. Now choose n such $\frac{b-a}{n} < \delta$.

(b) Let P be the partition $x_0 < \cdots < x_n$ and $m_i = \inf\{\, f(x) \mid x \in [x_{i-1}, x_i] \,\}$. Choose $x_i^* \in [x_{i-1}, x_i]$ such that $f(x_i^*) - m_i < \frac{\varepsilon}{2(b-a)}$. Then,

$$R(f,P^*) - L(f,P) = \sum_{i=1}^n (f(x_i^*) - m_i)(x_i - x_{i-1}) < \frac{\varepsilon}{2(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) = \frac{\varepsilon}{2}.$$

(c) By combining (a) and (b), we obtain an n such that

$$U(f,P) - L(f,P) < R(f,P^{**}) - R(f,P^*) + \varepsilon < 2\varepsilon.$$

B2. $$\underbrace{\sum_{i=1}^n m_i(x_i - x_{i-1})}_{L(f,P)} \leq \underbrace{\sum_{i=1}^n f(x_i^*)(x_i - x_{i-1})}_{R(f,P^*)} \leq \underbrace{\sum_{i=1}^n M_i(x_i - x_{i-1})}_{U(f,P)}.$$

B3.

(a) There are partitions Q_1, Q_2 such that $U(f, Q_1) - \int_a^b f(x)\,dx < \varepsilon/2$ and $\int_a^b f(x)\,dx - L(f, Q_2) < \varepsilon/2$. Now choose $Q = Q_1 \cup Q_2$.

(b) We shall prove that $U(f, P) \leq U(f, Q) + \frac{\varepsilon}{2}$. Let $P = \{x'_0, \ldots, x'_n\}$, $M'_i = \sup\{\, f(x) \mid x \in [x'_{i-1}, x'_i]\,\}$ and $M_i = \sup\{\, f(x) \mid x \in [x_{i-1}, x_i]\,\}$. We have:

$$M'_i \leq \begin{cases} M_j & \text{if } [x'_{i-1}, x_i] \subseteq [x_{j-1}, x_j] \text{ for some } j \\ M & \text{else} \end{cases}$$

There are at most m cases of the second type. Hence,

$$U(f, P) = \sum_{i=1}^{n} M'_i(x'_i - x'_{i-1}) \leq \sum_{j=1}^{m} M_j(x_j - x_{j-1}) + mM\frac{\varepsilon}{2Mm} = U(f, Q) + \frac{\varepsilon}{2}.$$

We can similarly prove that $L(f, Q) - \frac{\varepsilon}{2} \leq L(f, P)$.

(c) On combining B2 with B3(a) and (b), we see that $\triangle_P < \delta$ gives

$$L(f, Q) - \frac{\varepsilon}{2} \leq R(f, P^*) \leq U(f, Q) + \frac{\varepsilon}{2}.$$

Hence $|R(f, P^*) - \int_a^b f(x)\,dx| < \varepsilon$. So $\int_a^b f(x)\,dx$ fulfills the requirements to be the Riemann integral.

7 | Sequences and Series

7.1 Limit of a Sequence

Task 7.1.4. Suppose both L and L' satisfy the requirements for $\lim a_n$. Suppose $\varepsilon = |L - L'| \neq 0$. Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L|, |a_n - L'| < \varepsilon/2$. Hence, $n \geq N$ implies $|L - L'| < \varepsilon$, a contradiction.

Task 7.1.5. Given any $\varepsilon > 0$, Let $N = 1$. Then $n \geq N$ gives $|a_n - c| = |c - c| = 0 < \varepsilon$.

Task 7.1.7. Suppose $a_n \to L$. Consider $\varepsilon = 1$. Given any $N \in \mathbb{N}$, let $n = \max\{N, [L] + 2\}$. Then $n \geq N$, but $|a_n - L| > 1 = \varepsilon$.

Task 7.1.8. If N works for a_n and ε, it also works for b_n and ε.

If N works for b_n and ε, then $N + k$ works for a_n and ε.

Task 7.1.9. Suppose $\lim a_n = L > M$. Let $\varepsilon = L - M$. There is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Then $a_n > L - \varepsilon = M$, a contradiction.

Task 7.1.12. Note that $||a_n| - 0| = |a_n - 0|$. Hence, if N works for $|a_n|$ and ε, it also works for a_n and ε.

Task 7.1.14.

(a) $\displaystyle\lim_{n\to\infty} \frac{5n^2 - 1}{n^2 + 3n - 1000} = \lim_{n\to\infty} \frac{5 - 1/n^2}{1 + 3/n - 1000/n^2} = \frac{5 - 0}{1 + 0 - 0} = 5.$

(b) Apply the sandwich theorem: $\displaystyle -\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ gives $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = 0.$

Task 7.1.16.

(a) Use $N = M$.

(b) If N works for b_n and M, it also works for a_n and M.

(c) N works for a_n and $\varepsilon = 1/M$ if and only if it works for $|1/a_n|$ and M.

Task 7.1.17.

a_n	Bounded Above	Bounded Below	Bounded	Unbounded
n	✖	✔	✖	✔
$-n$	✔	✖	✖	✔
$(-1)^n$	✔	✔	✔	✖
$(-1)^n n$	✖	✖	✖	✔
$1/n$	✔	✔	✔	✖

Task 7.1.19.

a_n	Increasing	Decreasing	Monotone
n	✔	✖	✔
$-n$	✖	✔	✔
$(-1)^n$	✖	✖	✖
1	✔	✔	✔
$1/n$	✖	✔	✔

Task 7.1.27. Suppose $\lim a_n = L$. Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon$. Now, $n \geq N \implies 2n, 2n+1 > N \implies |a_{2n} - L|, |a_{2n+1} - L| < \varepsilon$. Hence $\lim a_{2n+1} = \lim a_{2n} = L$.

Next, suppose $\lim a_{2n+1} = \lim a_{2n} = L$. Let $\varepsilon > 0$. There are $N_o, N_e \in \mathbb{N}$ such that:

$$\begin{aligned} n \geq N_o &\implies |a_{2n+1} - L| < \varepsilon \\ n \geq N_e &\implies |a_{2n} - L| < \varepsilon \end{aligned}$$

Let $N = \max\{2N_o + 1, 2N_e\}$.

Task 7.1.28. Apply the previous Task:

$$\begin{aligned} \lim a_{2n} &= \lim \frac{2n}{2n+1} = 1 \\ \lim a_{2n+1} &= -\lim \frac{2n+1}{2n+2} = -1 \end{aligned}$$

Hence, $\lim \dfrac{(-1)^n n}{n+1}$ does not exist.

Exercises for §7.1

2.

(a) 0 (b) ∞ (c) 0 (d) ∞

4. Choose N such that $n \geq N \implies |a_n| < \varepsilon^2$.

6. Let $a_n/b_n \to L$. Then $a_n = (a_n/b_n)b_n \to L \cdot 0 = 0$.

8. Fix an R such that $L < R < 1$. There exists N such that $n \geq N \implies |a_n|^{1/n} < R \implies |a_n| < R^n$. By the sandwich theorem, $|a_n| \to 0$.

10. Apply the root test to a_n^n. Alternately, we have $|a_n| < 1$ for large n, hence $|a_n|^n < |a_n|$ for large n. Now apply the sandwich theorem.

7.2 Sequences and Functions

Task 7.2.3. Follows from the continuity of the exponential function.

Task 7.2.6. $\lim \log(1+\frac{2}{n})^n = \lim n\big(\log(\frac{1}{2}+\frac{1}{n}) - \log(\frac{1}{2})\big) = \log'(\frac{1}{2}) = 2$.

$\lim \log(1-\frac{1}{n})^n = \lim n\big(\log(-(-1+\frac{1}{n})) - \log(-(-1))\big) = \frac{d}{dx}\Big|_{x=-1} \log(-x) = -1$.

Task 7.2.7. False. Let $f(0) = 1$ and $f(x) = 0$ for $x \neq 0$. Let $a_n = 0$ for every n.

Task 7.2.8. Consider $a_n = \frac{1}{n\pi}$. Then $a_n \to 0$, but

$$\cos(1/a_n) + \sin(1/a_n) = \cos(n\pi) + \sin(n\pi) = (-1)^n$$

does not converge.

Task 7.2.13.

(a) ∞

(b) 0

Exercises for §7.2

2. Let $f(x) = (1+r/x)^x$. Then $\log f(x) = x\log(1+r/x)$, and

$$\lim_{x\to\infty} \log f(x) = \lim_{t\to 0+} \frac{\log(1+rt)}{t} = \lim_{t\to 0+} \frac{r}{1+rt} = r.$$

4.

(a) $-1/\sqrt{2}$

(b) $1/2$

(c) $\log a$

(d) rx^{r-1}

6.

(a) $1/e$

(b) 4

8. Let $f(x) = \cos x - x$. We have $f(0) = 1$ and $f(\pi/2) = -\pi/2$. Hence, the intermediate value theorem gives the existence of $c \in (0, \pi/2)$ such that $f(c) = 0$.

The Newton-Raphson iteration is $x_{n+1} = x_n + \dfrac{\cos x_n - x_n}{\sin x_n + 1}$. This gives:

$$x_1 = \pi/4 = 0.7854, \quad x_2 = 0.7395, \quad x_3 = 0.7391, \quad x_4 = 0.7391.$$

10.

(a) We are looking for a solution of $-c = c-(1+c^2)\arctan c$. Let $f(x) = 2x-(1+x^2)\arctan x$. Then $f(1) = 2-\pi/2 > 0$ and $f(2) = 4-5\arctan 2 < 0$.

(b) The Newton-Raphson iteration for seeking c is

$$x_{n+1} = x - \frac{2x_n - (x_n^2+1)\arctan x_n}{1-2x_n \arctan x_n}.$$

This gives $x_1 = 1.5$, $x_2 = 1.400$, $x_3 = 1.392$ and $x_4 = 1.392$.

7.3 Sum of a Series

Task 7.3.5. $\sum_{k=1}^{n} 1/k$ is an upper sum for $\int_1^{n+1} dx/x$.

Task 7.3.7.

(a) Converges (b) Converges (c) Diverges

Task 7.3.8. Let T_m be the mth partial sum of $\sum_{n=k}^{\infty} a_n$. That is, $T_m = \sum_{n=k}^{k+m-1} a_n$. Then, $T_m = S_{m+k-1} - S_{k-1}$. Hence T_m converges $\iff S_{m+k-1}$ converges $\iff S_m$ converges. Further,

$$\sum_{n=1}^{\infty} a_n = \lim S_m = \lim S_{m+k-1} = S_{k-1} + \lim T_m = S_{k-1} + \sum_{n=k}^{\infty} a_n.$$

Task 7.3.10.

(a) Let $a_n = \sin(n\pi/2)$. Then $a_{4n+1} = 1 \implies a_n \not\to 0$.

(b) Let $a_n = (-1)^n \frac{n-1}{n}$. Then $a_{2n} \to 1 \implies a_n \not\to 0$.

Exercises for §7.3

2. $\dfrac{1}{n(n+2)} = \dfrac{A}{n} + \dfrac{B}{n+2} \implies 1 = A(n+2) + Bn \implies A = \dfrac{1}{2},\ B = -\dfrac{1}{2}.$

Therefore,

$$\begin{aligned}\sum_{n=1}^{k} \frac{1}{n(n+2)} &= \frac{1}{2}\left(\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{k}-\frac{1}{k+2}\right)\right)\\ &= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{k+1}-\frac{1}{k+2}\right) \to \frac{3}{4}.\end{aligned}$$

4.

Convergent: (d), (f), (g), (h), (j).

Divergent: (a), (b), (c), (e), (i).

6. $\displaystyle\sum_{n=1}^{\infty} a_n$ converges $\implies a_n \to 0 \implies a_n^2 \le a_n$ for large n.

8.

(a) True. $\sum a_n$ converges $\implies$ $\sum a_{n+1}$ converges $\implies$ $\sum(a_n + a_{n+1})$ converges.

(b) False. Consider $\sum(-1)^n$.

10. The condition $a_n/b_n \to 0$ gives $a_n < b_n$ for large n. Apply the comparison test.

12. Let $S_k = \sum_{n=1}^{k} a_n$ and $T_k = \sum_{n=1}^{k} 2^n a_{2^n}$. These are increasing sequences, hence they are convergent if and only if they are bounded. Now,

$$\begin{aligned} S_{2^k-1} &= a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^{k-1}} + \cdots + a_{2^k-1}) \\ &\geq a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1}a_{2^k} = \tfrac{1}{2}T_k \end{aligned}$$

This shows that if $\sum a_n$ converges then $\sum 2^n a_{2^n}$ converges. For the converse, we note that

$$S_{2^k-1} \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{k-1}a_{2^{k-1}} = T_{k-1}.$$

14. We have seen that $\log(N+1) \leq \sum_{n=1}^{N} \frac{1}{n} \leq \log(N)+1$. Now $\log(x+1) = 100$ gives $x = e^{100} - 1 = 2.7 \times 10^{43}$, while $\log(x)+1 = 100$ gives $x = e^{99} = 9.9 \times 10^{42}$. So the number of terms to cross 100 is between 9.9×10^{42} and 2.7×10^{43}.

7.4 Absolute and Conditional Convergence

Task 7.4.6. We have $|a_n| = a_n^+ + a_n^-$. Therefore, the convergence of $\sum a_n^+$ and $\sum a_n^-$ implies the convergence of $\sum |a_n|$.

The converse follows from the inequalities $|a_n| \geq a_n^+$ and $|a_n| \geq a_n^-$.

Task 7.4.7. We are given that $\sum(a_n^+ - a_n^-)$ converges. If either $\sum a_n^+$ or $\sum a_n^-$ converges, then so does $\sum |a_n| = \sum(a_n^+ + a_n^-)$.

Exercises for §7.4

2. The required accuracy will be obtained after k terms, provided

$$\frac{0.5^{k+1}}{(2k+3)!} < 0.005.$$

This happens at $k = 1$. Hence $\sum_{n=0}^{\infty} \frac{(-0.5)^n}{(2n+1)!} \approx 1 - \frac{0.5}{3!} = 0.917$.

4.

(a) Diverges.

(b) Converges absolutely.

(c) Converges conditionally.

(d) Converges absolutely.

6. Let $S_k = \sum_{n=1}^{k} a_n$ and $T_k = \sum_{n=1}^{k} |a_n|$. The triangle inequality gives $|S_k| \leq T_k \leq \sum |a_n|$. Hence, $-\sum |a_n| \leq S_k \leq \sum |a_n|$. Now let $k \to \infty$.

8. Let $S_k = \sum_{n=1}^{k} a_n$, $O_k = \sum_{n=1}^{k} a_{2n-1}$, $E_k = \sum_{n=1}^{k} a_{2n}$. Then,

$$\begin{aligned} S_{2k} &= O_k + E_k \\ S_{2k+1} &= O_{k+1} + E_k \end{aligned}$$

These identities show:

- $\lim S_k$ exists if and only if both $\lim O_k$ and $\lim E_k$ exist.
- If these limits exist, then $\lim S_k = \lim O_k + \lim E_k$.

10.

(a) Proof by induction on m. For $m = 1$ the equality holds because $S_1 = a_1$. Assume the equality folds for m. Then,

$$\begin{aligned} \sum_{n=1}^{m+1} a_n b_n &= \sum_{n=1}^{m-1} (b_n - b_{n+1}) S_n + b_m S_m + a_{m+1} b_{m+1} \\ &= \sum_{n=1}^{m} (b_n - b_{n+1}) S_n + b_{m+1} S_m + a_{m+1} b_{m+1} \\ &= \sum_{n=1}^{m} (b_n - b_{n+1}) S_n + b_{m+1} S_{m+1}. \end{aligned}$$

(b) Apply the sandwich theorem, using the boundedness of S_m.

(c) Since b_n is a decreasing sequence,

$$\sum_{n=1}^{m-1} |(b_n - b_{n+1}) S_n| \leq M \sum_{n=1}^{m-1} (b_n - b_{n+1}) = M(b_1 - b_m) \leq M b_1$$

Now, (c) shows that $\sum_{n=1}^{m-1} (b_n - b_{n+1}) S_n$ is convergent. We also know that $b_m S_m$ converges. Hence, (a) shows that $\sum a_n b_n$ converges.

12.

(a) $$0 \leq \sum_{k=1}^{n} (x_k - y_k)^2 = 2(1 - \sum_{k=1}^{n} x_k y_k) \implies \sum_{k=1}^{n} x_k y_k \leq 1$$
$$0 \leq \sum_{k=1}^{n} (x_k + y_k)^2 = 2(1 + \sum_{k=1}^{n} x_k y_k) \implies \sum_{k=1}^{n} x_k y_k \geq -1$$

(b) We have $\left\| \frac{x}{||x||} \right\| = \left\| \frac{y}{||y||} \right\| = 1$. Therefore, $\left| \sum_{k=1}^{n} \frac{x_k}{||x||} \cdot \frac{y_k}{||y||} \right| \leq 1$.

Thematic Exercises

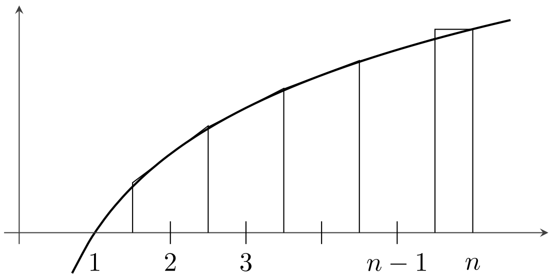
Stirling's Formula

A1. Since $1/x$ is a convex function, the tangent line to its graph at $(1 + \frac{1}{2n}, \frac{2n}{2n+1})$ lies below the graph. This gives $\log(1 + \frac{1}{n}) > \frac{1}{n} \cdot \frac{2n}{2n+1} = \frac{2}{2n+1}$. Hence,

$$1 + \tfrac{1}{n} > e^{\frac{2}{2n+1}} \quad \text{or} \quad (1 + \tfrac{1}{n})^{n+1/2} > e.$$

A2. $$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{((n+1)/e)^{n+1}\sqrt{n+1}} \cdot \frac{(n/e)^n\sqrt{n}}{n!} = \frac{e}{(1+\frac{1}{n})^{n+1/2}} < 1.$$

A3.



Since $\log x$ is a concave function, the tangent lines to its graph lie above the graph. Combined with the observation $\log 1.5 < 1$, this gives $1 + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n > \int_1^n \log x\,dx$.

A4.

$$\begin{aligned}\log a_n &= \sum_{k=2}^{n} \log k - (n+\tfrac{1}{2})\log n + n = \sum_{k=2}^{n-1} \log k - (n-\tfrac{1}{2})\log n + n \\ &> \int_1^n \log x\,dx + (n-1) - (n-\tfrac{1}{2})\log n = \tfrac{1}{2}\log n \geq 0.\end{aligned}$$

A5. For the second equality, apply the integral calculations from Example 5.3.11.

A6. The inequalities follow from $0 \leq \sin x \leq 1$ for $x \in [0, \pi/2]$. They lead to

$$\frac{\int_0^{\pi/2} \sin^{2n+2} x\,dx}{\int_0^{\pi/2} \sin^{2n} x\,dx} \leq \frac{\int_0^{\pi/2} \sin^{2n+1} x\,dx}{\int_0^{\pi/2} \sin^{2n} x\,dx} \leq 1.$$

Example 5.3.11 gives

$$\frac{\int_0^{\pi/2} \sin^{2n+2} x\,dx}{\int_0^{\pi/2} \sin^{2n} x\,dx} = \frac{4^{n+1}(n+1)!^2}{(2n+2)!} \cdot \frac{(2n)!}{4^n n!^2} = \frac{4(n+1)^2}{(2n+2)(2n+1)} \to 1.$$

A7. We know that $a_n = \dfrac{n!}{(n/e)^n\sqrt{n}}$ converges to some $L > 0$. Therefore $\dfrac{a_n^2}{a_{2n}} \to \dfrac{L^2}{L} = L$. But A5 and A6 show that $\dfrac{a_n^2}{a_{2n}} \to \sqrt{2\pi}$.

Gamma and Beta Functions

B1. Let $a < x < b$ be points in the domain of f and g. Then,

$$\begin{aligned}f(x) + g(x) &\leq f(a) + \frac{f(b)-f(a)}{b-a}(x-a) + g(a) + \frac{g(b)-g(a)}{b-a}(x-a) \\ &= (f(a)+g(a)) + \frac{(f(b)+g(b)) - (f(a)+g(a))}{b-a}(x-a)\end{aligned}$$

B2. Write the convexity inequality for $f_n(x)$ and let $n \to \infty$ on both sides.

B3. Let $a < x < b$ be points in the domain of f. By repeated bisection, we can find a sequence $c_n \to x$ such that $c_1 = a$, $c_2 = b$, and $c_{n+1} = \dfrac{c_n + c_{n-1}}{2}$ or $\dfrac{c_n + c_{n-2}}{2}$.

We can prove by strong induction that $f(c_n) \le f(a) + \dfrac{f(b) - f(a)}{b - a}(c_n - a)$ for $n \ge 3$.

For $n = 3$:

$$f(c_3) = f(\tfrac{a+b}{2}) \le \tfrac{1}{2}(f(a) + f(b)) = f(a) + \frac{f(b) - f(a)}{b - a}(c_3 - a).$$

Assuming the inequality holds up to n, we consider the two possibilities for the $n + 1$ case:

$$\begin{aligned} f(c_{n+1}) &= f\Big(\frac{c_n + c_{n-1}}{2}\Big) \le \frac{1}{2}(f(c_n) + f(c_{n-1})) \\ &\le f(a) + \frac{1}{2}\frac{f(b) - f(a)}{b - a}((c_n - a) + (c_{n-1} - a)) \\ &= f(a) + \frac{f(b) - f(a)}{b - a}(c_{n+1} - a), \end{aligned}$$

$$\begin{aligned} f(c_{n+1}) &= f\Big(\frac{c_n + c_{n-2}}{2}\Big) \le \frac{1}{2}(f(c_n) + f(c_{n-2})) \\ &\le f(a) + \frac{1}{2}\frac{f(b) - f(a)}{b - a}((c_n - a) + (c_{n-2} - a)) \\ &= f(a) + \frac{f(b) - f(a)}{b - a}(c_{n+1} - a). \end{aligned}$$

Now that the inequality is extablished, we prove our result by letting $n \to \infty$.

B4. We have $(\log \circ f_n)(x) \to (\log \circ f)(x)$ for every x. Apply B2.

B5. The problem statement should have also included the condition $a > 0$.

For $x = 0$, we have $Q(0, y) = ax^2 \ge 0$.

For $x \ne 0$, $Q(x, y) = x^2(a + 2c(y/x) + b(y/x)^2$. The sign of Q depends on the sign of $a + 2c(y/x) + b(y/x)^2$, which is a quadratic in y/x. Its discriminant is $-4(ab - c^2)$. Hence $ab - c^2 \ge 0$ if and only if the quadratic does not change sign. And $a > 0$ ensures the sign is positive.

B6. First, $\log f$ and $\log g$ are convex, hence continuous. So f, g are continuous. Therefore $\log(f + g)$ is continuous. Hence, it is enough to show that $\log(f + g)$ is weakly convex.

We note that

$$\begin{aligned} \log f \text{ is weakly convex} &\iff \log(f(\tfrac{x+y}{2})) \le \frac{\log(f(x)) + \log(f(y))}{2} \text{ for every } x, y \\ &\iff \log(f(\tfrac{x+y}{2})^2) \le \log(f(x)f(y)) \text{ for every } x, y \end{aligned}$$

$$\iff f(\tfrac{x+y}{2})^2 \leq f(x)f(y) \text{ for every } x, y.$$

B5 gives the following:

$$f(x)f(y) - f(\tfrac{x+y}{2})^2 \geq 0 \implies f(x)\alpha + 2f(\tfrac{x+y}{2})\alpha\beta + f(y)\beta \geq 0 \text{ for all } \alpha, \beta,$$
$$g(x)g(y) - g(\tfrac{x+y}{2})^2 \geq 0 \implies g(x)\alpha + 2g(\tfrac{x+y}{2})\alpha\beta + g(y)\beta \geq 0 \text{ for all } \alpha, \beta.$$

Hence,

$$(f(x) + g(x))\alpha + 2\Big(f(\tfrac{x+y}{2}) + g(\tfrac{x+y}{2})\Big)\alpha\beta + (f(y) + g(y))\beta \geq 0 \text{ for all } \alpha, \beta.$$

Again apply B5 to get:

$$(f(x) + g(x))(f(y) + g(y)) - \Big(f(\tfrac{x+y}{2}) + g(\tfrac{x+y}{2})\Big)^2 \geq 0.$$

B7.

(a) Every function ca^x is log convex. Hence $g_n(x)$ is log convex, as a sum of log convex functions.

(b) Each $g_n(x)$ is a Riemann sum for the integral $f(x)$, with mesh b/n. Therefore $g_n(x) \to f(x)$ for every x.

(c) $\displaystyle\int_0^\infty \phi(t)t^{x-1}\,dt = \lim_{n\to\infty} \int_0^n \phi(t)t^{x-1}\,dt.$

B8.

(a) Apply induction to prove that $f = \Gamma$ on every $(0, n]$. The $n = 1$ case holds by hypothesis. Assume the statement is true for n. Now let $x \in (n, n+1]$. Then $x - 1 \in (0, n]$, hence $f(x-1) = \Gamma(x-1)$. Therefore,

$$f(x) = (x-1)f(x-1) = (x-1)\Gamma(x-1) = \Gamma(x).$$

(b) Apply Exercises A1 and A2 of Chapter 4.

(c) We prove the first inequality:

$$\log(f(n)) - \log(f(n-1)) \leq \frac{\log(f(n+x)) - \log(f(n))}{x}$$
$$\implies x \log\left(\tfrac{f(n)}{f(n-1)}\right) \leq \log\left(\tfrac{f(n+x)}{f(n)}\right)$$
$$\implies (n-1)^x \leq \frac{(x+n-1)(x+n-2)\cdots x f(x)}{(n-1)!}$$

(d) Trivial.

(e) Sandwich theorem.

B9.

(a) We have $B(x, y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,dt + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}\,dt$. The first integral is improper (at 0) when $0 < x < 1$, while the second one is improper (at 1) when $0 < y < 1$. The convergence of the first integral is established by comparision with $\int_0^{1/2} 2t^{x-1}\,dt$, of the second by comparison with $\int_{1/2}^1 2(1-t)^{y-1}\,dt$.

(b) Substitute $s = 1 - t$.

(c) Substitute $t = \sin^2\theta$.

(d) Let $0 < a < b < 1$. Then,

$$\begin{aligned}
\int_a^b t^x(1-t)^{y-1}\,dt &= \int_a^b \left(\frac{t}{1-t}\right)^x (1-t)^{x+y-1}\,dt \\
&= -\frac{t^x(1-t)^y}{x+y}\Big|_a^b + \frac{x}{x+y}\int_a^b (1-t)^{y-1}t^{x-1}\,dt \\
&= \frac{a^x(1-a)^y - b^x(1-b)^y}{x+y} + \frac{x}{x+y}\int_a^b (1-t)^{y-1}t^{x-1}\,dt \\
&\to \frac{x}{x+y}B(x,y) \text{ as } a \to 0, b \to 1.
\end{aligned}$$

(Note: The more obvious integration by parts, based on the original factoring, gives only $B(x+1,y) = \frac{x}{y}B(x,y+1)$.)

B10.

(a) B7 shows that $B(x,y)$ and $\Gamma(x+y)$ are log convex, as functions of x. Hence, so is their product.

(b) $f(x+1) = B(x+1,y)\Gamma(x+y+1) = \frac{x}{x+y}B(x,y)(x+y)\Gamma(x+y) = xf(x)$.

(c) $f(1) = \left(\int_0^1 (1-t)^{y-1}\,dt\right)\Gamma(y+1) = \frac{1}{y}\Gamma(y+1) = \Gamma(y)$.

B11. From B9(c), we get $B(1/2,1/2) = 2\int_0^{\pi/2} 1\,d\theta = \pi$.

Hence, $\dfrac{\Gamma(1/2)^2}{\Gamma(1)} = \pi$, and $\Gamma(1/2) = \sqrt{\pi}$.

8 | Taylor and Fourier Series

8.1 Power Series

Task 8.1.11. $$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdots .$$

Task 8.1.13. Apply Abel's theorem to the Maclaurin series of $\arctan x$. At $x = 1$ it becomes $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, hence is convergent. Left continuity of the series and of $\arctan x$ at $x = 1$ gives

$$\frac{\pi}{4} = \arctan(1) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}.$$

Exercises for §8.1

2. $\sum_{n=1}^{\infty}\frac{x^n}{n^2}.$

4.

(a) $\frac{1}{1-x} = \frac{1}{-2-(x-3)} = -\frac{1}{2}\cdot\frac{1}{1+\left(\frac{x-3}{2}\right)}$. Now $|x-3| < 2$ gives

$$\frac{1}{1-x} = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x-3}{2}\right)^n.$$

(b) $e^x = e^3 e^{x-3} = e^3 \sum_{n=0}^{\infty}\frac{(x-3)^n}{n!}.$

6. The problem statement should have had $e = q/p$, so that $1/e = p/q$. We know that $2 < e < 3$, hence $q > 3$.

(a) $q!(\frac{1}{e} - s) = q!\Big(\frac{p}{q} - \sum_{n=2}^{q}\frac{(-1)^n}{n!}\Big) = (q-1)!p - \sum_{n=2}^{q}(-1)^n q(q-1)\cdots(n+1).$

(b) We apply Theorem 7.4.1(b):

$$|q!r| = q!\left|\frac{1}{e} - s\right| = q!\left|\frac{1}{e} - \sum_{n=2}^{q} \frac{(-1)^n}{n!}\right| \le \frac{q!}{(q+1)!} = \frac{1}{q+1}.$$

Further,

$$|q!r| = q!\left|\frac{1}{e} - s\right| \ge q!\left|\sum_{n=2}^{q+2} \frac{(-1)^n}{n!} - \sum_{n=2}^{q} \frac{(-1)^n}{n!}\right| = \frac{1}{q+2}.$$

The contradiction is that (a) makes $q!|r|$ an integer, while (b) puts it in $(0,1)$.

8. The series $1 - x + x^2 - x^3 \cdots$ equals $\dfrac{1}{1+x}$ on $(-1,1)$. Hence its left-hand limit at 1 is $1/2$. But it diverges at 1.

8.2 Taylor Series

Task 8.2.2. If $r = n \in \mathbb{W}$ then $\binom{r}{k} = 0$ for every $k > n$.

Conversely, $\binom{r}{n} = 0$ implies $r \in \{0, 1, \ldots, n-1\}$.

Task 8.2.4. We can follow the steps of Example 8.2.3. Alternately, we can combine the result of Example 8.2.3 with term-by-term differentiation.

Task 8.2.5. $|R_n(x)| = \left|\dfrac{f^{(n+1)}(\xi_n)}{(n+1)!}(x-c)^{n+1}\right| \le \dfrac{|M(x-c)|^{n+1}}{(n+1)!} \to 0.$

Task 8.2.8. Let the Maclaurin series of $\sec x$ be $b_0 + b_1 x + b_2 x^2 + \cdots$. Then

$$b_0 = \frac{1}{a_0} = 1, \quad b_1 = -\frac{1}{a_0} a_1 b_0 = 0,$$
$$b_2 = -\frac{1}{a_0}(a_1 b_1 + a_2 b_0) = -1 \cdot (0 - \frac{1}{2} \cdot 1) = \frac{1}{2}, \quad \text{etc.}$$

Exercises for §8.2

2.

(a) Apply the ratio test.

(b) Use term-by-term differentiation.

4.

(a) We have $(\sin^{-1})'(x) = \dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n x^{2n}.$

Therefore, $\sin^{-1} x = \sin^{-1} 0 + \displaystyle\int_0^x (\sin^{-1})'(t)\, dt = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} \frac{x^{2n+1}}{2n+1}.$

(b) We have $(\sinh^{-1})'(x) = \dfrac{1}{\sqrt{1+x^2}} = \displaystyle\sum_{n=0}^{\infty} \binom{-1/2}{n} x^{2n}.$

Therefore, $\sinh^{-1} x = \sinh^{-1} 0 + \int_0^x (\sinh^{-1})'(t)\, dt = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n+1}}{2n+1}$.

6.

(a) The Maclaurin series of the numerator is $\frac{1}{4}x^5 + \frac{13}{120}x^7 + \cdots$. Hence,

$$\lim_{x\to 0} \frac{2(\tan x - \sin x) - x^3}{x^5} = \lim_{x\to 0} \left(\frac{1}{4} + \frac{13}{120}x^2 + \cdots\right) = \frac{1}{4}.$$

(b) The Maclaurin series of the numerator and denominator give:

$$\begin{aligned}\lim_{x\to 0} \frac{\sinh(x^2) - x^2}{(\sin x - x)^2} &= \lim_{x\to 0} \frac{x^6/6 + x^{10}/120 + \cdots}{x^6/36 - x^8/360 + \cdots} \\ &= \lim_{x\to 0} \frac{1/6 + x^4/120 + \cdots}{1/36 - x^2/360 + \cdots} = 6.\end{aligned}$$

(c) $$\begin{aligned}\lim_{x\to 0} \frac{\log(1+x+x^2) + \log(1-x+x^2)}{x(e^x - 1)} &= \lim_{x\to 0} \frac{x^2 + x^4/2 + \cdots}{x^2 + x^3/2 + \cdots} \\ &= \lim_{x\to 0} \frac{1 + x^2/2 + \cdots}{1 + x/2 + \cdots} = 1.\end{aligned}$$

(d) $$\begin{aligned}\lim_{x\to\infty} \left(x - x^2 \log\left(1 + \frac{1}{x}\right)\right) &= \lim_{x\to 0} \frac{x - \log(1+x)}{x^2} \\ &= \lim_{x\to 0} \left(\frac{1}{2} - \frac{x}{3} + \cdots\right) = \frac{1}{2}.\end{aligned}$$

8.3 Fourier Series

Task 8.3.4. Use the following:

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m-n)x + \cos(m+n)x)\, dx$$
$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x)\, dx$$

Task 8.3.7. Put $x = \pi/4$ in the Fourier series of the square wave function.

Task 8.3.8.

Assume $0 < a < \pi$. Then,

$$\begin{aligned}a_0 &= 0, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{a-\pi} \cos nx\, dx - \frac{1}{\pi}\int_{a-\pi}^{a} \cos nx\, dx + \int_a^{\pi} \cos nx\, dx \\ &= \frac{2}{n\pi}\big(\sin n(a-\pi) - \sin na\big) = \frac{2}{n\pi}((-1)^n - 1)\sin na, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{a-\pi} \sin nx\, dx - \frac{1}{\pi}\int_{a-\pi}^{a} \sin nx\, dx + \int_a^{\pi} \sin nx\, dx \\ &= -\frac{2}{n\pi}\big(\cos n(a-\pi) - \cos na\big) = -\frac{2}{n\pi}((-1)^n - 1)\cos na\end{aligned}$$

Therefore,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n\pi}\sin na & \text{if } n \text{ is odd} \end{cases}, \quad b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{n\pi}\cos na & \text{if } n \text{ is odd} \end{cases}.$$

The value of the Fourier series at $x = 1$ is

$$\frac{4}{\pi}\Big(- \sum_{n\text{ odd}} \frac{1}{n} \sin na \cos na + \sum_{n\text{ odd}} \frac{1}{n} \cos na \sin na\Big) = 0.$$

Task 8.3.9.

$$\begin{aligned} 2\sin\theta/2 \cdot \Big(\frac{1}{2} + \sum_{n=1}^{m} \cos n\theta\Big) &= \sin\theta/2 + \sum_{n=1}^{m} 2\cos n\theta \sin\theta/2 \\ &= \sin\theta/2 + \sum_{n=1}^{m} \Big(\sin(n+\tfrac{1}{2})\theta - \sin(n-\tfrac{1}{2})\theta\Big) \\ &= \sin(m+\tfrac{1}{2})\theta. \end{aligned}$$

When $\theta = 2n\pi$, the right hand side is undefined, while the left hand side has value $m + 1/2$. Since these are the only points of disagreement, it follows that the right hand side has limit $m+1/2$ at these points. This is the sense in which the equality extends to all points.

Task 8.3.13. Put $x = 0$ in the Fourier series of Example 8.3.12.

Task 8.3.14.

(a) $\langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx \geq \frac{1}{\pi}\int_{-\pi}^{\pi} 0\,dx = 0$.

(b) Obvious.

(c) Apply the homogeneity property of integration.

(d) Apply the additivity property of integration.

(e) $||f+g||^2 = \langle f+g, f+g\rangle = \langle f, f\rangle + \langle f, g\rangle + \langle g, f\rangle + \langle g, g\rangle = ||f||^2 + ||g||^2$.

Task 8.3.16. Bessel's inequality shows that the series $\sum a_n^2$ and $\sum b_n^2$ converge. Hence $a_n, b_n \to 0$.

Exercises for §8.3

2. Recall that if g has period 2π then $\int_{-\pi}^{\pi} g(x)\,dx = \int_{T}^{T+2\pi} g(x)\,dx$ for any T.

$$\begin{aligned} a_0' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-a)\,dx = \frac{1}{\pi}\int_{-\pi-a}^{\pi-a} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0, \\ a_n' &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-a)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi-a}^{\pi-a} f(x)\cos n(x+a)\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos nx \cos na - \sin nx \sin na)\,dx \\ &= a_n \cos na - b_n \sin na. \end{aligned}$$

The b_n' calculation is similar.

4. Apply Exercise 10 of §3.4.

6. Using the hint, we have

$$\begin{aligned}\frac{\pi^4}{90} &= 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \Big(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\Big) + \Big(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4}\cdots\Big) \\ &= \Big(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\Big) + \frac{1}{2^4}\cdot\frac{\pi^4}{90}\end{aligned}$$

8. The series should have been given as $\sum_{n=1}^{\infty}\frac{\sin nx}{\sqrt{n}}$. To establish the convergence, we apply Dirichlet's test (Exercise 10 of §7.4), noting that:

- $1/\sqrt{n}$ is a decreasing sequence with limit 0.
- The partial sums $\sum_{k=1}^{m}\sin nx$ are bounded for fixed x, since:

$$\begin{aligned} x = 2k\pi &\implies \sum_{k=1}^{m}\sin nx = 0 \\ x \neq 2k\pi &\implies \Big|\sum_{k=1}^{m}\sin nx\Big| = \Big|\frac{\cos x/2 - \cos(m+\frac{1}{2})x}{2\sin x/2}\Big| \leq \frac{1}{|\sin x/2|}\end{aligned}$$

However, the series cannot be the Fourier series of an integrable function as it violates Bessel's inequality.

Remark: Can you see why $\sum_{n=1}^{\infty}\frac{\cos nx}{\sqrt{n}}$ should be replaced by $\sum_{n=1}^{\infty}\frac{\sin nx}{\sqrt{n}}$?

8.4 Complex Series

Task 8.4.1. Apply mathematical induction:

$$\begin{aligned} z^{n+1} &= z^n z = |z|^n(\cos n\theta + i\sin n\theta)|z|(\cos\theta + i\sin\theta) \\ &= |z|^{n+1}(\cos(n\theta+\theta) + i\sin(n\theta+\theta))\end{aligned}$$

Task 8.4.2. $1 + i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \implies (1+i)^{1000} = 2^{500}(\cos 250\pi + i\sin 250\pi) = 2^{500}$.

Task 8.4.3. $w^n = 1 \implies 1 - w^n = 0 \implies (1-w)(1+w+w^2+\cdots+w^{n-1}) = 0$.

Task 8.4.5.

(a) Use $|z_n - 0| = ||z_n| - 0|$.

(b) Use $|z_n - L| = |(z_n - L) - 0|$.

(c) Use $||z_n| - |L|| \leq |z_n - L|$.

Task 8.4.9. Use $z_n = \sum_{k=1}^{n} z_k - \sum_{k=1}^{n-1} z_k$.

Task 8.4.10. Mimic the proof of Theorem 7.3.6.

Task 8.4.11. Apply Theorem 8.4.6 to the partial sums of these series.

Task 8.4.13. Let $z_n = x_n + iy_n$. Then $|x_n|, |y_n| \le |z_n|$. If $\sum |z_n|$ converges, so do $\sum |x_n|$ and $\sum |y_n|$, hence also $\sum x_n$ and $\sum y_n$. Now apply Task 8.4.11.

Task 8.4.14. Apply the rearrangement theorem for real series to the real and imaginary parts.

Task 8.4.19. Same calculations as in the real case.

Task 8.4.21. Apply Theorem 8.4.20.

Task 8.4.22. $e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi) = e^{i(\theta+\phi)}$.

Task 8.4.24.

(a) Theorem 8.4.23 gives $e^z e^{-z} = e^{z-z} = e^0 = 1$. This shows $(e^z)^{-1} = e^{-z}$ as well as $e^z \neq 0$.

(b) Any z can be expressed as $|z|e^{i\theta}$. Since $|z| > 0$, we have $r \in \mathbb{R}$ such that $e^r = |z|$. Then $z = e^{r+i\theta}$.

(c) Let $z = x + iy$. Then $e^z = e^x \cos y + ie^x \sin y$. Therefore, $e^z = 1 \iff e^x \cos y = 1$ and $e^x \sin y = 0 \iff y \in 2\pi\mathbb{Z}$ and $x = 0$.

Task 8.4.26.

$$\begin{aligned}
||f+g||^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x)+g(x))(\overline{f(x)+g(x)})\,dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x)\overline{f(x)} + f(x)\overline{g(x)} + g(x)\overline{f(x)} + g(x)\overline{g(x)}\right) dx \\
&= ||f||^2 + \langle f,g\rangle + \overline{\langle f,g\rangle} + ||g||^2 = ||f||^2 + ||g||^2.
\end{aligned}$$

Task 8.4.27. Example 8.4.25 shows that $\langle e_k, e_n\rangle = 1$ if $k = n$, and is zero otherwise. Let $f_m(x) = \sum_{n=-m}^{m} \hat{f}(n)e^{inx}$. For $-m \le n \le m$, we have $\langle f - f_m, e_n\rangle = \langle f, e_n\rangle - \langle f_m, e_n\rangle = \hat{f}(n) - \hat{f}(n) = 0$. Hence, $\langle f - f_m, f_m\rangle = 0$. This implies

$$\sum_{n=-m}^{m} |\hat{f}(n)|^2 = ||f_m||^2 \le ||f - f_m||^2 + ||f_m||^2 = ||f||^2.$$

Bessel's inequality follows by letting $m \to \infty$.

Exercises for §8.4

2. The correct statement is $|z+w|^2 + |z-w|^2 = 2(|z|^2 + |w|^2)$. The proof is:

$$\begin{aligned}
|z+w|^2 + |z-w|^2 &= (z+w)(\overline{z}+\overline{w}) + (z-w)(\overline{z}-\overline{w}) \\
&= |z|^2 + z\overline{w} + w\overline{z} + |w|^2 + |z|^2 - z\overline{w} - w\overline{z} + |w|^2 \\
&= 2|z|^2 + 2|w|^2.
\end{aligned}$$

4. $$\sum_{k=1}^{n} e^{ikx} = e^{ix}\frac{e^{inx}-1}{e^{ix}-1} = e^{i(n+1)x/2}\frac{e^{inx/2}-e^{-inx/2}}{e^{ix/2}-e^{-ix/2}}$$
$$= (\cos(n+1)x/2 + i\sin(n+1)x/2)\frac{\sin(nx/2)}{\sin(x/2)}.$$

Now match the real and imaginary parts:

(a) $$\sum_{k=1}^{n}\cos(kx) = \frac{\cos(n+1)x/2\sin(nx/2)}{\sin(x/2)} = \frac{\sin(n+1/2)x - \sin(x/2)}{2\sin(x/2)}.$$

(b) $$\sum_{k=1}^{n}\sin(kx) = \frac{\sin(n+1)x/2\sin(nx/2)}{\sin(x/2)} = \frac{\cos(x/2) - \cos(n+1/2)x}{2\sin(x/2)}.$$

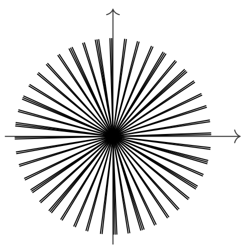
6. $$\frac{1}{(1-z)^2} = \frac{1}{((1-i)-(z-i))^2} = \frac{1}{(1-i)^2}\sum_{n=0}^{\infty}\frac{n+1}{(1-i)^n}(z-i)^n.$$

The radius of convergence is $R = \lim_{n\to\infty}\dfrac{|1-i|}{(n+1)^{1/n}} = \sqrt{2}$. So the series converges on the open disc $\{\, z \mid |z-i| < \sqrt{2}\,\}$, and diverges when $|z-i| > \sqrt{2}$. It diverges on the circle $|z-i| = \sqrt{2}$, since

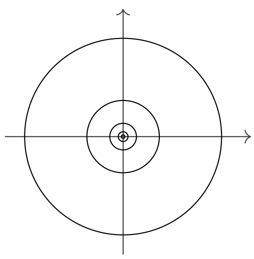
$$\Big|\frac{n+1}{(1-i)^n}(z-i)^n\Big| = \Big|\frac{n+1}{(1-i)^n}\sqrt{2}^n\Big| = |n+1| \not\to 0.$$

8.

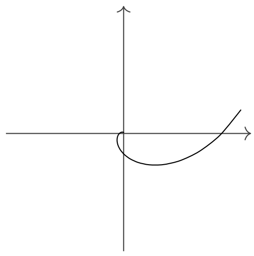
(a) The points e^{iy}, $y \in \mathbb{Z}$, are densely distributed on the unit circle. Multiplying by e^x gives rays from the origin through these points. The set can't be completely drawn but a reasonable impression can be given as follows:



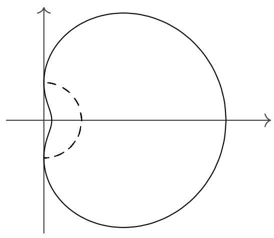
(b) The points e^{iy} form the unit circle. Scaling by e^x, $x \in \mathbb{Z}$, creates concentric circles:



(c) A spiral:



(d) $z = \frac{\pi}{2}e^{i\theta} \implies e^z = e^{(\pi/2)\cos\theta}e^{i(\pi/2)\sin\theta}$. The values $e^{i(\pi/2)\sin\theta}$ twice traverse a semicircle (shown as a dashed curve below). The scaling by $e^{(\pi/2)\cos\theta}$ affects each location twice, once pulling it towards the origin, and once pushing it out. This creates the following heart-shaped curve.



10. Follows directly from the definitions.

Thematic Exercises

Uniform Convergence

A1.

(a) Obvious.

(b) Obvious.

(c) $||cf|| = \sup\{\,|cf(x)| \mid x \in A\,\} = \sup\{\,|c|\,|f(x)| \mid x \in A\,\}$
$= |c|\sup\{\,|f(x)| \mid x \in A\,\} = |c|\,||f||$.

(d) For any $x \in A$, $|f(x) + g(x)| \le |f(x)| + |g(x)| \le ||f|| + ||g||$.

A2. Apply A1.

A3.

(a) $0 \le |f_n(x) - f(x)| \le ||f_n - f|| \to 0$.

(b) Let $f_n(x) = x^n$ on $[0,1]$. Then $f_n \xrightarrow{pw} f$, where $f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{else} \end{cases}$.

And, $||f_n - f|| \ge |f_n(1 - \frac{1}{n}) - f(1 - \frac{1}{n})| = (1 - \frac{1}{n})^n \to \dfrac{1}{e} > 0$.

A4. We have N such that $n \ge N \implies ||f_n - f|| < 1$. Define

$$M = \max\left\{||f_1||,\ ||f_2||, \ldots,\ ||f_{N-1}||, ||f|| + 1\right\}.$$

A5.

(a) Follows from the uniform convergence of f_n to f.

(b) Let $n \geq N$. We have $f_n(x) - \frac{\varepsilon}{b-a} \leq f(x) \leq f_n(x) + \frac{\varepsilon}{b-a}$. Let P be a partition with points $\{x_0, x_1, \ldots, x_n\}$. Define
$$\begin{aligned} m_i &= \inf\{\, f(x) \mid x \in [x_{i-1}, x_i]\,\}, \\ M_i &= \sup\{\, f(x) \mid x \in [x_{i-1}, x_i]\,\}, \\ m_i' &= \inf\{\, f_n(x) \mid x \in [x_{i-1}, x_i]\,\}, \\ M_i' &= \inf\{\, f_n(x) \mid x \in [x_{i-1}, x_i]\,\}. \end{aligned}$$
We have $m_i' - \frac{\varepsilon}{b-a} \leq m_i \leq M_i \leq M_i' + \frac{\varepsilon}{b-a}$. Let $\triangle x_i = x_i - x_{i-1}$. Then

$$\sum_{i=1}^{n} m_i' \triangle x_i - \varepsilon \leq \sum_{i=1}^{n} m_i \triangle x_i \leq \sum_{i=1}^{n} M_i \triangle x_i \leq \sum_{i=1}^{n} M_i' \triangle x_i + \varepsilon.$$

(c) Choose N as in (a). Choose a partition P such that $U(f_N, P) - L(f_N, P) < \varepsilon$. Then (b) gives $U(f, P) - L(f, P) < 3\varepsilon$. Hence f is integrable. And,

$$\Big| \int_a^b f_n - \int_a^b f \Big| \leq \int_a^b |f_n - f| \leq ||f_n - f||(b-a) \to 0.$$

A6.

(a) Follows from the uniform convergence of f_n to f.

(b) Follows from the continuity of f_N at x.

(c) $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \varepsilon$.

A7.

(a) Apply A5 to the sequence of partial sums.

(b) Apply A6 to the sequence of partial sums.

A8.

(a) Use $|f_n(x)| \leq ||f_n|| \leq M_n$.

(b) $\Big| f(x) - \sum_{n=1}^{m} f_n(x) \Big| = \Big| \sum_{n=m+1}^{\infty} f_n(x) \Big| \leq \sum_{n=m+1}^{\infty} |f_n(x)| \leq \sum_{n=m+1}^{\infty} M_n$. Hence, $\Big\| f - \sum_{n=1}^{m} f_n \Big\| \leq \sum_{n=m+1}^{\infty} M_n \to 0$ as $m \to \infty$.

A9. Apply the Weierstrass M-test.

A10.

(a) Apply A9. (The hint is not needed.)

(b) Apply A9. (This observation is interesting because, in general, the behaviour at the endpoints of the interval of convergence can be different. But now we see that absolute convergence at one yields absolute convergence at the other.)

A11.

(a) Since f' is continuous, it is integrable. By Exercise 3 of §8.3, the Fourier coefficients of f' are na_n and nb_n. By Bessel's inequality, $\sum n^2 a_n^2$ and $\sum n^2 b_n^2$ converge.

(b) If $|a_n| > 1/n$ then $n|a_n|^2 > |a_n|$. Hence $|a_n| \le \max\{\frac{1}{n^2}, n^2 a_n^2\}$. The convergence of $\sum 1/n^2$ and $\sum n^2 a_n^2$ therefore gives the convergence of $\sum |a_n|$.

(c) Use $|a_n \cos nx| \le |a_n|$ and $|b_n \cos nx| \le |b_n|$.

Irrationality of Some Numbers

B1. Suppose that $r = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then $q_n p - p_n q \neq 0$ and $q_n p - p_n q \to 0$. But this is impossible, since the $q_n p - p_n q$ are integers.

B2. Integration by parts.

B3. Proof by induction, using B2.

B4.

(a) $0 \le x \le a \implies 0 \le a - x \le a \implies 0 \le \dfrac{x^n (a-x)^n}{n!} \le \dfrac{a^{2n}}{n!}$.

(b) From (a) we see that $e^{-x} f_n(x)$ converges uniformly to 0 on $[0, a]$. Apply Exercise A5 of this chapter.

B5. We shall give the proof for $x = 0$. Since $a \in \mathbb{N}$, we have $(a-x)^n = \sum_{j=0}^{n} c_j x^j$ with $c_j \in \mathbb{Z}$. Hence, $f_n(x) = \sum_{j=0}^{n} \frac{c_j}{n!} x^{j+n}$. By considering the Taylor coefficients of $f_n(x)$ we obtain

$$f_n^{(k)}(0) = \begin{cases} 0 & \text{if } k < n \text{ or } k > 2n \\ \dfrac{c_{k-n}}{n!} k! & \text{if } n \le k \le 2n \end{cases} .$$

B6. Let $a \in \mathbb{N}$. Then B3, B4, B5 give sequences p_n, q_n of integers such that $q_n e^{-a} - p_n \neq 0$ and $q_n e^{-a} - p_n \to 0$. By B1, e^{-a} is irrational. Hence e^a is also irrational. So e^a is irrational for every non-zero integer.

Let $r = p/q$ with $p, q \in \mathbb{Z}$. If e^r is rational, so is $(e^r)^q = e^p$, a contradiction.

B7. If $\log r$ is rational then $r = e^{\log r}$ is irrational.

Errata

Chapter 2

§2.2, Exercise 14(b) The hint that this case can be covered by considering shifts, is not helpful. An alternate approach has beeen provided.

Chapter 5

§5.4, Exercise 6: The given substitution should have been $t = \tan(x/2)$.

Exercise A3 The initial conditions should have been $f(0) = B$ and $f'(0) = A$.

Chapter 6

§6.3, Exercise 2: There is an error in the problem statement. The equality to be proved is $T_{\int f,n}(x) = \int_a^x T_{f,n-1}(t)\,dt$.

§6.4, Exercise 6: The problem statement should have $S(f, P_n)$ instead of $L(f, P_n)$. The given inequalities also need slight corrections, which have been provided in the solution.

§6.5, Task 6.5.5: The last two requirements need to be corrected to $q'(a) = y'_a$ and $q'(b) = y'_b$.

Chapter 7

Exercise B5 The problem statement should have the additional condition $a > 0$.

Chapter 8

§8.1, Exercise 6: The problem statement should have $e = q/p$ (instead of $e = p/q$).

§8.3, Exercise 8: The given series should be $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ (instead of $\sum_{n=1}^{\infty} \frac{\cos nx}{\sqrt{n}}$).

§8.4, Exercise 2: The correct statement of the parallelogram identity is $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.