MIDTERM example 4

SI-MKS

Speed of light in free space	$c = 2.99792458 \times 10^{8}\ \mathrm{m\ s^{-1}}$
Planck's constant	$\hbar = 6.5821188926 \times 10^{-16}\ \mathrm{eV\ s}$
	$\hbar = 1.054571596 \times 10^{-34}\ \mathrm{J\ s}$
Electron charge	$e = 1.602176462 \times 10^{-19}\ \mathrm{C}$
Electron mass	$m_0 = 9.10938188 \times 10^{-31}\ \mathrm{kg}$
Neutron mass	$m_n = 1.67492716 \times 10^{-27}\ \mathrm{kg}$
Proton mass	$m_p = 1.67262158 \times 10^{-27}\ \mathrm{kg}$
Boltzmann	$k_B = 1.3806503 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
constant	$k_B = 8.617342 \times 10^{-5}\ \mathrm{eV\ K^{-1}}$
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}\ \mathrm{F\ m^{-1}}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\ m^{-1}}$
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0 \mu_0}$
Avagadro's number	$N_A = 6.02214199 \times 10^{23}\ \mathrm{mol^{-1}}$
Bohr radius	$a_B = 0.52917721 \times 10^{-10}\mathrm{m}$

$$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$$

Inverse fine-structure constant $\alpha^{-1} = 137.0359976$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

Problem 1

(a) Derive the current operator $\mathbf{J}$ for an electron of mass m constrained to move in a real one-dimensional potential $V(x)$ and described by wave function $\psi(x)$. (20%)

(b) An electron moving from left-to-right in the x-direction with energy E encounters a potential step. The potential is $V(x) = V_0$ for $x \geq 0$ and $V(x) = 0$ for $x < 0$. On one side of the potential step the electron has mass m_1 and on the other side it has mass m_2. Calculate the transmission and reflection for current flow in the presence of this potential and discuss any assumptions or approximations you make. (60%)

(c) Sketch the wave function $\psi(x)$ when particle energy $E = V_0$ and calculate its contribution to transmission current. (20%)

Problem 2

(a) What is the definition of a Hermitian operator and an anti-Hermitian operator? (20%)

(b) Show that the eigenvalues of a Hermitian operator are real and that the eigenvalues of an anti-Hermitian operator are pure imaginary. (40%)

(c) Show that the eigenfunctions of a Hermitian operator are orthogonal. (10%)

(d) Show that $-i\hbar\frac{\partial}{\partial r}$ is not a Hermitian operator in radial coordinates. (30%)

Problem 3

In classical mechanics, the Hamiltonian for a one-dimensional harmonic oscillator with motion in the x-direction at frequency ω is

$$H = \frac{p^2}{2m_0} + \frac{m_0\omega^2}{2}x^2$$

where m_0 is the mass of the particle and p is the particle momentum.

(a) Introduce the operator $\hat{b} = \left(\frac{m_0\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i\hat{p}}{m_0\omega}\right)$ and show that in quantum mechanics the Hamiltonian may be symmetrized so that $\hat{H} = \frac{\hbar\omega}{2}(\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b})$. (20%)

(b) Obtain the four commutation relations for $\hat{b}$ and $\hat{b}^\dagger$ and show that $\hat{H} = \hbar\omega(\hat{b}^\dagger\hat{b} + 1/2)$. (20%)

(c) Show that the $n = 0$ ground-state $|0\rangle$ is defined by $\hat{b}|0\rangle = 0$. (20%)

(d) Derive the standard deviation in position and momentum of each harmonic oscillator state $|n\rangle$ and show that they satisfy the Heisenberg uncertainty relation. (20%)

(e) If the standard deviation in position is $\Delta x = 1$ nm, what is the value of the standard deviation in momentum for the state $|0\rangle$ and the state $|4\rangle$? (20%)

Problem 4

(a) Find the one, two, and three-dimensional density of states for electrons of mass m. (20%)

(b) Show that at low temperatures one-dimensional electron conductance can be quantized. (50%)

(c) Estimate the maximum frequency of operation of a one-dimensional conductance device with capacitance 10 fF (10^{-14} F)? (20%)

(d) How might one increase the maximum frequency response in (c)? (10%)