Solutions to exercises

Exercise 1.1 First we need to convert the distance to SI units:

$$\begin{aligned}68\text{ light-years} &= 68 \times 9.461 \times 10^{15}\,\text{m}\\ &= 6.5 \times 10^{17}\,\text{m}.\end{aligned}$$

Then we can obtain the time taken by dividing the distance by the speed of travel:

$$\begin{aligned}\text{time} &= \frac{\text{distance}}{\text{speed}}\\ &= \frac{6.5 \times 10^{17}\,\text{m}}{12 \times 10^{3}\,\text{m}\,\text{s}^{-1}}\\ &= 5.4 \times 10^{13}\,\text{s}\\ &= \frac{5.4 \times 10^{13}\,\text{s}}{365.25\text{ day yr}^{-1} \times (24 \times 3600)\,\text{s day}^{-1}}\\ &= 1.7 \times 10^{6}\,\text{yr} \approx 2\,\text{Myr}.\end{aligned}$$

(Myr indicates 10^6 yr, or a megayear.)

Thus the time taken for Pioneer 10 to reach Aldebaran is almost 2 million years.

Exercise 1.2 In interpreting Figure 1.8, we note that both axes are logarithmic.

(a) The most favourable contrast ratio occurs when the vertical distance between Jupiter's curve and the Sun's is minimized. The two curves are converging as they disappear off the right-hand side of the figure, so the most favourable contrast ratio occurs for wavelengths around (or greater than) $100\,\mu$m. The value of the contrast ratio at $100\,\mu$m is approximately 10 000.

(b) The spectral energy distribution of Jupiter peaks at around $0.5\,\mu$m. This is very different (by a factor of around 200) to the wavelength of the most favourable contrast ratio. The reason for the discrepancy is that the Sun's spectral energy distribution also peaks at around $0.5\,\mu$m, and the second, thermal emission component of Jupiter's spectral energy distribution gives a more favourable contrast ratio.

(c) Wavelengths around $20\,\mu$m are at the peak of Jupiter's thermal emission. Though the contrast ratio (almost 100 000) is less favourable than at longer wavelengths, the flux from Jupiter is over 20 times higher at $20\,\mu$m than it is at $100\,\mu$m. There is limited value in having a favourable contrast ratio if the flux from both objects is immeasurably small!

Exercise 1.3 Kepler's third law in the form used for planetary orbits is

$$\frac{a^3}{P_{\text{orb}}^2} = \frac{G(M_* + M_{\text{P}})}{4\pi^2}. \qquad \text{(Eqn 1.1)}$$

To make the estimate, we will consider the star as a small mass in orbit around a much larger mass positioned at the centre of the Galaxy. We can therefore replace $M_* + M_{\text{P}}$ with M_{total}, where this is the mass of the Galaxy. We will use the Galactocentric distance as the value for a, and this will allow us to make an estimate of P_{orb}, the time taken for a complete orbit around the Galaxy. Thus we have

$$P_{\text{orb}} = \left(\frac{4\pi^2 a^3}{GM_{\text{total}}}\right)^{1/2}. \tag{S1.1}$$

To use this we need to convert all quantities into SI units (which we can accomplish using the information in the Appendix):

$$M_{\text{total}} = 10^{12}\,\text{M}_\odot = 1.99 \times 10^{30} \times 10^{12}\,\text{kg} \approx 2 \times 10^{42}\,\text{kg},$$
$$a = 8\,\text{kpc} = 8 \times 10^3 \times 3.086 \times 10^{16}\,\text{m} \approx 2 \times 10^{20}\,\text{m}.$$

In each case we have retained only one significant figure as we are making a rough estimate. Thus we have

$$\begin{aligned}
P_{\text{orb}} &\approx \left(\frac{4\pi^2(2 \times 10^{20}\,\text{m})^3}{7 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2} \times 2 \times 10^{42}\,\text{kg}}\right)^{1/2} \\
&\approx \left(\frac{4 \times 10 \times 8 \times 10^{60}\,\text{m}^3}{7 \times 2 \times 10^{31}\,\text{kg}\,\text{m}\,\text{s}^{-2}\,\text{m}^2\,\text{kg}^{-2}\,\text{kg}}\right)^{1/2} \\
&\approx \left(\frac{3 \times 10^{62}\,\text{m}^3}{1 \times 10^{32}\,\text{m}^3\,\text{s}^{-2}}\right)^{1/2} \\
&\approx \left(3 \times 10^{30}\,\text{s}^2\right)^{1/2} \\
&\approx 2 \times 10^{15}\,\text{s}.
\end{aligned}$$

It's difficult to grasp how long 10^{15} s is, so we will convert the answer to years:

$$\begin{aligned}
P_{\text{orb}} &\approx \frac{2 \times 10^{15}\,\text{s}}{3600\,\text{s}\,\text{h}^{-1} \times 24\,\text{h}\,\text{day}^{-1} \times 365.25\,\text{day}\,\text{yr}^{-1}} \\
&\approx 6 \times 10^7\,\text{yr}.
\end{aligned}$$

Thus we have deduced that it takes about 60 Myr for the Sun to complete its orbit around the Galaxy. (Note: this is a very rough answer due to the initial approximations and the accumulated rounding errors.)

Exercise 1.4 (a) Equation 1.12 has a simple dependence on i:

$$V \propto \sin i.$$

The function $\sin i$ has a maximum value of 1, when $i = 90°$; this corresponds to the line of sight to the system being exactly in the plane of the orbit, as shown in Figure S1.1a. The minimum value of the radial velocity corresponds to $\sin i = 0$, which occurs when $i = 0°$; this corresponds to the plane of the orbit coinciding with the plane of the sky as viewed by the observer, as shown in Figure S1.1b. The orbital velocities of the star and its planet are always orthogonal to the line of sight, and zero radial velocity variation is observed. At intermediate orientations, $0° < i < 90°$, a finite radial velocity variation with amplitude less than that of the true orbital velocity, $\boldsymbol{v}_*$, of the star is observed, as shown in Figure S1.1c: $V = |\boldsymbol{v}_*| \sin i$.

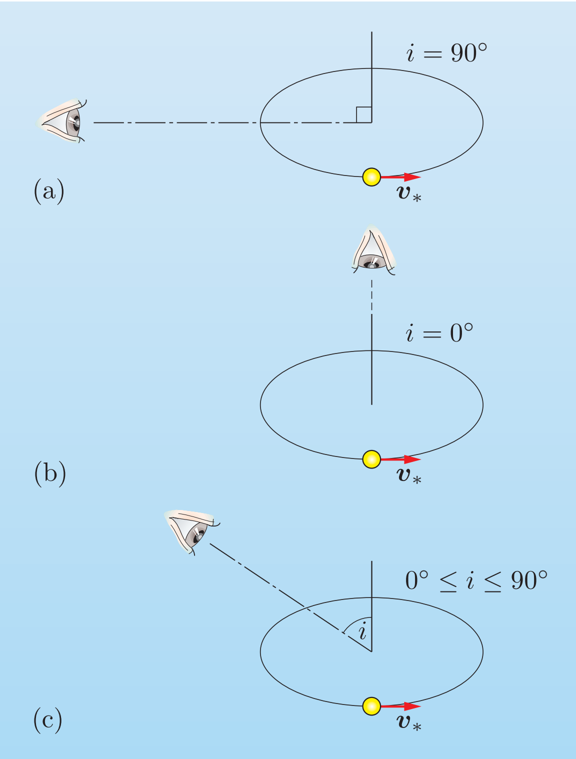

Figure S1.1 (a) An elliptical orbit viewed from an orbital inclination of $i = 90°$; the z-axis lies in the plane of the orbit. (b) The same orbit viewed from $i = 0°$; the plane of the orbit coincides with the plane of the sky, and there is no component of the orbital velocity in the direction towards or away from the observer. (c) At intermediate orientations, $0° < i < 90°$, only the component $\boldsymbol{v}_* \sin i$ of the orbital motion is in the direction towards or away from the observer.

(b) The eccentricity appears twice in Equation 1.12. Inside the brackets in the numerator, it multiplies a constant ($\cos \omega_{\mathrm{OP}}$, where ω_{OP} is a constant parameter of the orbit). The radial velocity variability is exclusively in the $\cos(\theta(t) + \omega_{\mathrm{OP}})$ term, so this first occurrence of the eccentricity does not affect the radial velocity variations. The second appearance of the eccentricity is in the denominator of the terms that determine the amplitude multiplying the time-variable cosine term. Here it contributes to the $\sqrt{1-e^2}$ term, which has the value 0 for $e = 1$ and the value 1 for $e = 0$. As the eccentricity approaches 1, this term in the denominator approaches 0, and therefore the amplitude of the radial velocity variation approaches infinity. As the eccentricity increases, the amplitude of the radial velocity variations increases.

(c) The observed radial velocity is given by Equation 1.12:

$$V(t) = V_{0,z} + \frac{2\pi a M_{\mathrm{P}} \sin i}{(M_{\mathrm{P}} + M_*)P\sqrt{1-e^2}} \left(\cos(\theta(t) + \omega_{\mathrm{OP}}) + e \cos \omega_{\mathrm{OP}}\right).$$

The variable part of this is

$$\frac{2\pi a M_{\mathrm{P}} \sin i}{(M_{\mathrm{P}} + M_*)P\sqrt{1-e^2}} \cos(\theta(t) + \omega_{\mathrm{OP}}),$$

and $\cos(\theta(t) + \omega_{\mathrm{OP}})$ varies cyclicly between -1 and $+1$. The radial velocity amplitude is therefore

$$A_{\mathrm{RV}} = \frac{2\pi a M_{\mathrm{P}} \sin i}{(M_{\mathrm{P}} + M_*) P \sqrt{1 - e^2}}. \qquad \text{(Eqn 1.13)}$$

For the specific case of Jupiter orbiting around the Sun, we substitute the appropriate subscripts:

$$A_{\mathrm{RV}} = \frac{2\pi a \mathrm{M_J} \sin i}{(\mathrm{M_J} + \mathrm{M_\odot}) \mathrm{P_J} \sqrt{1 - \mathrm{e_J^2}}}. \qquad \text{(S1.2)}$$

The data were conveniently all given in SI units except for $\mathrm{P_J}$, which we must convert from years to seconds:

$$\begin{aligned}\mathrm{P_J} &= 12\,\mathrm{yr} \\ &= 12\,\mathrm{yr} \times 365.25\,\mathrm{days\,yr^{-1}} \times 24\,\mathrm{h\,day^{-1}} \times 60 \times 60\,\mathrm{s\,h^{-1}} \\ &= 3.8 \times 10^8\,\mathrm{s},\end{aligned}$$

where we have given our converted value to two significant figures, while performing the intermediate steps in the conversion to a higher precision.

Substituting values into Equation S1.2, we obtain

$$\begin{aligned}A_{\mathrm{RV}} &= \frac{2\pi \times 8 \times 10^{11}\,\mathrm{m} \times 2 \times 10^{27}\,\mathrm{kg} \times \sin i}{2 \times 10^{30}\,\mathrm{kg} \times 3.8 \times 10^8\,\mathrm{s} \times \sqrt{1 - 0.0025}} \\ &= 13 \sin i\,\mathrm{m\,s^{-1}}.\end{aligned}$$

Exercise 1.5 (a) No. Equation 1.13 also contains a factor P in the denominator. Planetary orbits obey Kepler's third law, so $P^2 \propto a^3$. This means that $P \propto a^{3/2}$, so $A_{\mathrm{RV}} \propto a^{1-3/2} = a^{-1/2}$. The radial velocity amplitude decreases as the planet's orbital semi-major axis increases.

(b) Planet mass appears in both the denominator and the numerator of Equation 1.13, giving a dependence

$$A_{\mathrm{RV}} \propto \frac{M_{\mathrm{P}}}{M_{\mathrm{P}} + M_*}.$$

The quantity on the right-hand side will increase with M_{P} if M_* is held fixed. This proportionality also includes the only appearance of the stellar mass, M_*, in Equation 1.13. If M_{P} is held fixed, the right-hand side will decrease as M_* increases.

The eccentricity contributes to Equation 1.13 solely through the factor $\sqrt{1 - e^2}$ in the denominator, so

$$A_{\mathrm{RV}} \propto \frac{1}{\sqrt{1 - e^2}}.$$

As e increases, $(1 - e^2)$ decreases, so the right-hand side of the proportionality increases with increasing e.

We have already shown that the radial velocity amplitude decreases as the planet's orbital semi-major axis increases, so summarizing our analysis of Equation 1.13, we see that the radial velocity amplitude is highest for massive planets in close-in eccentric orbits around low-mass host stars.

Exercise 2.1 (a) Equation 1.21 is

$$\text{geometric transit probability} \approx \frac{R_*}{a}. \qquad \text{(Eqn 1.21)}$$

To use this we need a value for the semi-major axis, a. Kepler's third law (Equation 1.1) can be expressed as

$$a^3 = G(M_* + M_\mathrm{P})\frac{P^2}{4\pi^2}, \qquad \text{(S2.1)}$$

so the orbital semi-major axis is

$$a = (G(M_* + M_\mathrm{P}))^{1/3}\left(\frac{P}{2\pi}\right)^{2/3} \approx \left(GM_*\left(\frac{P}{2\pi}\right)^2\right)^{1/3}, \qquad \text{(S2.2)}$$

where the approximation lies in neglecting the planet's mass compared with that of the star. Since Jupiter's mass is $\sim 10^{-3}\,\mathrm{M_\odot}$ and other giant planets and stars have masses of the same order of magnitude, to the precision at which we are working this is a good approximation, so we will revert to using an equals sign in the subsequent working. In SI units, the quantities that we need to substitute into Equation S2.2 are

$$G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}},$$
$$M_* = 1.12\,\mathrm{M_\odot} = 1.12 \times 1.99 \times 10^{30}\,\mathrm{kg} = 2.23 \times 10^{30}\,\mathrm{kg},$$
$$P = 3.52\ \text{days} = 3.52 \times 24 \times 3600\,\mathrm{s} = 3.04 \times 10^5\,\mathrm{s}.$$

Consequently, we have

$$\begin{aligned} a &= \left(6.67 \times 10^{-11} \times 2.23 \times 10^{30}\left(\frac{3.04 \times 10^5}{6.28}\right)^2 \mathrm{N\,m^2\,kg^{-1}\,s^2}\right)^{1/3} \\ &= \left(3.485 \times 10^{29}\,\mathrm{kg\,m\,s^{-2}\,m^2\,kg^{-1}\,s^2}\right)^{1/3} \\ &= \left(3.485 \times 10^{29}\,\mathrm{m^3}\right)^{1/3} \\ &= 7.04 \times 10^9\,\mathrm{m}. \end{aligned} \qquad \text{(S2.3)}$$

We are given the radius, R_*, and simply need to convert this to metres:

$$R_* = 1.146\,\mathrm{R_\odot} = 1.146 \times 6.96 \times 10^8\,\mathrm{m} = 7.98 \times 10^8\,\mathrm{m}. \qquad \text{(S2.4)}$$

Consequently, substituting the values from Equations S2.3 and S2.4 into Equation 1.21, we have

$$\text{geometric transit probability} \approx \frac{R_*}{a} \approx \frac{7.98 \times 10^8\,\mathrm{m}}{7.04 \times 10^9\,\mathrm{m}} \approx 0.113.$$

The probability of a planet in a circular orbit like HD 209458 b's transiting from any random line of sight is approximately 11%, i.e. better than 1 in 10!

(b) The assumptions implicitly made by adopting Equation 1.21 are (i) that the orbit is randomly oriented, and (ii) that the orbit is circular. Since HD 209458 b was discovered by the radial velocity technique, whose sensitivity to a given planet decreases steadily as the orbital inclination decreases, the probability of HD 209458 b transiting was actually slightly higher than suggested by Equation 1.21. For a non-circular orbit, the planet spends time at a variety of distances, which will affect the probability of transiting. The probability depends on e and ω_{OP} as well as a. A final subtlety for eccentric orbits is that the planet moves more quickly when it is closer to the star (as prescribed by Kepler's second law, or equivalently the conservation of angular momentum), so factoring in the finite observational coverage renders the transits slightly less likely to be caught. To quantitatively assess the relative importance of these three factors requires more information than is given in the question.

Exercise 2.2 (a) We have

$$A_{\mathrm{RV}} = \frac{2\pi a M_{\mathrm{P}} \sin i}{(M_{\mathrm{P}} + M_*) P \sqrt{1 - e^2}} \qquad \text{(Eqn 1.13)}$$

and

$$a \approx \left(G(M_* + M_{\mathrm{P}}) \left(\frac{P}{2\pi} \right)^2 \right)^{1/3}. \qquad \text{(Eqn S2.1)}$$

Assuming a circular orbit, $e = 0$, and making the approximation $M_{\mathrm{P}} \ll M_*$, these become

$$A_{\mathrm{RV}} = \frac{2\pi a M_{\mathrm{P}} \sin i}{M_* P} \qquad \text{(S2.5)}$$

and

$$a = \left(G M_* \left(\frac{P}{2\pi} \right)^2 \right)^{1/3}. \qquad \text{(Eqn S2.2)}$$

Rearranging Equation S2.5, we have

$$M_{\mathrm{P}} \sin i = \frac{A_{\mathrm{RV}} M_* P}{2\pi a},$$

and substituting in for a, we obtain

$$\begin{aligned} M_{\mathrm{P}} \sin i &= \frac{A_{\mathrm{RV}} M_* P}{2\pi} \left(\frac{2\pi}{P} \right)^{2/3} \left(\frac{1}{G M_*} \right)^{1/3} \\ &= A_{\mathrm{RV}} \left(\frac{M_*^2 P}{2\pi G} \right)^{1/3}. \end{aligned} \qquad \text{(S2.6)}$$

A_{RV} and P are observables, and everything else on the right-hand side except for M_* is a constant, so this is the expression that we seek.

(b) The values for HD 209458 are

$$\begin{aligned} A_{\mathrm{RV}} &= 84.67 \pm 0.70\,\mathrm{m\,s^{-1}}, \\ M_* &= 1.12\,\mathrm{M_\odot} = 1.12 \times 1.99 \times 10^{30}\,\mathrm{kg} = 2.23 \times 10^{30}\,\mathrm{kg}, \\ P &= 3.52\ \text{days} = 3.52 \times 24 \times 3600\,\mathrm{s} = 3.04 \times 10^5\,\mathrm{s}, \end{aligned}$$

so we have

$$
\begin{aligned}
M_{\rm P} \sin i &= A_{\rm RV} \left(\frac{M_*^2 P}{2\pi G} \right)^{1/3} \\
&= 84.67\,{\rm m\,s^{-1}} \left(\frac{(2.23 \times 10^{30})^2\,{\rm kg}^2 \times 3.04 \times 10^5\,{\rm s}}{6.28 \times 6.67 \times 10^{-11}\,{\rm N\,m^2\,kg^{-2}}} \right)^{1/3} \\
&= 84.67 \times \left(3.609 \times 10^{75} \frac{{\rm kg}^2\,{\rm s}}{{\rm kg\,m\,s^{-2} m^2\,kg^{-2}}} \right)^{1/3} {\rm m\,s^{-1}} \\
&= 84.67 \times 1.53 \times 10^{25} \left({\rm kg^3\,m^{-3}\,s^3}\right)^{1/3} {\rm m\,s^{-1}} \\
&= 1.30 \times 10^{27}\,{\rm kg}.
\end{aligned}
$$

In the above, we explicitly kept the units of all the quantities when we substituted in, which allowed us to check that we (i) had got a dimensionally correct expression in part (a), and (ii) were using an appropriate choice of units for each quantity. If we had tried to use $M_* = 1.12\,{\rm M_\odot}$ in the expression, our final units would have been ${\rm M_\odot^{2/3}\,kg^{1/3}}$, alerting us that there was something amiss. Of course, assuming that the expression that we begin with is correct, if we use SI units throughout then we should always obtain an answer in SI units. Using this fact without working the units through wastes a valuable check of our working.

(c) We are given $i = 86.71° \pm 0.05°$, so $\sin i = 0.9984$. Consequently,

$$M_{\rm P} = \frac{1.30 \times 10^{27}\,{\rm kg}}{\sin i} = \frac{1.30 \times 10^{27}\,{\rm kg}}{0.9984} = 1.30 \times 10^{27}\,{\rm kg}.$$

Converting this to the other mass units requested:

$$M_{\rm P} = \frac{1.30 \times 10^{27}\,{\rm kg}}{1.90 \times 10^{27}\,{\rm kg\,M_J^{-1}}} = 0.684\,{\rm M_J}$$

and

$$M_{\rm P} = \frac{1.30 \times 10^{27}\,{\rm kg}}{1.99 \times 10^{30}\,{\rm kg\,M_\odot^{-1}}} = 6.53 \times 10^{-4}\,{\rm M_\odot}.$$

(d) Thus we can work out the ratio between $M_{\rm P}$ and M_* for the HD 209458 system:

$$\frac{M_{\rm P}}{M_*} = \frac{6.53 \times 10^{-4}\,{\rm M_\odot}}{1.12\,{\rm M_\odot}} = 5.83 \times 10^{-4}.$$

The planet's mass is a factor of 2 less than a thousandth that of the star. Since we have worked to a precision of only three significant figures, the approximation $M_{\rm P} \ll M_*$ was applicable.

(e) There are two basic approaches that could be used.

To tackle the problem *analytically*, the full versions of Kepler's third law (Equation 1.1) and Equation 1.13 would be used, and the algebra would need to carry through all the instances of $M_{\rm P}$. This would result in a much more complex expression, but it could still be solved.

To tackle the problem *iteratively*, we could replace the expression $(M_{\rm P} + M_*)$ with $M_{\rm total}$ in Kepler's third law (Equation 1.1) and Equation 1.13. Then the method in parts (a)–(d) could be used to evaluate M_{P} using the approximation $M_{\text{total}} \approx M_*$. This would be our first estimate for M_{P}, and we could then refine our estimate by using $M_{\text{total}} \approx M_* + M_{\mathrm{P}}$ and calculating a new value for M_{P}. This technique should be repeated until, at the precision required, the new value of M_{P} does not differ from the value used to estimate it.

Exercise 2.3 (a) The bandpass is broad and centred on 700 nm, so we will use the fiducial flux density value for the R band, which is centred on 700 nm. Since the value of the sky brightness is already given in pixel units in Table 2.1, we can slightly simplify the procedure outlined in the text. The sky flux density per pixel is given by applying the standard conversion from magnitudes:

$$m_1 - m_2 = 2.5 \log_{10}(F_2/F_1), \tag{S2.7}$$

where $m_1 = 6.8$ (from Table 2.1), $m_2 = 0.0$, the fiducial magnitude, F_1 is the flux density corresponding to m_1, and $F_2 = 1.74 \times 10^{-11}\,\mathrm{W\,m^{-2}\,nm^{-1}}$ (the R band value from Table A.4 in the Appendix). Making F_1 the subject of Equation S2.7, we have

$$\begin{aligned} F_1 &= F_2 \times 10^{-(m_1-m_2)/2.5} \\ &= 1.74 \times 10^{-11}\,\mathrm{W\,m^{-2}\,nm^{-1}} \times 10^{-6.8/2.5} \\ &= 1.74 \times 10^{-11} \times 1.91 \times 10^{-3}\,\mathrm{W\,m^{-2}\,nm^{-1}} \\ &= 3.32 \times 10^{-14}\,\mathrm{W\,m^{-2}\,nm^{-1}}. \end{aligned}$$

We will take 700 nm to be the typical wavelength of the radiation. Thus the typical photon energy is

$$\begin{aligned} E_{\text{ph}} = h\nu &= \frac{hc}{\lambda} \\ &= \frac{6.63 \times 10^{-34}\,\mathrm{J\,s} \times 3.00 \times 10^{8}\,\mathrm{m\,s^{-1}}}{700 \times 10^{-9}\,\mathrm{m}} = 2.84 \times 10^{-19}\,\mathrm{J}. \end{aligned}$$

This allows us to convert from flux to photon rate, l_{sky}:

$$\begin{aligned} l_{\text{sky}} = \frac{F_1}{E_{\text{ph}}} &= \frac{3.32 \times 10^{-14}\,\mathrm{W\,m^{-2}\,nm^{-1}}}{2.84 \times 10^{-19}\,\mathrm{J}} \\ &= 1.169 \times 10^{5}\,\mathrm{s^{-1}\,m^{-2}\,nm^{-1}}. \end{aligned}$$

We expect, therefore, $l_{\text{sky}} = 1.17 \times 10^5$ sky photons per second per square metre of telescope aperture per nanometre included in the bandpass to fall on each pixel of the PASS survey CCD.

Using the values that we are given: the telescope aperture is of diameter 2.5 cm, so the collecting area is $A = \frac{\pi}{4} \times 0.025^2\,\mathrm{m^2} = 4.91 \times 10^{-4}\,\mathrm{m^2}$. The bandpass is wide ($\Delta\lambda = 300$ nm), so the photon rate per pixel is

$$\begin{aligned} \frac{\mathrm{d}n_{\text{sky}}}{\mathrm{d}t} &= 1.169 \times 10^{5}\,\mathrm{s^{-1}\,m^{-2}\,nm^{-1}} \times A \times \Delta\lambda \\ &= 1.169 \times 10^{5}\,\mathrm{s^{-1}\,m^{-2}\,nm^{-1}} \times 4.91 \times 10^{-4}\,\mathrm{m^2} \times 300\,\mathrm{nm} \\ &= 1.72 \times 10^{4}\,\mathrm{s^{-1}}. \end{aligned}$$

We expect almost 20 000 sky photons per pixel per second for the PASS survey.

(b) The expected number of sky photons, n_{sky}, per pixel in a 10 s exposure is

$$n_{\text{sky}} = 1.72 \times 10^{4}\,\mathrm{s^{-1}} \times 10\,\mathrm{s} = 1.72 \times 10^{5},$$

or almost 200 000 sky photons per pixel per exposure.

Exercise 2.4 We have

$$F = \frac{L}{4\pi r^2}, \qquad \text{(Eqn 2.2)}$$

and we know that sources can be detected if $F \geq S$. This means that a source will be detected if

$$S \leq \frac{L}{4\pi r^2},$$

where we have substituted $S \leq F$ in Equation 2.2. The limiting distance will be when the flux exactly equals S, so that the limiting distance, d_{max}, corresponds to

$$S = \frac{L}{4\pi d_{\text{max}}^2}.$$

Making the distance the subject of the equation, this becomes

$$d_{\text{max}} = \left(\frac{L}{4\pi S}\right)^{1/2}, \qquad \text{(Eqn 2.3)}$$

as required

Exercise 2.5

(a) The arc length is $2a\theta$, where θ is the angle indicated in Figure S2.1. From this figure, we see that

$$\sin\theta = \frac{R_*}{a},$$

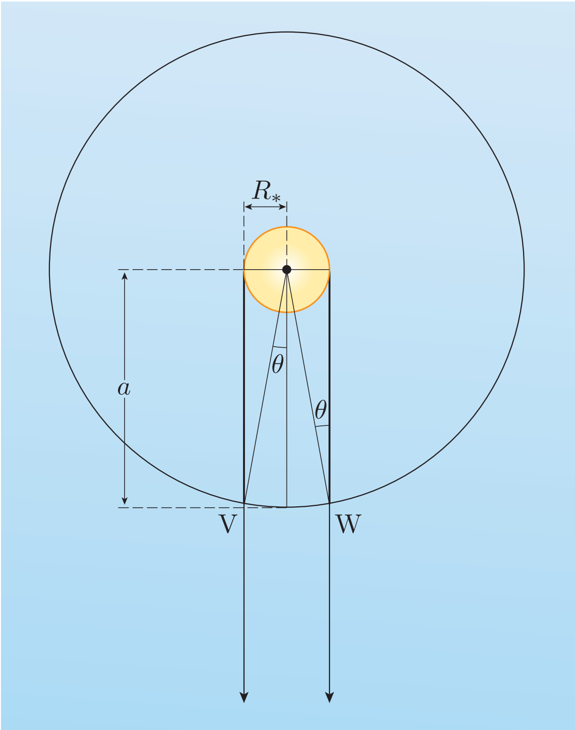

Figure S2.1 The two outer vertical lines indicate the path taken by light rays on each side of the star to a distant observer. The rays are parallel. Points V and W are the intersections of these lines with the planet's orbit.

so

$$\theta = \sin^{-1}\left(\frac{R_*}{a}\right).$$

Hence we have

$$\begin{aligned} T_{\text{dur}} &= P \times \frac{\text{length of arc from V to W}}{2\pi a} \\ &= P \times \frac{2a\sin^{-1}(R_*/a)}{2\pi a} \\ &= \frac{P}{\pi}\sin^{-1}\left(\frac{R_*}{a}\right). \end{aligned} \qquad \text{(S2.8)}$$

(b) The deviation between the exact and approximated values for T_{dur} is simply the difference between the right-hand sides of Equations 2.6 and S2.8:

$$\Delta T_{\text{dur}} = \frac{P}{\pi}\left(\sin^{-1}\left(\frac{R_*}{a}\right) - \frac{R_*}{a}\right). \qquad \text{(S2.9)}$$

The fractional deviation is simply ΔT_{dur} divided by T_{dur}, i.e.

$$\frac{\Delta T_{\text{dur}}}{T_{\text{dur}}} = \frac{\frac{P}{\pi}\left(\sin^{-1}\left(\frac{R_*}{a}\right) - \frac{R_*}{a}\right)}{\frac{P}{\pi}\sin^{-1}\left(\frac{R_*}{a}\right)} = 1 - \frac{\frac{R_*}{a}}{\sin^{-1}\left(\frac{R_*}{a}\right)}.$$

(c) As we saw in the solutions to Exercises 2.1 and 2.2, Kepler's third law tells us that

$$a = \left(\frac{P}{2\pi}\right)^{2/3}(GM_*)^{1/3}.$$

(d) To evaluate $\Delta T_{\text{dur}}/T_{\text{dur}}$, we need to use our expression for a for the specific values of $M_* = 1\,\mathrm{M_\odot}$ and orbital period $P = 1$ day. Generally, it is advisable to substitute in expressions to obtain the answer algebraically, before substituting in numerical values. In this case, however, the quantity that we seek depends only on the ratio R_*/a, and the algebraic expression for a is significantly more complicated than this.

So, evaluating a with

$$\begin{aligned} P &= 1 \text{ day} = 86\,400\,\mathrm{s}, \\ M_* &= 1.99 \times 10^{30}\,\mathrm{kg}, \\ G &= 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}, \end{aligned}$$

we have

$$\begin{aligned} a &= \left(\frac{P}{2\pi}\right)^{2/3}(GM_*)^{1/3} \\ &= \left(1.375 \times 10^4\,\mathrm{s}\right)^{2/3}\left(1.327 \times 10^{20}\,\mathrm{N\,m^2\,kg^{-1}}\right)^{1/3} \\ &= 5.740 \times 10^2 \times 5.101 \times 10^6\,\mathrm{s^{2/3}\,N^{1/3}\,m^{2/3}\,kg^{-1/3}} \\ &= 2.928 \times 10^9\,\mathrm{s^{2/3}\,(kg\,m\,s^{-2})^{1/3}\,m^{2/3}\,kg^{-1/3}} \\ &= 2.93 \times 10^9\,\mathrm{m}. \end{aligned}$$

In SI units, $1\,\mathrm{R}_\odot = 6.96 \times 10^8$ m, so

$$\frac{R_*}{a} = \frac{6.96 \times 10^8}{2.93 \times 10^9} = 0.238$$

and

$$\sin^{-1}\left(\frac{R_*}{a}\right) = \sin^{-1}(0.238) = 0.240,$$

so

$$\frac{\Delta T_{\mathrm{dur}}}{T_{\mathrm{dur}}} = 1 - \frac{0.238}{0.240} = 0.0083 \approx 1\%.$$

(e) Assuming that the star's properties remain constant, the semi-major axis will increase as $P^{2/3}$ with the increasing orbital period of the planet. The ratio R_*/a will therefore decrease as $P^{-2/3}$. The small angle approximation $\sin\theta \approx \theta$ becomes increasingly accurate as the angle, θ, decreases (where $\theta = R_*/a$ in this case). So for longer orbital periods, the approximation will be better, and the fractional deviation will be less than 1%. Demonstrating this numerically, for $P = 5.2$ days,

$$a(5.2\text{ days}) = a(1\text{ day}) \times \left(\frac{5.2}{1}\right)^{2/3} = 2.93 \times 10^9\text{ m} \times 3.00 = 8.79 \times 10^9\text{ m},$$

so

$$\frac{R_*}{a} = \frac{6.96 \times 10^8}{8.79 \times 10^9} = 0.0792$$

and

$$\sin^{-1}\left(\frac{R_*}{a}\right) = \sin^{-1}(0.0792) = 0.0793.$$

Consequently,

$$\frac{\Delta T_{\mathrm{dur}}}{T_{\mathrm{dur}}} = 1 - \frac{0.0792}{0.0793} \approx 0.1\%.$$

The approximation becomes ever more exact as the planet's orbital period is increased.

(f) The approximation used in Equation 2.6 is good to a precision of about 1% or better for the calculation of the durations of transits of planets in orbits $P > 1$ day around main sequence host stars. This precision should be sufficient for an estimate.

Exercise 2.6 We have

$$T_{\mathrm{dur}} \approx \frac{PR_*}{\pi a}. \qquad \text{(Eqn 2.6)}$$

Converting the values that we are given to metres and hours, we have

$$P = 1.3382 \times 24\,\mathrm{h} = 32.12\,\mathrm{h},$$

$$R_* = 1.15 \times 6.96 \times 10^8\text{ m} = 8.00 \times 10^8\text{ m},$$

$$a = 0.023 \times 1.5 \times 10^{11}\text{ m} = 3.45 \times 10^9\text{ m},$$

and substituting in these values gives

$$T_{\text{dur}} \approx \frac{32.12 \times 8.00 \times 10^8\,\text{h m}}{\pi \times 3.45 \times 10^9\,\text{m}} \approx 2.37\,\text{h}.$$

So the transit duration is 2.4 hours (to 2 s.f.).

Exercise 2.7 (a) Equation 2.34 tells us that the number of transiting planet discoveries predicted scales as $L_*^{3/2}$. If stars were more luminous, the signal received would be stronger, and the signal-to-noise ratio at any given distance would be better. This $N \propto L^{3/2}$ dependency is exactly the same as derived in Equation 2.5, and arises from the dependency of the survey volume on the luminosity of the sources (assuming a fixed flux threshold for detection).

(b) L_{SN} is the limiting signal-to-noise ratio and is proportional to the flux from the planet host star at the faint limit. This limiting host star brightness is an example of the limiting flux, S, in Equation 2.5. The number of planets is proportional to $L_{\text{SN}}^{-3/2}$ according to Equation 2.34, and consequently this equation expresses the same $N \propto S^{-3/2}$ relationship as Equation 2.5.

Exercise 2.8 Assuming, for simplicity, that the 16 cameras have been constantly operational since 2004, we can estimate the number of months of operation to date. At the time of writing (March 2009), this is about 5 years, or 60 months. Thus the number of data points is of the order of

$$\begin{aligned}\text{number of points} &= N_{\text{months}} \times n_{\text{cameras}} \times 5 \times 10^8 \\ &= 60 \times 16 \times 5 \times 10^8 \\ &= 4.8 \times 10^{11} \\ &= 5 \times 10^{11} \text{ (to 1 s.f.)}.\end{aligned}$$

Thus the archive needs to organize and store on the order of 10^{12} unique photometric data points.

Exercise 2.9 The minimum value of the ratio is almost zero: the value that occurs for a barely detectable grazing transit.

The maximum value will correspond to a central transit, i.e. impact parameter of $b = 0.0$. From the results of Exercise 2.5 we know that we can adopt the approximate expression for the transit duration

$$T_{\text{dur}}(b = 0.0) \approx \frac{PR_*}{\pi a}, \qquad \text{(Eqn 2.6)}$$

so the ratio that we must calculate is

$$\frac{T_{\text{dur}}(b = 0.0)}{P} \approx \frac{R_*}{\pi a}. \qquad \text{(S2.10)}$$

Substituting for a from Kepler's third law (Equation 1.1), i.e.

$$a = (G(M_* + M_{\text{P}}))^{1/3} \left(\frac{P}{2\pi}\right)^{2/3}$$

into Equation S2.10, we obtain

$$\begin{aligned}\frac{T_{\text{dur}}(b = 0.0)}{P} &\approx \frac{R_*}{\pi} \frac{2^{2/3}\pi^{2/3}}{P^{2/3}G^{1/3}M_*^{1/3}} \\ &\approx R_* \left(\frac{4}{\pi P^2 G M_*}\right)^{1/3}.\end{aligned}$$

From this we can see that for a given star, the ratio will be a maximum for the minimum value of the orbital period as

$$\frac{T_{\text{dur}}(b=0.0)}{P} \propto P^{-2/3}.$$

At the time of writing (October 2009) the shortest-period known planet is CoRoT-7 b with $P = 0.85$ days. Adopting this period and the mass and radius of the Sun, we can obtain an estimate for the maximum value for the ratio. The input values are

$$P = 0.85\ \text{days} = 0.85 \times 24 \times 3600\,\text{s} = 7.3 \times 10^4\,\text{s},$$
$$R_* = 6.96 \times 10^8\,\text{m},$$
$$M_* = 1.99 \times 10^{30}\,\text{kg},$$

so we have

$$\begin{aligned}\frac{T_{\text{dur}}(b=0.0)}{P} &\approx R_* \left(\frac{4}{\pi P^2 G M_*}\right)^{1/3} \\ &\approx 6.96 \times 10^8\,\text{m} \times \left(\frac{4}{\pi \times (7.3\times 10^4)^2 \times 6.67 \times 10^{-11} \times 1.99 \times 10^{30}\,\text{s}^2\,\text{N}\,\text{m}^2\,\text{kg}^{-2}\,\text{kg}}\right)^{1/3} \\ &\approx 0.085.\end{aligned}$$

The transit duration varies between 0% and 8.5% of a planet's orbital period.

Exercise 2.10 The radial velocity semi-amplitude is given by Equation 1.13 as

$$A_{\text{RV}} = \frac{2\pi a M_{\text{P}} \sin i}{(M_{\text{P}} + M_*)P(1-e^2)^{1/2}},$$

where the semi-major axis is given by Kepler's third law (Equation 1.1) as

$$a = \left[\frac{GP^2(M_* + M_{\text{P}})}{4\pi^2}\right]^{1/3}.$$

Substituting the second equation into the first, we have

$$A_{\text{RV}} = \left(\frac{2\pi G}{P}\right)^{1/3} \frac{M_{\text{P}} \sin i}{(M_* + M_{\text{P}})^{2/3}(1-e^2)^{1/2}},$$

and rearranging to make M_{P} the subject of the equation, we get

$$M_{\text{P}} = \frac{A_{\text{RV}}(M_* + M_{\text{P}})^{2/3}(1-e^2)^{1/2}}{\sin i} \left(\frac{P}{2\pi G}\right)^{1/3}.$$

Now, if we assume that the radial velocities measured in the two spectra represent the extremes of the reflex orbital motion of the star, the minimum orbital period of the putative planet must be 12 hours, and the minimum value of the radial velocity amplitude is given by the difference between the two radial velocity measurements. Since the star has the same spectral type as the Sun, clearly $M_* = \mathrm{M}_\odot$, and the mass term on the right-hand side has a minimum value of $M_*^{2/3}$. Also, we note that the maximum value of $\sin i$ is 1, so setting this at its limit will also give a minimum value for M_{P}. For such a short orbital period, the orbit would circularize, so we may assume $e \sim 0$, and the equation therefore reduces to

$$M_{\text{P}} \geq A_{\text{RV}} M_*^{2/3} \left(\frac{P}{2\pi G}\right)^{1/3}.$$

So, if the observed radial velocity changes are due to the reflex motion caused by an unseen orbiting body, the minimum mass of this object is

$$\begin{aligned}
M_{\mathrm{P}} &\geq 10^4\,\mathrm{m\,s^{-1}} \times (1.99 \times 10^{30}\,\mathrm{kg})^{2/3} \times \left(\frac{12 \times 3600\,\mathrm{s}}{2\pi \times 6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}}\right)^{1/3} \\
&\geq 7.4 \times 10^{28}\,\mathrm{m\,s^{-1}\,kg^{2/3}\,s^{1/3}\,N^{-1/3}\,m^{-2/3}\,kg^{2/3}} \\
&\geq 7.4 \times 10^{28}\,\mathrm{m\,s^{-1}\,kg^{2/3}\,s^{1/3}\,(kg\,m\,s^{-2})^{-1/3}\,m^{-2/3}\,kg^{2/3}} \\
&\geq 7.4 \times 10^{28}\,\mathrm{kg} \\
&\geq 39\,\mathrm{M_J}.
\end{aligned}$$

Since the putative planet has a mass in excess of almost 40 times the mass of Jupiter (about $0.037\,\mathrm{M_\odot}$), this object may be a brown dwarf, but cannot be a planet, and so can be eliminated from further follow-up.

Exercise 3.1 (a) The semi-major axis of the Earth's orbit is $a = 1$ AU and the period of the Earth's orbit is $P = 1$ year. So, from Equation 3.1, the Earth's orbital speed is

$$\begin{aligned}
v &= \frac{2\pi a}{P} \\
&= \frac{2\pi \times 1\,\mathrm{AU}}{1\ \mathrm{year}} \\
&= \frac{2\pi \times 1.496 \times 10^{11}\,\mathrm{m}}{365.25 \times 24 \times 3600\,\mathrm{s}} \\
&= 2.98 \times 10^4\,\mathrm{m\,s^{-1}}.
\end{aligned}$$

So the orbital speed of the Earth is about $30\,\mathrm{km\,s^{-1}}$.

(b) Although this planet is at the same distance from its star as the Earth is from the Sun, the orbital periods will differ because the star has a mass different to that of the Sun. We can calculate the orbital period from Kepler's third law (Equation 1.1) as

$$\begin{aligned}
P &= \left(\frac{4\pi^2 a^3}{G(M_* + M_{\mathrm{P}})}\right)^{1/2} \\
&\approx \left(\frac{4\pi^2 a^3}{GM_*}\right)^{1/2} \\
&\approx \left(\frac{4\pi^2 \times (1.496 \times 10^{11}\,\mathrm{m})^3}{6.673 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}} \times 0.5 \times 1.99 \times 10^{30}\,\mathrm{kg}}\right)^{1/2} \\
&\approx 4.46 \times 10^7\,\mathrm{s}.
\end{aligned}$$

This orbital period is about 1.41 years. So, using Equation 3.1 once again, the orbital speed of the planet is

$$\begin{aligned}
v &= \frac{2\pi a}{P} = \frac{2\pi \times 1\,\mathrm{AU}}{4.46 \times 10^7\,\mathrm{s}} \\
&= \frac{2\pi \times 1.496 \times 10^{11}\,\mathrm{m}}{4.46 \times 10^7\,\mathrm{s}} = 2.11 \times 10^4\,\mathrm{m\,s^{-1}}.
\end{aligned}$$

So the orbital speed of the planet is about $21\,\mathrm{km\,s^{-1}}$.

Exercise 3.2 (a) We start from Equation 3.4:

$$T_{\text{dur}} = \frac{P}{\pi} \sin^{-1}\left(\frac{\sqrt{(R_* + R_{\text{P}})^2 - a^2\cos^2 i}}{a}\right).$$

This can be rewritten as

$$T_{\text{dur}} = \frac{P}{\pi} \sin^{-1}\left[\left(\frac{R_*^2}{a^2} + \frac{2R_*R_{\text{P}}}{a^2} + \frac{R_{\text{P}}^2}{a^2} - \cos^2 i\right)^{1/2}\right].$$

Since $a \gg R_* \gg R_{\text{P}}$, the second and third terms inside the brackets are much smaller than the first, so

$$T_{\text{dur}} \approx \frac{P}{\pi} \sin^{-1}\left(\frac{R_*^2}{a^2} - \cos^2 i\right)^{1/2}.$$

Since the term in the argument of the inverse sine function will be small, and $\sin x \approx x$ for small values of x in radians, we have

$$T_{\text{dur}} \approx \frac{P}{\pi}\left(\frac{R_*^2}{a^2} - \cos^2 i\right)^{1/2},$$

as required.

(b) To use this equation to determine i, we need to first obtain a value for a. Kepler's third law tells us that

$$\frac{a^3}{P^2} \approx \frac{GM_*}{4\pi^2}.$$

Since we have a Sun-like star, the right-hand side is the same for the planet in the question and the Earth, so

$$\frac{a^3}{P^2} = \frac{\mathrm{a}_\oplus^3}{\mathrm{P}_\oplus^2},$$

where $\mathrm{a}_\oplus$ and $\mathrm{P}_\oplus$ are the semi-major axis and the orbital period of the Earth, respectively. Hence

$$a^3 = \mathrm{a}_\oplus^3\left(\frac{P}{\mathrm{P}_\oplus}\right)^2,$$

thus

$$\begin{aligned} a &= \mathrm{a}_\oplus\left(\frac{P}{\mathrm{P}_\oplus}\right)^{2/3} \\ &= 1\,\text{AU}\left(\frac{6\ \text{days}}{365.25\ \text{days}}\right)^{2/3} \\ &= 1.496 \times 10^{11}\ \text{m} \times (0.0164)^{2/3} \\ &= 9.67 \times 10^9\ \text{m}. \end{aligned}$$

Now we must recast the equation to solve for i:

$$T_{\text{dur}} \approx \frac{P}{\pi}\left(\frac{R_*^2}{a^2} - \cos^2 i\right)^{1/2},$$

so

$$\frac{T_{\text{dur}}^2 \pi^2}{P^2} \approx \left(\frac{R_*}{a}\right)^2 - \cos^2 i,$$

thus

$$\cos^2 i \approx \left(\frac{R_*}{a}\right)^2 - \frac{T_{\text{dur}}^2 \pi^2}{P^2},$$

giving

$$\cos i \approx \left[\left(\frac{R_*}{a}\right)^2 - \left(\frac{T_{\text{dur}}\pi}{P}\right)^2\right]^{1/2}$$

and finally

$$i \approx \cos^{-1}\left[\left(\frac{R_*}{a}\right)^2 - \left(\frac{T_{\text{dur}}\pi}{P}\right)^2\right]^{1/2}.$$

Now, substituting in $R_* = 1\,\mathrm{R}_\odot = 6.96 \times 10^8$ m, $a = 9.67 \times 10^9$ m, $T_{\text{dur}} = 2$ hours and $P = 6$ days $= 144$ hours, we have

$$\begin{aligned}
i &\approx \cos^{-1}\left[\left(\frac{6.96 \times 10^8\,\text{m}}{9.67 \times 10^9\,\text{m}}\right)^2 - \left(\frac{2\pi\,\text{h}}{144\,\text{h}}\right)^2\right]^{1/2} \\
&\approx \cos^{-1}\left[\left(\frac{6.96}{96.7}\right)^2 - \left(\frac{\pi}{72}\right)^2\right]^{1/2} \\
&\approx \cos^{-1}(0.0573) \\
&\approx 86.7^\circ.
\end{aligned}$$

(c) If $T_{\text{dur}} = 4$ h and all other parameters of the system are as above, then we have

$$\begin{aligned}
i &\approx \cos^{-1}\left[\left(\frac{6.96 \times 10^8\,\text{m}}{9.67 \times 10^9\,\text{m}}\right)^2 - \left(\frac{4\pi\,\text{h}}{144\,\text{h}}\right)^2\right]^{1/2} \\
&\approx \cos^{-1}\left[\left(\frac{6.96}{96.7}\right)^2 - \left(\frac{\pi}{36}\right)^2\right]^{1/2} \\
&\approx \cos^{-1}(-0.002\,43)^{1/2}.
\end{aligned}$$

Since the square root of a negative number is not a real number, this equation is invalid. That is, the transit duration is too long to be possible in such a system. We would be forced to conclude that the measurements are erroneous or that the candidate is an astrophysical mimic: probably it is either a blended eclipsing binary or a grazing eclipse binary.

Exercise 3.3 (a) The signal is $\Delta F = 0.0164F$, where we have used the depth as given in the caption of Figure 3.5. The noise is $1.1 \times 10^{-4}F$. So

$$\frac{\text{Signal}}{\text{Noise}} = \frac{0.0164F}{1.1 \times 10^{-4}F} = 149.$$

(b) The curvature in the transit floor causes a drop from approximately $0.9870F$ at second contact to approximately $0.9835F$ at mid-transit. The 'signal' level is therefore $(0.9870 - 0.9835)F = 0.0035F$.

If the scatter was more than about 0.3% (or $3 \times 10^{-3}F$), it would be difficult to detect the curvature.

Exercise 3.4 As the light emerging from close to the limb of the planet has $\gamma = 80°$, so $\mu = \cos\gamma = 0.174$.

The linear limb darkening law (Equation 3.9) gives

$$\frac{I(\mu)}{I(1)} = 1 - u(1-\mu),$$

so

$$\begin{aligned}\frac{I(0.174)}{I(1)} &= 1 - 0.215 \times (1 - 0.174)\\ &= 0.82.\end{aligned}$$

The logarithmic limb darkening law (Equation 3.10) gives

$$\frac{I(\mu)}{I(1)} = 1 - u_l(1-\mu) - \nu_l \mu \ln \mu,$$

so

$$\begin{aligned}\frac{I(0.174)}{I(1)} &= 1 - 0.14 \times (1 - 0.174) + 0.12 \times 0.174 \times \ln 0.174\\ &= 1 - 0.116 + 0.036\\ &= 0.92.\end{aligned}$$

The quadratic limb darkening law (Equation 3.11) gives

$$\frac{I(\mu)}{I(1)} = 1 - u_\mathrm{q}(1-\mu) - \nu_\mathrm{q}(1-\mu)^2,$$

so

$$\begin{aligned}\frac{I(0.174)}{I(1)} &= 1 - 0.29 \times (1 - 0.174) + 0.13 \times (1 - 0.174)^2\\ &= 1 - 0.240 + 0.089\\ &= 0.85.\end{aligned}$$

Clearly, these three limb darkening prescriptions give markedly different amounts of limb darkening. The form of limb darkening law adopted *does* make a difference.

Exercise 3.5 The impact parameter is $b = 3\,\mathrm{R}_\odot/4$, and the inclination angle is $i = 86.5°$, so the semi-major axis of the orbit is

$$a = b/\cos i = 3\,\mathrm{R}_\odot/4\cos 86.5° = 12.3\,\mathrm{R}_\odot$$

or

$$a = 12.3 \times 6.96 \times 10^8\,\mathrm{m} = 8.56 \times 10^9\,\mathrm{m}.$$

Using Kepler's third law (Equation 1.1), the period of the orbit is

$$\begin{aligned} P_{\text{orb}} &= \left(\frac{4\pi^2 a^3}{GM_*}\right)^{1/2} \\ &= \left(\frac{4\pi^2 \times (8.56 \times 10^9\,\text{m})^3}{6.673 \times 10^{-11}\,\text{N}\,\text{m}^2\,\text{kg}^{-2} \times 1.99 \times 10^{30}\,\text{kg}}\right)^{1/2} \\ &= 4.32 \times 10^5\,\text{s}. \end{aligned}$$

The orbital period is therefore 120 hours (or 5 days).

1 hour before or after mid-transit therefore corresponds to a phase angle of

$$\pm\omega t = \frac{\pm 2\pi t}{P_{\text{orb}}} = \frac{\pm(2\pi \times 1\,\text{h})}{120\,\text{h}} = \pm 0.052\ \text{radians}.$$

We therefore need to work out the position of the planet in three cases:
(i) $\omega t = 2\pi - 0.052 = 6.231$ radians, (ii) $\omega t = 0$ radians, (iii) $\omega t = 0.052$ radians.

(i) In this case, the horizontal component of the planet's position, with respect to the centre of the star's disc, is

$$a \sin \omega t = 12.3\,\text{R}_\odot \times \sin(6.231\ \text{radians}) = -0.64\,\text{R}_\odot.$$

The vertical component of the planet's position, with respect to the centre of the star's disc, is

$$a \cos i \cos \omega t = 12.3\,\text{R}_\odot \times \cos 86.5^\circ \times \cos(6.231\ \text{radians}) = 0.75\,\text{R}_\odot.$$

(ii) In this case, the horizontal component of the planet's position, with respect to the centre of the star's disc, is

$$a \sin \omega t = 12.3\,\text{R}_\odot \times \sin(0\ \text{radians}) = 0.$$

The vertical component of the planet's position, with respect to the centre of the star's disc, is

$$a \cos i \cos \omega t = 12.3\,\text{R}_\odot \times \cos 86.5^\circ \times \cos(0\ \text{radians}) = 0.75\,\text{R}_\odot.$$

(iii) In this case, the horizontal component of the planet's position, with respect to the centre of the star's disc, is

$$a \sin \omega t = 12.3\,\text{R}_\odot \times \sin(0.052\ \text{radians}) = 0.64\,\text{R}_\odot.$$

The vertical component of the planet's position, with respect to the centre of the star's disc, is

$$a \cos i \cos \omega t = 12.3\,\text{R}_\odot \times \cos 86.5^\circ \times \cos(0.052\ \text{radians}) = 0.75\,\text{R}_\odot.$$

The curvature of the locus of the planet is too small to notice when working at this level of precision, and the locus of the transit is as shown in Figure S3.1.

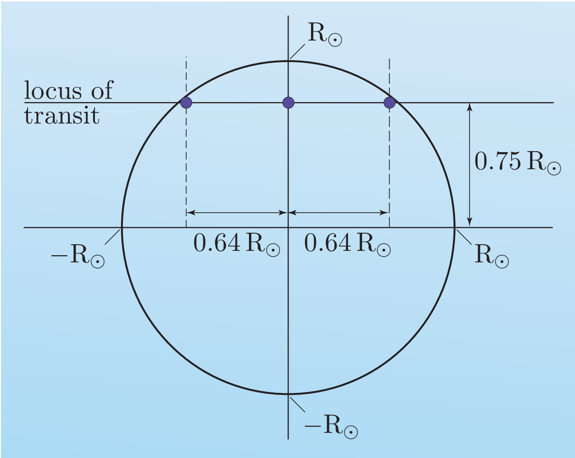

Figure S3.1 The locus of the planetary transit.

Exercise 3.6 (a) Since we are told that the star has negligible limb darkening, we may use the relationships appropriate to a stellar disc of uniform brightness. The observed flux is given by

$$\begin{aligned} F(t) &= F_{\text{unocculted}} - \Delta F \\ &= \pi I_0 R_*^2 - I_0\, A_{\text{e}}(t) \\ &= I_0\left(\pi R_*^2 - A_{\text{e}}(t)\right), \end{aligned} \tag{S3.1}$$

where

$$A_{\text{e}} = \begin{cases} 0 & \text{if } 1+p<\xi, \\ R_*^2\left(p^2\alpha_1+\alpha_2-\dfrac{\sqrt{4\xi^2-(1+\xi^2-p^2)^2}}{2}\right) & \text{if } 1-p<\xi\leq 1+p, \\ \pi p^2 R_*^2 & \text{if } 1-p\geq\xi, \end{cases}$$

(Eqn 3.27)

with

$$p=\frac{R_{\text{P}}}{R_*},\quad \cos\alpha_1=\frac{p^2+\xi^2-1}{2\xi p},\quad \cos\alpha_2=\frac{1+\xi^2-p^2}{2\xi},$$

$$\xi=\frac{a}{R_*}\left(\sin^2\omega t+\cos^2 i\cos^2\omega t\right)^{1/2}\quad\text{and}\quad \omega=\frac{2\pi}{P_{\text{orb}}}.$$

(b) For $p = 0.1$ and $\xi = 0.2$, clearly $1 - p \geq \xi$, so the third case of Equation 3.27 is appropriate, which corresponds to the planet falling entirely within the stellar disc. In this case, $A_{\text{e}} = \pi p^2 R_*^2 = \pi R_{\text{P}}^2$, so the change in flux is $\Delta F = I_0 A_{\text{e}} = I_0 \pi R_{\text{P}}^2$. The relative change in flux is therefore

$$\frac{\Delta F}{F}=\frac{I_0\pi R_{\text{P}}^2}{I_0\pi R_*^2}=\frac{R_{\text{P}}^2}{R_*^2},$$

as required.

(c) The orbital inclination, i, is known from the transit duration, T_{dur}. First contact, at time $t = t_1$, occurs when $\xi_1 = 1 + p$, and second contact, at time $t = t_2$, occurs when $\xi_2 = 1 - p$. Knowing these two times, and knowing the transit duration, T_{dur}, and the orbital period, P, we can evaluate ξ for the two contact points:

$$\xi_1 = 1 + p = \frac{a}{R_*}\left(\sin^2\omega t_1 + \cos^2 i \cos^2\omega t_1\right)^{1/2} = \frac{a}{R_*}\Gamma_1,$$

$$\xi_2 = 1 - p = \frac{a}{R_*}\left(\sin^2\omega t_2 + \cos^2 i \cos^2\omega t_2\right)^{1/2} = \frac{a}{R_*}\Gamma_2,$$

where Γ_1 and Γ_2 have been introduced as shorthand for the known, but complicated, expressions involving the measured times and the determined orbital inclination. Consequently, we have two expressions relating p, a and R_*. We can therefore treat these as simultaneous equations, eliminate a, and make p the subject of the resulting equation:

$$p = \frac{\Gamma_1 - \Gamma_2}{\Gamma_1 + \Gamma_2}. \tag{S3.2}$$

We recall that $p = R_{\text{P}}/R_*$, so Equation S3.2 gives us the ratio of the star and planet radii in terms of the measured contact times and the inclination.

(d) The method illustrated in part (b) depends on measuring the depth of the transit, ΔF, and relating this to the fraction of the stellar flux occulted, which is assumed to be proportional to the area of the planet's disc. If the star is limb darkened, the definition of ΔF is complicated: if we use the depth at the centre of the transit, this will probably result in an erroneously large planet size. At mid-transit the occulted fraction of the total stellar flux is likely to be greater than $(R_{\text{P}}/R_*)^2$ because the bright central regions of the stellar disc are being occulted. Limb darkening seriously affects the planet size deduced from this method.

On the other hand, the method illustrated in part (c) depends only on measuring the timing at which the contact points occur. So long as the second and third contacts can be clearly identified in the light curve, this method is unaffected by the limb darkening.

Exercise 3.7

(a) A large planet will give rise to a prolonged ingress and egress, and the transit will be relatively deep; a large impact factor means that the duration of the transit is relatively short; and very little limb darkening implies that the transit floor will be flat. A sketch of such a transit is shown in Figure S3.2a.

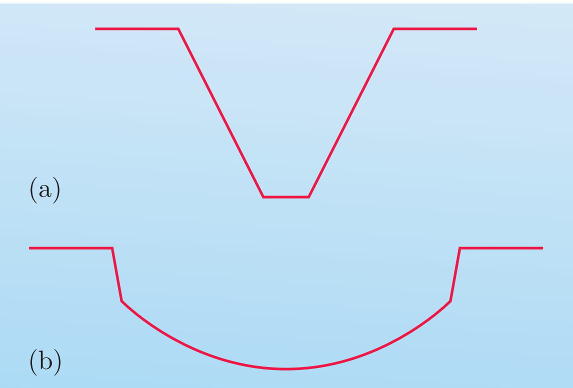

Figure S3.2 (a) A transit light curve for a large planet with a large impact factor and very little limb darkening. (b) A transit light curve for a small planet with a small impact factor and significant limb darkening.

(b) A small planet will give rise to a short ingress and egress, and the transit will be relatively shallow; a small impact factor means that the duration of the transit is relatively long; and significant limb darkening implies that the transit floor will be curved. A sketch of such a transit is shown in Figure S3.2b.

Exercise 4.1 We need to use Kepler's third law, but recognize that a planet in orbit around a solar-mass star with an orbital period of 1 year will have a semi-major axis of 1 AU. Hence we can write

$$\frac{a^3}{M_* P^2} = \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}.$$

(a) (i) In this case, the orbital period is given by

$$\begin{aligned} P &= \left(\frac{a^3}{M_*} \times \frac{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\ &= \left(\frac{(1\,\mathrm{AU})^3}{0.7\,\mathrm{M}_\odot} \times \frac{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\ &= \left(\frac{1}{0.7}\right)^{1/2} \mathrm{yr} = 1.195\,\mathrm{yr} = 437\ \mathrm{days}. \end{aligned}$$

(ii) In this case, the orbital period is given by

$$\begin{aligned} P &= \left(\frac{a^3}{M_*} \times \frac{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\ &= \left(\frac{(1\,\mathrm{AU})^3}{1.5\,\mathrm{M}_\odot} \times \frac{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\ &= \left(\frac{1}{1.5}\right)^{1/2} \mathrm{yr} = 0.816\,\mathrm{yr} = 298\ \mathrm{days}. \end{aligned}$$

(b) (i) In this case, the semi-major axis of the orbit is given by

$$\begin{aligned} a &= \left(M_* P^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}\right)^{1/3} \\ &= \left(0.7\,\mathrm{M}_\odot \times (500\ \mathrm{days})^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}\right)^{1/3} \\ &= (0.7)^{1/3} \times \left(\frac{500}{365.25}\right)^{2/3} \mathrm{AU} = 1.095\,\mathrm{AU}. \end{aligned}$$

(ii) In this case, the semi-major axis of the orbit is given by

$$\begin{aligned} a &= \left(M_* P^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}\right)^{1/3} \\ &= \left(1.5\,\mathrm{M}_\odot \times (500\ \mathrm{days})^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M}_\odot \times (1\,\mathrm{yr})^2}\right)^{1/3} \\ &= (1.5)^{1/3} \times \left(\frac{500}{365.25}\right)^{2/3} \mathrm{AU} = 1.411\,\mathrm{AU}. \end{aligned}$$

(c) (i) For a planet in a 1 AU orbit around a late M type star with $M_* = 0.2\,\mathrm{M_\odot}$, the orbital period would be

$$\begin{aligned}
P &= \left(\frac{a^3}{M_*} \times \frac{1\,\mathrm{M_\odot} \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\
&= \left(\frac{(1\,\mathrm{AU})^3}{0.2\,\mathrm{M_\odot}} \times \frac{1\,\mathrm{M_\odot} \times (1\,\mathrm{yr})^2}{(1\,\mathrm{AU})^3}\right)^{1/2} \\
&= \left(\frac{1}{0.2}\right)^{1/2} \mathrm{yr} = 2.236\,\mathrm{yr} = 817\ \mathrm{days}.
\end{aligned}$$

(ii) For a planet in a 500-day orbit around a late M type star with $M_* = 0.2\,\mathrm{M_\odot}$, the semi-major axis would be

$$\begin{aligned}
a &= \left(M_* P^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M_\odot} \times (1\,\mathrm{yr})^2}\right)^{1/3} \\
&= \left(0.2\,\mathrm{M_\odot} \times (500\ \mathrm{days})^2 \times \frac{(1\,\mathrm{AU})^3}{1\,\mathrm{M_\odot} \times (1\,\mathrm{yr})^2}\right)^{1/3} \\
&= (0.2)^{1/3} \times \left(\frac{500}{365.25}\right)^{2/3} \mathrm{AU} = 0.721\,\mathrm{AU}.
\end{aligned}$$

(d) Exoplanets have been discovered around stars with a range of masses, from late M type with $M_* \sim 0.2\,\mathrm{M_\odot}$ to stars of mass significantly greater than that of the Sun. A planetary orbit with a given semi-major axis can therefore correspond to a range of orbital periods, spanning a factor of almost 3 in the examples above. Similarly, a planetary orbit with a given period can correspond to a range of orbital semi-major axes, spanning a factor of around 2 in the examples above.

Exercise 4.2 (a) Figure 1.4 in the box on 'The nearest stars and planets' shows that of the roughly 330 known stars within 10 pc, almost 240 are M stars, i.e. about 72% of the stars (including the white dwarfs) are M stars. Of the A–M spectral type stars, about 80% are M stars. We are not told how many of these stars are main sequence stars, and we will assume that they all are; this is justified because stars spend most of their lifetimes on the main sequence. Reading from Figure 4.8c, the fraction of M stars hosting one or more planets is $\sim 2 \pm 1\%$. Thus we expect the number of M stars within 10 pc hosting RV planets to be

$$N_{\mathrm{P,M}} \approx 240 \times (0.02 \pm 0.01) \approx 4.8 \pm 2.4.$$

Similarly, there are 71 F, G, K stars within 10 pc, and reading off the graph, the fraction of these hosting planets is $\sim 4.1 \pm 0.6\%$. Thus we expect the number of F, G, K stars hosting RV planets to be

$$N_{\mathrm{P,FGK}} \approx 71 \times (0.041 \pm 0.006) \approx 2.9 \pm 0.4.$$

Finally, there are 4 A stars within 10 pc in the RECONS census, and for the highest-mass F stars and the A stars, Figure 4.8c suggests that $\sim 9 \pm 3\%$ host giant planets. Thus

$$N_{\mathrm{P,A}} \approx 4 \times (0.09 \pm 0.03) \approx 0.36 \pm 0.12 \approx 0.4 \pm 0.1.$$

Gathering this together, the total number of systems hosting RV-detectable giant planets within 10 pc is expected to be about

$$N_{\mathrm{P,RV}} \approx (4.8 \pm 2.4) + (2.9 \pm 0.4) + (0.4 \pm 0.1) \approx 8.1 \pm 2.9.$$

Thus the fraction of stars within 10 pc hosting RV-detectable planets is expected to be

$$\text{fraction hosting planets} = \frac{\text{number hosting planets}}{\text{total number}} \approx \frac{8.1 \pm 2.9}{330} \approx 0.02 \pm 0.009.$$

(b) Over the mass range shown in Figure 4.8c, i.e. 0.1–$1.9\,\mathrm{M}_\odot$, the main sequence luminosity increases from $10^{-3}\,\mathrm{L}_\odot$ to $10\,\mathrm{L}_\odot$, i.e. by a factor of 10^4. Equation 2.2 tells us that the flux from a source decreases as the inverse square of the distance, and consequently the distance at which a star of luminosity L is brighter than V ~ 10 varies as

$$d_{\text{max}} = \left(\frac{L}{4\pi S_{10}}\right)^{1/2}, \qquad \text{(Eqn 2.3)}$$

where we have used the notation S_{10} for the flux corresponding to magnitude V ~ 10. Thus the most massive A stars in the range are visible at distances about 100 times the limiting distance for the least massive M stars in the range. Since the volume of space included within a limiting distance d_{max} is given by

$$\text{volume} \propto d_{\text{max}}^3,$$

the magnitude limit includes A stars in a volume 10^6 times the volume for M stars. If we assume that our local volume is representative of the relative numbers of A stars and M stars, then Figure 1.4 suggests that the ratio of A stars to M stars will be roughly

$$\frac{N_{\text{A}}}{N_{\text{M}}} \approx \frac{4 \times 10^6}{239} \approx 10^4. \qquad \text{(S4.1)}$$

Thus the magnitude-limited sample is strongly biased in favour of the more luminous, more massive stars. The number that we obtained in Equation S4.1 is an overestimate of the ratio of the numbers in the lowest-mass bin and the highest-mass bin in Figure 4.8c since we used the extremes ($M_* = 0.1\,\mathrm{M}_\odot$ and $M_* = 1.9\,\mathrm{M}_\odot$) rather than the average masses within the bins for the estimate, but it illustrates the point.

Since the magnitude-limited sample is dominated by the highest-mass bin, we expect that the percentage of stars with RV-detectable planets will be more or less the value for that bin, i.e. we expect that the percentage of stars with planets will be $\sim 9 \pm 3\%$, where we have read the number from Figure 4.8c.

(c) The two estimates are different. The answer in part (a) is effectively the fraction for the lowest-mass bin in Figure 4.8c because the volume-limited sample is dominated by these lowest-mass stars. The answer in part (b) is effectively the fraction for the highest-mass bin in Figure 4.8c because the flux-limited sample is dominated by these luminous high-mass stars.

(d) The answer in part (a) suggests that we should expect roughly 8 RV-detectable planets within 10 pc. Figure 1.4 shows 18 planets, but this includes the 8 in the Solar System, so it shows 10 exoplanets. This is consistent with our estimate, which is not surprising as both Figure 1.4 and Figure 4.8c draw on the known census of exoplanets. We should expect more exoplanets to be discovered within 10 pc, however, as the RV precision of available instrumentation improves, and longer time bases are sampled. These factors will allow previously undiscovered exoplanets to be detected.

Exercise 4.3 The reflex RV amplitude is given by Equation 1.13 as

$$A_{\mathrm{RV}} = \frac{2\pi a M_{\mathrm{P}} \sin i}{(M_{\mathrm{P}} + M_*) P \sqrt{1-e^2}}.$$

In all the cases here, the planetary mass is negligible compared to that of the star, so $M_{\mathrm{P}} + M_* \approx M_*$.

Hence for GJ 581 b, the reflex RV amplitude of the star is

$$\begin{aligned} A_{\mathrm{RV,b}} &= \frac{2\pi \times (0.041 \times 1.496 \times 10^{11}\,\mathrm{m}) \times (15.64 \times 5.97 \times 10^{24}\,\mathrm{kg})}{(0.31 \times 1.99 \times 10^{30}\,\mathrm{kg}) \times (5.369 \times 24 \times 3600\,\mathrm{s})} \\ &= 12.6\,\mathrm{m\,s^{-1}}. \end{aligned}$$

Similarly, for GJ 581 c

$$\begin{aligned} A_{\mathrm{RV,c}} &= \frac{2\pi \times (0.07 \times 1.496 \times 10^{11}\,\mathrm{m}) \times (5.36 \times 5.97 \times 10^{24}\,\mathrm{kg})}{(0.31 \times 1.99 \times 10^{30}\,\mathrm{kg}) \times (12.929 \times 24 \times 3600\,\mathrm{s}) \times (1-0.17)^{1/2}} \\ &= 3.4\,\mathrm{m\,s^{-1}}, \end{aligned}$$

for GJ 581 d

$$\begin{aligned} A_{\mathrm{RV,d}} &= \frac{2\pi \times (0.22 \times 1.496 \times 10^{11}\,\mathrm{m}) \times (7.09 \times 5.97 \times 10^{24}\,\mathrm{kg})}{(0.31 \times 1.99 \times 10^{30}\,\mathrm{kg}) \times (66.8 \times 24 \times 3600\,\mathrm{s}) \times (1-0.38)^{1/2}} \\ &= 3.1\,\mathrm{m\,s^{-1}}, \end{aligned}$$

and for GJ 581 e

$$\begin{aligned} A_{\mathrm{RV,e}} &= \frac{2\pi \times (0.03 \times 1.496 \times 10^{11}\,\mathrm{m}) \times (1.94 \times 5.97 \times 10^{24}\,\mathrm{kg})}{(0.31 \times 1.99 \times 10^{30}\,\mathrm{kg}) \times (3.149 \times 24 \times 3600\,\mathrm{s})} \\ &= 1.9\,\mathrm{m\,s^{-1}}. \end{aligned}$$

If, at a given time, the contributions to the reflex RV amplitude from each planet are all in phase with each other, then the maximum total amplitude that may be observed is simply $(12.6 + 3.4 + 3.1 + 1.9)\,\mathrm{m\,s^{-1}} = 21\,\mathrm{m\,s^{-1}}$.

Exercise 4.4 In Worked Example 4.1 we calculated that the pressure in the core of Jupiter is approximately 10^{13} Pa, or 10^8 bar. So Figure 4.12 indicates that the core temperature of Jupiter is roughly $\log(T/\mathrm{K}) = 4.4$, or $T = 25\,000\,\mathrm{K}$. The degeneracy parameter is $\theta = 0.03$. We can therefore rearrange Equation 4.18 to give the number density of electrons as

$$\begin{aligned} n &= \left(\frac{2\pi m k T_{\mathrm{F}}}{h^2}\right)^{3/2} \\ &= \left(\frac{2\pi m k T}{\theta h^2}\right)^{3/2} \\ &= \left(\frac{2\pi \times (9.109 \times 10^{-31}\,\mathrm{kg}) \times (1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}) \times (25\,000\,\mathrm{K})}{0.03 \times (6.626 \times 10^{-34}\,\mathrm{J\,s})^2}\right)^{3/2} \\ &= 1.8 \times 10^{30}\,(\mathrm{kg\,J^{-1}\,s^{-2}})^{3/2} \\ &= 1.8 \times 10^{30}\,(\mathrm{kg\,kg^{-1}\,m^{-2}\,s^{2}\,s^{-2}})^{3/2} \\ &= 1.8 \times 10^{30}\,\mathrm{m^{-3}}. \end{aligned}$$

Exercise 4.5 (a) The energy radiated away is $E_{\rm rad} = L\,\Delta t$, and the change in gravitational energy is $\Delta E_{\rm GR} = ({\rm d}E_{\rm GR}/{\rm d}t)\,\Delta t$. We have

$$L = -\frac{\zeta - 1}{\zeta}\frac{{\rm d}E_{\rm GR}}{{\rm d}t}, \qquad \text{(Eqn 4.25)}$$

so

$$E_{\rm rad} = L\,\Delta t = -\frac{\zeta - 1}{\zeta}\frac{{\rm d}E_{\rm GR}}{{\rm d}t}\,\Delta t = -\frac{\zeta - 1}{\zeta}\,\Delta E_{\rm GR}, \qquad \text{(S4.2)}$$

which is the general expression that we require.

(b) In the case of an ideal diatomic gas, we know that $\zeta = 3.2$, so using Equation S4.2 we have

$$E_{\rm rad} = -\frac{\zeta - 1}{\zeta}\,\Delta E_{\rm GR} = -\frac{2.2}{3.2}\,\Delta E_{\rm GR} = -0.69\,\Delta E_{\rm GR}.$$

(c) As the planet contracts, its gravitational energy decreases, and as calculated above, around 70% of this energy can be radiated away. The remainder of the 'lost' gravitational energy goes to increasing the internal energy of the material of which the planet is composed. Hence conservation of energy is maintained.

(d) The temperature, pressure and density of the H_2 gas during the collapse must all increase. As noted above, the internal energy of the gas will increase, and this will raise the temperature. Since the same amount of gas is contained within a smaller volume, an increase in temperature implies an increase in pressure, via the ideal gas law $PV = nkT$. Similarly, since the mass is constant but the volume is decreasing, the density of the gas must also increase.

Exercise 5.1 (a) Using Equation 5.5 we have

$$T_{\rm eq,\oplus} = \frac{1}{2}\left[\frac{(1 - {\rm A}_\oplus)\,{\rm L}_\odot}{\sigma\pi {\rm a}_\oplus^2}\right]^{1/4},$$

and we are told that ${\rm A}_\oplus = 0.30$. The other constants that we require are

$${\rm L}_\odot = 3.83 \times 10^{26}\,{\rm J\,s^{-1}},$$
$${\rm a}_\oplus = 1\,{\rm AU} = 1.496 \times 10^{11}\,{\rm m},$$
$$\sigma = 5.671 \times 10^{-8}\,{\rm J\,m^{-2}\,K^{-4}\,s^{-1}}.$$

Substituting in, we have

$$\begin{aligned} T_{\rm eq,\oplus} &= \frac{1}{2}\left[\frac{(1 - 0.3)3.83 \times 10^{26}\,{\rm J\,s^{-1}}}{5.67 \times 10^{-8}\,{\rm J\,m^{-2}\,K^{-4}\,s^{-1}} \times 3.14 \times (1.50 \times 10^{11}\,{\rm m})^2}\right]^{1/4} \\ &= \frac{1}{2}\left[\frac{0.7 \times 3.83 \times 10^{26}\,{\rm K}^4}{5.67 \times 10^{-8} \times 3.14 \times (1.50 \times 10^{11})^2}\right]^{1/4} \\ &= 255\,{\rm K}, \end{aligned}$$

where we have restricted ourselves to three significant figures in the calculations as we were given ${\rm A}_\oplus$ to only two significant figures.

(b) The temperature that we have calculated for the Earth is about $-19°{\rm C}$, which is in reasonable agreement with the actual temperature that we experience here on Earth. It is, of course, slightly cooler than the actual temperature, but there are a number of assumptions that are not completely valid. For example, the Earth does not have a uniform temperature and it does not radiate as a perfect black body.

Exercise 5.2 We have

$$T_{\mathrm{eq}} = \frac{1}{2}\left[\frac{(1-A)L_*}{\sigma\pi a^2}\right]^{1/4}. \qquad \text{(Eqn 5.5)}$$

Since we are just asked for an approximate numerical value, we will not consider the error range that we have been given for L_*, and will simply adopt $L_* \approx 1.61\,\mathrm{L}_\odot$. Substituting in the values that we are given, therefore, we have

$$\begin{aligned}
T_{\mathrm{eq}} &\approx \frac{1}{2}\left[\frac{1.61 \times 3.83 \times 10^{26}\,\mathrm{J\,s^{-1}}}{5.67 \times 10^{-8}\,\mathrm{J\,m^{-2}\,K^{-4}\,s^{-1}} \times 3.14(0.0471)^2(1.50 \times 10^{11}\,\mathrm{m})^2}\right]^{1/4} (1-A)^{1/4} \\
&\approx \tfrac{1}{2}\left[6.98 \times 10^{13}\,\mathrm{K}^4\right]^{1/4}(1-A)^{1/4} \\
&\approx \tfrac{1}{2}\,[2890]\,(1-A)^{1/4}\,\mathrm{K} \\
&\approx 1400(1-A)^{1/4}\,\mathrm{K},
\end{aligned}$$

as required.

Exercise 5.3 (a) Referring to the figure, the transit depth is around 15%, i.e. $\Delta F/F = 0.15$.

(b) We know that the transit depth is related to the ratio of the stellar and planetary radii by

$$\frac{\Delta F}{F} = \frac{R_\mathrm{P}^2}{R_*^2}. \qquad \text{(Eqn 1.18)}$$

Consequently, the transit depth of HD 209458 b at the wavelength of Lyman α implies that

$$\frac{R_\mathrm{P}^2}{R_*^2} = 0.15$$

and so

$$\begin{aligned}
R_\mathrm{P} &= \sqrt{0.15}R_* \\
&= 0.387 \times 1.15 \times 6.96 \times 10^8\,\mathrm{m} \\
&= 3.1 \times 10^8\,\mathrm{m} \\
&\approx \tfrac{1}{2}\,\mathrm{R}_\odot \\
&\approx 4\,\mathrm{R_J}.
\end{aligned}$$

This inferred value for R_P is approximately three times bigger than the currently accepted value for the radius of the planet, as it must be since the transit depth in Lyman α is approximately ten times deeper than the depth in the optical continuum. As we saw in Section 4.5, no known giant planets have radii this big, nor do models predict planetary radii of this size (cf. Section 4.4).

(c) The occulted area of the star in Lyman α is far larger than the area of the planet's disc as inferred from the optical continuum. This suggests that the planet has a large cloud of hydrogen surrounding it, so the area that absorbs in the spectral lines of hydrogen is far larger than the area of the planet's disc. The cloud of hydrogen will be transparent except at the wavelengths absorbed by hydrogen atoms. The most obvious explanation for the existence of this cloud surrounding the planet is that it is being evaporated off the surface of the planet by the intense irradiation of the nearby host star.

Exercise 5.4 (a) Equation 1.13 is

$$A_{\rm RV} = \frac{2\pi a M_{\rm P} \sin i}{(M_{\rm P} + M_*)P\sqrt{1-e^2}}.$$

Comparing this with the expression that we require, it is clear that we need to eliminate a from the equation. We can do this by using Kepler's third law

$$\frac{a^3}{P^2} = \frac{G(M_* + M_{\rm P})}{4\pi^2}, \qquad \text{(Eqn 1.1)}$$

which yields

$$a = \left(\frac{P^2 G(M_* + M_{\rm P})}{4\pi^2}\right)^{1/3},$$

and substituting this into Equation 1.13, we obtain

$$A_{\rm RV} = \frac{2\pi}{P}\,\frac{P^{2/3}G^{1/3}(M_* + M_{\rm P})^{1/3}}{2^{2/3}\pi^{2/3}}\,\frac{M_{\rm P}\sin i(1-e^2)^{-1/2}}{M_{\rm P} + M_*}.$$

Collecting terms,

$$A_{\rm RV} = \left(\frac{2\pi G}{P}\right)^{1/3} \frac{M_{\rm P}\sin i}{(M_{\rm P} + M_*)^{2/3}}(1-e^2)^{-1/2},$$

as required.

(b) If $M_{\rm P} \ll M_*$ and $e \approx 0.0$, then the expression for $A_{\rm RV}$ simplifies to

$$A_{\rm RV} \approx \left(\frac{2\pi G}{P}\right)^{1/3} \frac{M_{\rm P}\sin i}{M_*^{2/3}}.$$

We need to express each of the variables P, $M_{\rm P}$ and M_* as a normalized quantity. To do this we need to multiply and divide through by the appropriate quantities:

$$\begin{aligned}
A_{\rm RV} &\approx (2\pi G)^{1/3}\left(\frac{P}{\rm yr}\right)^{-1/3}\left(\frac{M_{\rm P}\sin i}{\rm M_\oplus}\right)\left(\frac{M_*}{\rm M_\odot}\right)^{-2/3}\frac{\rm M_\oplus}{{\rm yr}^{1/3}\,{\rm M}_\odot^{2/3}}\\
&\approx \left(\frac{2\pi G}{\rm yr\,M_\odot^2}\right)^{1/3}{\rm M_\oplus}\left(\frac{P}{\rm yr}\right)^{-1/3}\left(\frac{M_{\rm P}\sin i}{\rm M_\oplus}\right)\left(\frac{M_*}{\rm M_\odot}\right)^{-2/3}\\
&\approx \left(\frac{2\pi\times 6.67\times 10^{-11}\,{\rm N\,m^2\,kg^{-2}}}{365.25\times 24\times 3600\,{\rm s}\times(1.99\times 10^{30})^2\,{\rm kg}^2}\right)^{1/3}\\
&\qquad \times 5.97\times 10^{24}\,{\rm kg}\left(\frac{P}{\rm yr}\right)^{-1/3}\left(\frac{M_{\rm P}\sin i}{\rm M_\oplus}\right)\left(\frac{M_*}{\rm M_\odot}\right)^{-2/3}\\
&\approx \left(\frac{3.353\times 10^{-78}\,{\rm kg\,m\,s^{-2}\,m^2\,kg^{-2}}}{\rm s\,kg^2}\right)^{1/3}\\
&\qquad \times 5.97\times 10^{24}\,{\rm kg}\left(\frac{P}{\rm yr}\right)^{-1/3}\left(\frac{M_{\rm P}\sin i}{\rm M_\oplus}\right)\left(\frac{M_*}{\rm M_\odot}\right)^{-2/3}\\
&\approx 0.0894\,{\rm m\,s^{-1}}\left(\frac{P}{\rm yr}\right)^{-1/3}\left(\frac{M_{\rm P}\sin i}{\rm M_\oplus}\right)\left(\frac{M_*}{\rm M_\odot}\right)^{-2/3}.
\end{aligned}$$

Exercise 5.5 (a) We have

$$A_{\mathrm{S}} = V_{\mathrm{S}} \sin i_{\mathrm{S}} \left(\frac{R_{\mathrm{P}}^2}{R_*^2 - R_{\mathrm{P}}^2} \right), \qquad \text{(Eqn 5.27)}$$

and for an exo-Earth $R_{\mathrm{P}} \ll R_*$, so

$$A_{\mathrm{S}} \approx V_{\mathrm{S}} \sin i_{\mathrm{S}} \frac{R_{\mathrm{P}}^2}{R_*^2}.$$

We are required to express this in terms of normalized variables, so again we multiply and divide through by the appropriate quantities:

$$\begin{aligned}
A_{\mathrm{S}} &\approx \left(\frac{\mathrm{R}_\oplus}{\mathrm{R}_\odot} \right)^2 \times 5 \times 10^3\,\mathrm{m\,s^{-1}} \left(\frac{V_{\mathrm{S}} \sin i_{\mathrm{S}}}{5\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R_{\mathrm{P}}}{\mathrm{R}_\oplus} \right)^2 \left(\frac{R_*}{\mathrm{R}_\odot} \right)^{-2} \\
&\approx \left(\frac{6.37 \times 10^6\,\mathrm{m}}{6.96 \times 10^8\,\mathrm{m}} \right)^2 \times 5 \times 10^3\,\mathrm{m\,s^{-1}} \times \left(\frac{V_{\mathrm{S}} \sin i_{\mathrm{S}}}{5\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R_{\mathrm{P}}}{\mathrm{R}_\oplus} \right)^2 \left(\frac{R_*}{\mathrm{R}_\odot} \right)^{-2} \\
&\approx 0.42\,\mathrm{m\,s^{-1}} \times \left(\frac{V_{\mathrm{S}} \sin i_{\mathrm{S}}}{5\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R_{\mathrm{P}}}{\mathrm{R}_\oplus} \right)^2 \left(\frac{R_*}{\mathrm{R}_\odot} \right)^{-2},
\end{aligned}$$

as required.

(b) We are asked to express the ratio $A_{\mathrm{S}}/A_{\mathrm{RV}}$ for an exo-Earth in terms of normalized parameters. We already obtained a suitable expression for A_{RV} in our work for Exercise 5.4:

$$A_{\mathrm{RV}} \approx 0.0894\,\mathrm{m\,s^{-1}} \left(\frac{P}{\mathrm{yr}} \right)^{-1/3} \left(\frac{M_{\mathrm{P}} \sin i}{\mathrm{M}_\oplus} \right) \left(\frac{M_*}{\mathrm{M}_\odot} \right)^{-2/3}, \qquad \text{(Eqn 5.17)}$$

which we can divide into the expression that we just obtained for A_{S}:

$$\begin{aligned}
\frac{A_{\mathrm{S}}}{A_{\mathrm{RV}}} &\approx \frac{0.42\,\mathrm{m\,s^{-1}}}{0.0894\,\mathrm{m\,s^{-1}}} \left(\frac{V_{\mathrm{S}} \sin i_{\mathrm{S}}}{5\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R_{\mathrm{P}}}{\mathrm{R}_\oplus} \right)^2 \left(\frac{R_*}{\mathrm{R}_\odot} \right)^{-2} \left(\frac{P}{\mathrm{yr}} \right)^{1/3} \left(\frac{M_{\mathrm{P}} \sin i}{\mathrm{M}_\oplus} \right)^{-1} \left(\frac{M_*}{\mathrm{M}_\odot} \right)^{2/3} \\
&\approx 4.69 \left(\frac{V_{\mathrm{S}} \sin i_{\mathrm{S}}}{5\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R_{\mathrm{P}}}{\mathrm{R}_\oplus} \right)^2 \left(\frac{M_{\mathrm{P}} \sin i}{\mathrm{M}_\oplus} \right)^{-1} \left(\frac{P}{\mathrm{yr}} \right)^{1/3} \left(\frac{\rho_*}{\rho_\odot} \right)^{2/3},
\end{aligned}$$

where we gathered the MR^{-3} terms for the star in question and the Sun to form the ratio of densities.

(c) Exo-Earths will exhibit transits of depth $\Delta F \sim 10^{-4} F$ once a year. Performing photometry of the required precision over several years will be challenging, so this is not a terribly promising prospect. The reflex radial velocity amplitude of exo-Earths is $\sim 10\,\mathrm{cm\,s^{-1}}$, which is beyond the capabilities of the best currently available instruments (November 2009), and may be less than the typical intrinsic radial velocity variability of main sequence stars (cf. Figure 4.1). It is possible that exo-Earths may be detected by their radial velocity variations, as the intrinsic stellar variability can probably be characterized and largely removed, but this will be challenging. The amplitude of the Rossiter–McLaughlin effect is about 5 times greater than the reflex radial velocity amplitude, and at $0.42\,\mathrm{m\,s^{-1}}$ it is within the capabilities of the best current instrumentation, e.g. HARPS.

Exercise 6.1 Before we begin any calculations, we will express the sizes and distances that we are given in Table 6.1 in a uniform set of units. We note that the sizes and distances always appear in the relevant equations as ratios of two lengths: they must do so, as neither temperature nor secondary eclipse depth has dimensions of length. Consequently, it doesn't really matter which unit we choose to express all the lengths in, but the simplest thing to do is to convert all three lengths to SI units, as the conversion factors in each case are given in the Appendix. Applying the conversion factors, we obtain Table S6.1.

Table S6.1 Selected parameters of HD 189733 b and its host star for use in Exercises 6.1 and 6.3.

Quantity	Value	Units	Value	Units
a	0.030 99	AU	4.64×10^9	m
R_P	1.14	R_J	8.15×10^7	m
R_*	0.788	$R_\odot$	5.48×10^8	m
T_{eff}	5000	K		
$T_{bright}(16\,\mu m)$	4315	K		

The first step in calculating the secondary eclipse depth, $\Delta F_{SE}/F$, is to calculate the day side temperature of the planet, T_{day}. The appropriate equation is

$$T_{day}^4 = (1-P)(1-A)\,\frac{R_*^2}{2a^2}\,T_{eff}^4, \qquad \text{(Eqn 6.4)}$$

and we are told to assume that no heat is redistributed to the night side of the planet, so $P = 0$, and that $A = 0.05$. First rearranging Equation 6.4, then substituting these values and the appropriate values from Table S6.1, we obtain

$$\left(\frac{T_{day}}{T_{eff}}\right)^4 = (1-P)(1-A)\,\frac{R_*^2}{2a^2}$$

so

$$\begin{aligned}\left(\frac{T_{day}}{5000\,\mathrm{K}}\right)^4 &= (1)(0.95)\,\frac{(5.48\times10^8\,\mathrm{m})^2}{2\times(4.64\times10^9\,\mathrm{m})^2}\\ &= 0.95\times0.006\,97\\ &= 6.63\times10^{-3}\end{aligned}$$

thus

$$\begin{aligned}T_{day} &= 5000\,\mathrm{K}\times(6.63\times10^{-3})^{1/4}\\ &= 5000\,\mathrm{K}\times0.285\\ &= 1427\,\mathrm{K}.\end{aligned}$$

We are not given any information on the star HD 189733 beyond the radius, the effective temperature, and the brightness temperature at $16\,\mu$m. We are told to assume that the star emits as a black body (though we know that it doesn't because the brightness temperature at $16\,\mu$m differs from the effective temperature) to estimate the fractional depth of the secondary eclipse. Without further information, we cannot comment on how justified we are in making this approximation for the stellar flux at $24\,\mu$m.

The simplest form of the equations holds if we can use the Rayleigh–Jeans law rather than the unapproximated Planck function. To assess whether this is a good approximation, we need to evaluate $hc/(k\,T_{\rm day})$ and compare this to the wavelength in question, $\lambda_{\rm c} = 24\,\mu$m. Using our value of $T_{\rm day}$, we have

$$\begin{aligned}\frac{hc}{k\,T_{\rm day}} &= \frac{6.626\times10^{-34}\,{\rm J\,s}\times 2.998\times10^{8}\,{\rm m\,s^{-1}}}{1.381\times10^{-23}\,{\rm J\,K^{-1}}\times 1427\,{\rm K}}\\ &= 1.01\times10^{-5}\,{\rm m}\\ &= 10.1\,\mu{\rm m}.\end{aligned} \qquad \text{(S6.1)}$$

The essence of the Rayleigh–Jeans law is the replacement of $\exp(hc/(\lambda k\,T_{\rm day})) - 1$ with $hc/(\lambda k\,T_{\rm day})$. With the value that we obtained in Equation S6.1, the Rayleigh–Jeans approximation uses

$$\frac{hc}{\lambda k\,T_{\rm day}} \approx \frac{10\,\mu{\rm m}}{24\,\mu{\rm m}} \approx 0.42,$$

while the full Planck function would use

$$\exp\left(\frac{hc}{\lambda k\,T_{\rm day}}\right) - 1 = 1.52 - 1 = 0.52.$$

Thus we can see that the approximation is good to within 20%, which is probably adequate given that we have already approximated both the planet and star as black body emitters. We will use the Rayleigh–Jeans law, but note that this is perhaps overestimating the planet flux by 20%.

Adopting Equation 6.15, which is appropriate to the Rayleigh–Jeans regime and the black body approximations discussed above, we have

$$\begin{aligned}\frac{\Delta F_{\rm SE}}{F} &\approx \left[\frac{(1-P)(1-A)}{2a^2}\right]^{1/4} \frac{R_{\rm P}^2}{R_*^{3/2}}\\ &\approx \left[\frac{(1)(0.95)}{2(4.64\times10^{9}\,{\rm m})^2}\right]^{1/4} \frac{(8.15\times10^{7}\,{\rm m})^2}{(5.48\times10^{8}\,{\rm m})^{3/2}}\\ &\approx 0.0063,\end{aligned}$$

i.e. the secondary eclipse depth is predicted to be 0.6%.

Exercise 6.2

(a) We have

$$\Delta f_{{\rm P},\lambda} = \left(\frac{R_{\rm P}}{d}\right)^2 \left[B_\lambda(T_{\rm day}) - B_\lambda(T_{\rm night})\right], \qquad \text{(Eqn 6.12)}$$

which gives us the amplitude $\Delta f_{{\rm P},\lambda}$ in flux units, *assuming that the orbit is edge-on*. If the orbit is not edge-on, we will always see part of both the day side and the night side hemispheres. The amplitude given in Equation 6.12 is a maximum value, attained only for exactly edge-on orbits. Having noted this caveat, to obtain contrast units we simply divide through by the total flux of the system:

$$\text{peak to peak amplitude in contrast units} = \frac{\Delta f_{{\rm P},\lambda}}{f_{*,\lambda} + f_{{\rm P},\lambda}}.$$

Since $f_{*,\lambda} \gg f_{{\rm P},\lambda}$ we can make the approximation

$$\text{peak to peak amplitude in contrast units} \approx \frac{\Delta f_{{\rm P},\lambda}}{f_{*,\lambda}}.$$

To find the amplitude in a bandpass centred on wavelength λ_c, we need to integrate over the bandpass (just as we did in Worked Example 6.2):

$$\frac{\Delta F_P}{F} = \frac{\int_{\lambda_l}^{\lambda_u} \Delta f_{P,\lambda}\, Q_\lambda \,\mathrm{d}\lambda}{\int_{\lambda_l}^{\lambda_u} f_{*,\lambda}\, Q_\lambda \,\mathrm{d}\lambda},$$

where, as in Worked Example 6.2, Q_λ is the weighting as a function of wavelength. If we assume that we can take the fluxes at the central wavelengths as proportional to the integrated flux over the bandpass, then

$$\begin{aligned}\frac{\Delta F_P}{F} &= \frac{\Delta f_{P,\lambda_c}}{f_{*,\lambda_c}}\\ &= \left(\frac{R_P}{d}\right)^2 \left[B_{\lambda_c}(T_{\text{day}}) - B_{\lambda_c}(T_{\text{night}})\right] \times \left(\frac{d}{R_*}\right)^2 \frac{1}{B_{\lambda_c}(T_{\text{bright}})}\\ &= \frac{B_{\lambda_c}(T_{\text{day}}) - B_{\lambda_c}(T_{\text{night}})}{B_{\lambda_c}(T_{\text{bright}})} \left(\frac{R_P}{R_*}\right)^2,\end{aligned}$$

which is the expression that we were asked for.

(b) For a transiting hot Jupiter, the orbit is within a few degrees of edge-on, so Equation 6.18 gives the amplitude of the phase function more or less exactly. Since the system is transiting, the quantity R_P/R_* can be determined empirically from the transit depth. The flux from the star can be measured, and consequently the only unknowns in Equation 6.18 are the day side and night side fluxes. The amplitude of the phase function in flux units can also be measured, for use in Equation 6.12. Assuming that the distance to the star is known, e.g. from the star's spectral type and apparent brightness, we have two equations and everything but the day side and night side fluxes is empirically known. In the case of a transiting planet, therefore, it is possible to deduce the temperature difference between the day side and night side hemispheres, so long as we assume that both hemispheres emit as black bodies.

(c) In the case of a hot Jupiter planet that does not transit, the orbital orientation will be unknown, and the full amplitude as described in Equation 6.18 would not be seen. The ratio R_P/R_* would be unconstrained, so with these unknown factors it would not be possible to deduce the temperature difference between the two hemispheres.

Exercise 6.3 (a) In Figure 6.5 the base of the secondary eclipse is one quarter of the way up from 0.994 to 0.996 on the vertical axis. Therefore the value of the relative intensity during the secondary eclipse is 0.9945. The values shown are normalized so that the out-of-eclipse level is 1.0000. The deficit during the secondary eclipse is therefore

$$\Delta F_{\text{SE}} = (1.0000 - 0.9945)F = 0.0055F.$$

In contrast units, therefore,

$$\frac{\Delta F_{\text{SE}}}{F} = 0.0055.$$

The secondary eclipse depth is just over one half of one per cent.

(b) In terms of the full Planck function, the secondary eclipse depth is given by

$$\frac{\Delta F_{\text{SE}}}{F} \approx p_{\lambda_c}\left(\frac{R_P}{a}\right)^2 + \frac{B_{\lambda_c}(T_{\text{day}})}{B_{\lambda_c}(T_{\text{bright}})}\left(\frac{R_P}{R_*}\right)^2, \qquad \text{(Eqn 6.13)}$$

and we are told that we can ignore the reflection component, so we have

$$\frac{\Delta F_{SE}}{F} = \frac{B_{\lambda_c}(T_{day})}{B_{\lambda_c}(T_{bright})} \left(\frac{R_P}{R_*}\right)^2,$$

where we have reverted to an equality, but note that we have approximated the integral over the bandpass and neglected the reflection term. The full Planck function is rather unwieldy, but we can simplify it by noting that we can express the equation with a ratio of Planck functions as the subject:

$$\frac{B_\lambda(T_{day})}{B_\lambda(T_{bright})} = \frac{\Delta F_{SE}}{F} \left(\frac{R_*}{R_P}\right)^2.$$

Substituting in for the Planck function, this becomes

$$\frac{\exp(hc/(\lambda_c k T_{bright})) - 1}{\exp(hc/(\lambda_c k T_{day})) - 1} = \frac{\Delta F_{SE}}{F} \left(\frac{R_*}{R_P}\right)^2, \qquad \text{(Eqn 6.19)}$$

which is the first expression that we were asked to derive. Manipulating this to isolate T_{day}, we have

$$\left(\exp\left(\frac{hc}{\lambda_c k T_{bright}}\right) - 1\right) \frac{F}{\Delta F_{SE}} \left(\frac{R_P}{R_*}\right)^2 = \exp\left(\frac{hc}{\lambda_c k T_{day}}\right) - 1$$

so

$$\left(\exp\left(\frac{hc}{\lambda_c k T_{bright}}\right) - 1\right) \frac{F}{\Delta F_{SE}} \left(\frac{R_P}{R_*}\right)^2 + 1 = \exp\left(\frac{hc}{\lambda_c k T_{day}}\right)$$

thus

$$\log_e\left[\left(\exp\left(\frac{hc}{\lambda_c k T_{bright}}\right) - 1\right) \frac{F}{\Delta F_{SE}} \left(\frac{R_P}{R_*}\right)^2 + 1\right] = \frac{hc}{\lambda_c k T_{day}}.$$

Making T_{day} the subject of the equation, we have

$$T_{day} = \frac{hc}{\lambda_c k}\left[\log_e\left[\left(\exp\left(\frac{hc}{\lambda_c k T_{bright}}\right) - 1\right) \frac{F}{\Delta F_{SE}} \left(\frac{R_P}{R_*}\right)^2 + 1\right]\right]^{-1}. \qquad \text{(Eqn 6.20)}$$

This equation, the second that we were asked to derive, gives us a (complicated) expression for T_{day}.

(c) Either of the expressions in part (b) can be used. Since Equation 6.20 has T_{day} as its subject, we will use that one. The expression is complicated, so we will break down the evaluation using the specific values for the 16 μm secondary eclipse of HD 189733 b. First we note that the quantity $hc/\lambda_c k$ appears twice, so we will evaluate that. The wavelength is the central wavelength, λ_c, of the bandpass is 16 μm. Consequently,

$$\begin{aligned}\frac{hc}{\lambda_c k} &= \frac{6.626 \times 10^{-34}\,\mathrm{J\,s} \times 2.998 \times 10^8\,\mathrm{m\,s^{-1}}}{16 \times 10^{-6}\,\mathrm{m} \times 1.381 \times 10^{-23}\,\mathrm{J\,K^{-1}}} \\ &= 899.0\,\mathrm{K}.\end{aligned}$$

Similarly, using the values in the question and Table S6.1, we can evaluate

$$\begin{aligned}\frac{F}{\Delta F_{SE}} \left(\frac{R_P}{R_*}\right)^2 &= \frac{1}{0.0055} \left(\frac{8.15 \times 10^7\,\mathrm{m}}{5.48 \times 10^8\,\mathrm{m}}\right)^2 \\ &= 4.0 \quad \text{(to 2 s.f.)},\end{aligned}$$

where we quote the result to only two significant figures because our estimate of $\Delta F_{\rm SE}$ from the eclipse depth is no better than this. The exponential function on the right-hand side is

$$\exp\left(\frac{hc}{\lambda_{\rm c} k\, T_{\rm bright}}\right) = \exp\left(\frac{899.0\,{\rm K}}{4315\,{\rm K}}\right) = \exp(0.20835) = 1.2316.$$

Substituting these values into Equation 6.20, we have

$$\begin{aligned} T_{\rm day} &= 899.0\,{\rm K}\left[\log_{\rm e}\left[(1.2316-1)\times 4.0+1\right]\right]^{-1} \\ &= 899.0\,{\rm K}\left[\log_{\rm e}\left[0.9265+1\right]\right]^{-1} \\ &= 1.525\times 899.0\,{\rm K} \\ &= 1371\,{\rm K} = 1400\,{\rm K} \quad \text{(to 2 s.f.)}, \end{aligned}$$

where in the first line we used the convention that multiplication takes precedence to avoid yet another set of brackets surrounding the terms '$(1.2316 - 1) \times 4.0$'.

(d) The essence of the Rayleigh–Jeans law is to replace $\exp(hc/\lambda_{\rm c}kT) - 1$ with $hc/\lambda_{\rm c}kT$. For the star, we have $T_{\rm bright}(16\,\mu{\rm m}) = 4315$, which means that

$$\frac{hc}{\lambda_{\rm c}kT} = 0.2084,$$

while

$$\exp\left(\frac{hc}{\lambda_{\rm c}kT}\right) - 1 = 0.2316.$$

These are in the same ballpark, but differ by more than 10%. Since we can read the most uncertain of the input quantities, $\Delta F_{\rm SE}$, from the graph at a better precision than this, it would probably not be justified to use the Rayleigh–Jeans law instead of the full Planck function to evaluate the $16\,\mu$m flux from the star HD 189733 b.

Comment: In Exercise 6.1 we made a bigger approximation, but in that exercise, we were making an estimate; here we are deriving results from state-of-the-art observations.

The planet has a day side temperature of $1400\,{\rm K}$, which gives

$$\frac{hc}{\lambda_{\rm c}kT} = 0.6422,$$

while

$$\exp\left(\frac{hc}{\lambda_{\rm c}kT}\right) - 1 = 0.9006.$$

In this case it is clear that the Rayleigh–Jeans law would introduce a discrepancy of about 30% in the flux from the planet. It would not be a justifiable approximation.

(e) If the wavelength of the observation were shorter, the quantity $hc/\lambda_{\rm c}kT$ would become bigger, and the approximation, which is valid for $hc/\lambda_{\rm c}kT \ll 1$, would be worse. The Rayleigh–Jeans law is most applicable at long wavelengths.

(f) If the brightness temperature of the star were higher, the quantity $hc/\lambda_{\rm c}kT$ would become smaller, and the approximation, which is valid for $hc/\lambda_{\rm c}kT \ll 1$, would be better. The Rayleigh–Jeans law is most applicable at high temperatures.

Exercise 7.1 The magnitude of the force of gravity due to the Sun acting on the Earth is $F_\odot = G\,M_\odot M_\oplus/(1^2\,AU^2)$, while the magnitude of the force of gravity due to Jupiter acting on the Earth is $F_J = GM_J M_\oplus/((5-1)^2\,AU^2)$ when they are at their closest separation. The ratio of these forces is

$$\frac{F_\odot}{F_J} = \frac{M_\odot \times 16}{10^{-3}\,M_\odot \times 1} = 1.6 \times 10^4.$$

So the gravitational force due to the Sun is about sixteen thousand times stronger than the gravitational force due to Jupiter, and this is of course independent of the mass of the Earth.

Exercise 7.2 Using Equation 7.22,

$$\begin{aligned}\tau_{\rm circ} &= \frac{2}{21}\frac{Q_P}{k_{dP}}\left(\frac{a^3}{GM_*}\right)^{1/2}\frac{M_P}{M_*}\left(\frac{a}{R_P}\right)^5\\ &= \frac{2}{21}\times 10^5 \times \left(\frac{(7.63\times 10^9\,{\rm m})^3}{(6.673\times 10^{-11}\,{\rm N\,m^2\,kg^{-2}})\times(2.39\times 10^{30}\,{\rm kg})}\right)^{1/2}\\ &\quad\times \frac{9.31\times 10^{26}\,{\rm kg}}{2.39\times 10^{30}\,{\rm kg}}\times\left(\frac{7.63\times 10^9\,{\rm m}}{1.24\times 10^8\,{\rm m}}\right)^5\\ &= 1.73\times 10^{14}\,{\rm s},\end{aligned}$$

which is around 5 million years.

Exercise 7.3 The equilibrium temperature of a planet is given by

$$T_{\rm eq} = \frac{1}{2}\left(\frac{(1-A)L_*}{\sigma\pi a^2}\right)^{1/4}. \qquad \text{(Eqn 5.5)}$$

Rearranging this to make a the subject,

$$a = \frac{1}{(2T_{\rm eq})^2}\left(\frac{(1-A)L_*}{\sigma\pi}\right)^{1/2}.$$

(a) So, for the A type star, the inner boundary of the habitable zone, where $T_{\rm eq} = 373\,{\rm K}$, is given by

$$\begin{aligned}a &= \frac{1}{(2\times 373\,{\rm K})^2}\left(\frac{(1-0)\times 10\times 3.83\times 10^{26}\,{\rm J\,s^{-1}}}{5.671\times 10^{-8}\,{\rm J\,m^{-2}\,K^{-4}\,s^{-1}}\times\pi}\right)^{1/2}\\ &= 2.635\times 10^{11}\,{\rm m}\\ &= 1.76\,{\rm AU}.\end{aligned}$$

Similarly, the outer boundary of the habitable zone for the A type star, where $T_{\rm eq} = 273\,{\rm K}$, is given by

$$\begin{aligned}a &= \frac{1}{(2\times 273\,{\rm K})^2}\left(\frac{(1-0)\times 10\times 3.83\times 10^{26}\,{\rm J\,s^{-1}}}{5.671\times 10^{-8}\,{\rm J\,m^{-2}\,K^{-4}\,s^{-1}}\times\pi}\right)^{1/2}\\ &= 4.918\times 10^{11}\,{\rm m}\\ &= 3.29\,{\rm AU}.\end{aligned}$$

So the width of the habitable zone of the A type star is estimated as about 1.5 AU.

(b) Now, since $a \propto L_*^{1/2}$, and the luminosity of the K type star is 100 times smaller than that of the A type star, the inner and outer limits of the habitable zone for the K type star will be 10 times smaller than those for the A type star (all else being equal). As a result, the inner edge is at about 0.18 AU and the outer edge is at about 0.33 AU. The width of the habitable zone of the K type star is therefore estimated as about 0.15 AU.

Exercise 7.4 An exoplanet might exist in a stable orbit around any one of the six component stars (e.g. Figure S7.1a or b). Alternatively, an exoplanet could have a stable orbit around any one of the three binary pairs that make up the system (e.g. Figure S7.1c or d), or around the quadruple system that constitutes the 'visual binary' (Figure S7.1e). Finally, one might even have an exoplanet in a stable orbit around the entire sextuple system (Figure S7.1f), although such a planet would likely be only loosely bound to the system as it would be necessarily very distant from the centre of mass.

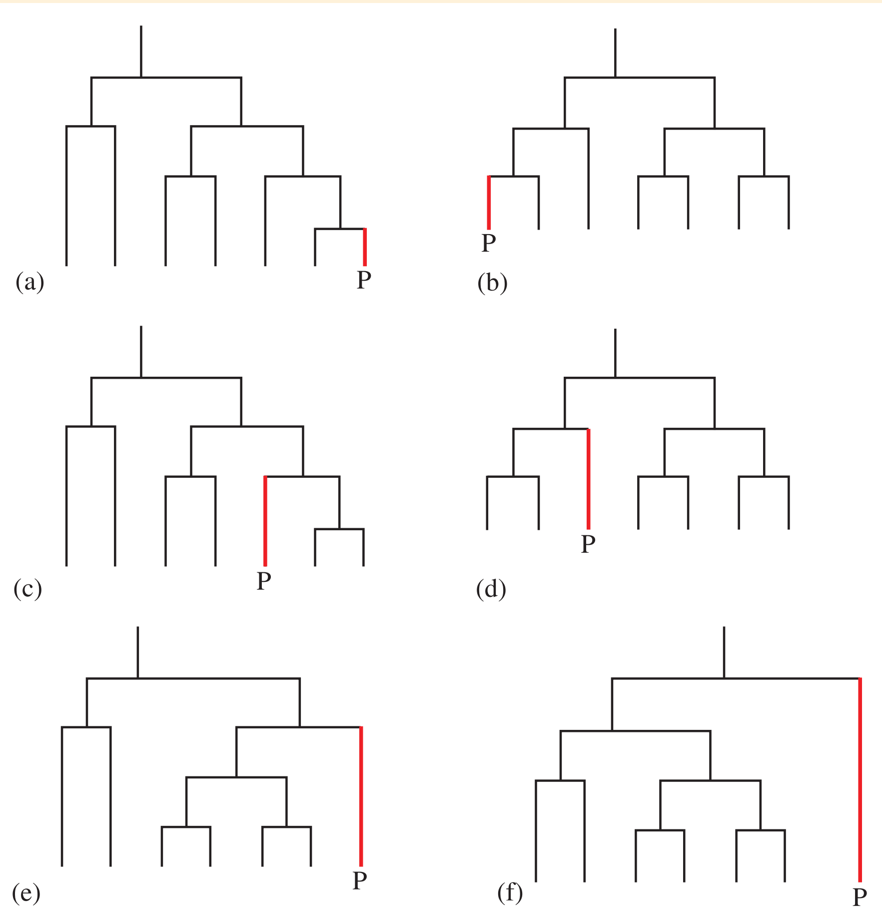

Figure S7.1 See solution to Exercise 7.4. 'P' indicates the possible location of a planet.

Exercise 7.5 (a) The maximum offset of the transit time occurs when the sine function has a value of ± 1. In this case, the offset is $|\delta t_2| \approx P_2 a_1 \mu_1 / 2\pi a_2$.

Now, the reduced mass is $\mu_1 \approx M_1/M_*$, which in this case is $\mu_1 \approx 3 \times 10^{-6}$. So

$$|\delta t_2| \approx \frac{30\ \text{days} \times 0.072\ \text{AU} \times 3 \times 10^{-6}}{2\pi \times 0.19\ \text{AU}}.$$

The maximum offset is therefore 5.4×10^{-6} days or 0.47 seconds.

(b) If the mass of the inner planet were 10 times larger, the maximum offset in the transit time would also be 10 times larger, i.e. 4.7 seconds.

Exercise 7.6 The maximum transit timing offset will occur when the cosine term in the numerator of the right-hand side of Equation 7.28 is ± 1 and when the sine term in the denominator is 1. The timing offset is then

$$|\delta t_2| \approx \frac{P_2 \mu_1 a_1 \sqrt{1 - e_2^2}}{2\pi a_2 (1 - e_2)}.$$

(a) When $e_2 = 0.1$, this becomes

$$|\delta t_2| \approx \frac{30\ \text{days} \times 3 \times 10^{-6} \times 0.072 \times \sqrt{0.99}}{2\pi \times 0.19 \times 0.9} \approx 6.0 \times 10^{-6}\ \text{days} = 0.52\ \text{s}.$$

(b) When $e_2 = 0.5$, this becomes

$$|\delta t_2| \approx \frac{30\ \text{days} \times 3 \times 10^{-6} \times 0.072 \times \sqrt{0.75}}{2\pi \times 0.19 \times 0.5} \approx 9.4 \times 10^{-6}\ \text{days} = 0.81\ \text{s}.$$

Exercise 7.7 Using Equation 7.29, we first note that the reduced mass is $\mu_2 \approx M_2/M_* \approx 0.5 \times 10^{-3}$. The TTV is therefore

$$\begin{aligned} \delta t_1 &\approx (0.5 \times 10^{-3} \times 0.5 \times 100\ \text{days}) \times \left(\frac{0.05}{0.42}\right)^3 \\ &\approx 4.2 \times 10^{-5}\ \text{days} \\ &= 3.6\ \text{seconds}. \end{aligned}$$

Exercise 7.8 (a) Using Equation 7.30, the maximum TTV is

$$\delta t_2 \approx \frac{12\ \text{days}}{4.5 \times 3} \times \frac{3 \times 10^{-6}}{10^{-3}} \approx 2.7 \times 10^{-3}\ \text{days} \approx 4\ \text{minutes}.$$

(b) Using Equation 7.31, the libration period of this TTV is

$$P_{\text{lib}} \approx 0.5 \times 3^{-4/3} \times (10^{-3})^{-2/3} \times 12\ \text{days} \approx 139\ \text{days}.$$

So the maximum TTV would recur roughly every 11 or 12 transits.

Exercise 7.9 (a) Rearranging Equation 7.32, we have

$$\begin{aligned} a_{\text{M}} M_{\text{M}} &\approx \frac{2\pi a \times 2^{1/2} \times M_{\text{P}}\, \delta t_{\text{M}}}{P} \\ &\approx \frac{2 \times \pi \times 0.05\ \text{AU} \times 2^{1/2} \times 10^{-3}\ \text{M}_\odot \times 30\ \text{s}}{4 \times 24 \times 3600\ \text{s}} \\ &\approx 3.86 \times 10^{-8}\ \text{AU}\ \text{M}_\odot. \end{aligned}$$

Rearranging Equation 7.33, we have

$$\begin{aligned}
a_{\mathrm{M}}^{-1/2} M_{\mathrm{M}} &\approx \frac{2^{1/2}[M_{\mathrm{P}}(M_{\mathrm{P}}+M_*)]^{1/2}\,\delta T_{\mathrm{M}}}{T_{\mathrm{dur}}\,a^{1/2}} \\
&\approx \frac{2^{1/2}\times[10^{-3}\times(10^{-3}+1)]^{1/2}\,\mathrm{M}_\odot\times 15\,\mathrm{s}}{3\times 3600\,\mathrm{s}\times 0.05^{1/2}\,\mathrm{AU}^{1/2}} \\
&\approx 2.78\times 10^{-4}\,\mathrm{AU}^{-1/2}\,\mathrm{M}_\odot.
\end{aligned}$$

Now, dividing the first of these by the second, we get

$$a_{\mathrm{M}}^{3/2} \approx 1.39\times 10^{-4}\,\mathrm{AU}^{3/2},$$

so

$$a_{\mathrm{M}} \approx 2.68\times 10^{-3}\,\mathrm{AU} = 4.0\times 10^{8}\,\mathrm{m}$$

(which is around four hundred thousand kilometres).

So, if the TTV and the transit duration variation are due to the presence of an exomoon, they imply that this moon must be orbiting fairly close to the planet. (The radius of Jupiter is about 7×10^7 m, and the radius of a hot Jupiter exoplanet may well be somewhat larger.)

Substituting this orbital radius back into the first equation,

$$M_{\mathrm{M}} \approx \frac{3.86\times 10^{-8}}{2.68\times 10^{-3}}\,\mathrm{M}_\odot \approx 1.44\times 10^{-5}\,\mathrm{M}_\odot,$$

which is about five times the mass of the Earth.

(b) Kepler's third law applied to the orbit of the putative exomoon around the exoplanet may be written as

$$\begin{aligned}
P_{\mathrm{M}} &= \left(\frac{4\pi^2 a_{\mathrm{M}}^3}{G(M_{\mathrm{P}}+M_{\mathrm{M}})}\right)^{1/2} \\
&\approx \left(\frac{4\pi^2\times(4.0\times 10^8\,\mathrm{m})^3}{6.67\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}\times(10^{-3}+1.4\times 10^{-5})\times 1.99\times 10^{30}\,\mathrm{kg}}\right)^{1/2} \\
&\approx 1.37\times 10^{5}\,\mathrm{s}.
\end{aligned}$$

The orbital period of the exomoon would therefore be about 38 hours.

Exercise 7.10 Using Equation 7.32,

$$\begin{aligned}
\delta t_{\mathrm{M}} &\approx \frac{a_{\mathrm{M}} M_{\mathrm{M}}}{2^{1/2} M_{\mathrm{P}}}\,\frac{P}{2\pi a} \\
&\approx \frac{10^9\,\mathrm{m}\times 3\times 10^{-6}}{2^{1/2}\times 10^{-3}}\times\frac{46\times 24\times 3600\,\mathrm{s}}{2\pi\times 0.2\times 1.5\times 10^{11}\,\mathrm{m}} \\
&\approx 45\,\mathrm{s}.
\end{aligned}$$

The maximum TTV is therefore less than 1 minute.

Using Equation 7.33,

$$\begin{aligned}
\delta T_{\mathrm{M}} &\approx \left(\frac{a}{a_{\mathrm{M}}}\right)^{1/2}\frac{M_{\mathrm{M}}}{[M_{\mathrm{P}}(M_{\mathrm{P}}+M_*)]^{1/2}}\,\frac{T_{\mathrm{dur}}}{2^{1/2}} \\
&\approx \left(\frac{0.2\times 1.5\times 10^{11}\,\mathrm{m}}{10^9\,\mathrm{m}}\right)^{1/2}\times\frac{3\times 10^{-6}}{[10^{-3}\times(10^{-3}+0.5)]^{1/2}}\times\frac{5\times 3600\,\mathrm{s}}{2^{1/2}} \\
&\approx 9.3\,\mathrm{s}.
\end{aligned}$$

The maximum transit duration variation is therefore of order 10 seconds.

Exercise 7.11 The Hill radius for the planet is given by Equation 7.24 as

$$\begin{aligned} R_{\mathrm{H}} &= a\left(\frac{M_{\mathrm{P}}}{3M_*}\right)^{1/3} \\ &= 0.05\,\mathrm{AU} \times \left(\frac{10^{-3}\,\mathrm{M}_\odot}{3\,\mathrm{M}_\odot}\right)^{1/3} \\ &= 0.0035\,\mathrm{AU}. \end{aligned}$$

Hence the Hill radius is about 5.2×10^8 m or just over $7\,\mathrm{R_J}$.

Exercise 7.12 From Equation 7.16, the Roche limit is

$$\begin{aligned} d_{\mathrm{R}} &= R_{\mathrm{M}}\left(\frac{2M_{\mathrm{P}}}{M_{\mathrm{M}}}\right)^{1/3} \\ &= 6.4 \times 10^6\,\mathrm{m} \times \left(\frac{2 \times 10^{-3}}{3 \times 10^{-6}}\right)^{1/3} \\ &= 5.6 \times 10^7\,\mathrm{m}. \end{aligned}$$

The Roche limit for this planet and moon is therefore about 56 thousand kilometres.

Exercise 7.13 We first note that the reduced mass is

$$\mu_{\mathrm{T}} \approx \frac{M_{\mathrm{T}}}{M_{\mathrm{P}}} \approx \frac{3 \times 10^{-6}\,\mathrm{M}_\odot}{10^{-3}\,\mathrm{M}_\odot} \approx 3 \times 10^{-3}.$$

The maximum TTV is given by Equation 7.34 as

$$\delta t \approx \frac{3 \times 10^{-3} \times 4\ \text{days} \times 25^\circ}{360^\circ} \approx 8.3 \times 10^{-4}\ \text{days} = 72\ \text{seconds}.$$

Exercise 8.1 We need to apply

$$\text{geometric transit probability} = \frac{R_* + R_{\mathrm{P}}}{a} \approx \frac{R_*}{a} \qquad \text{(Eqn 1.21)}$$

and substitute in the appropriate solar and terrestrial values. Thus

$$\begin{aligned} \text{geometric transit probability} &\approx \frac{\mathrm{R}_\odot}{1\,\mathrm{AU}} \\ &\approx \frac{6.96 \times 10^8\,\mathrm{m}}{1.50 \times 10^{11}\,\mathrm{m}} \\ &\approx 0.0047. \end{aligned}$$

We are told to assume that Kepler will observe 5000 such stars, so the number of transiting exo-Earths detected will be

$$\begin{aligned} \text{total exo-Earths detected} &= \text{geometric transit probability} \times 5000 \\ &= 0.0047 \times 5000 = 23. \end{aligned}$$

Kepler is predicted to find just over 23 exo-Earths, with the (optimistic) assumption that each star hosts a single planet in an orbit like Earth's. The actual anticipated number is less than this, by a factor (of perhaps roughly 10 or so) that depends on the planeticity and the value of $\alpha(a, M_\mathrm{P})$ over the range within which planets would be classed as exo-Earths (cf. Subsection 4.2.2).

Exercise 8.2 (a) The density is simply the mass divided by the volume, so

$$\begin{aligned}\rho &= \frac{3\,M_\mathrm{P}}{4\pi\,R_\mathrm{P}^3}\\ &= \frac{3}{4\pi}\,\frac{6.55\,\mathrm{M}_\oplus}{(2.68\,\mathrm{R}_\oplus)^3}.\end{aligned}\tag{S8.1}$$

This gives

$$\begin{aligned}\rho &= \frac{0.75\times 6.55\times 5.97\times 10^{24}\,\mathrm{kg}}{\pi\times(2.68\times 6.37\times 10^{6}\,\mathrm{m})^3}\\ &= 1876\,\mathrm{kg\,m^{-3}}.\end{aligned}$$

In terms of the density of Earth, Equation S8.1 tells us that

$$\rho = \frac{6.55}{19.25}\rho_\oplus = 0.340\rho_\oplus.$$

Thus we see that GJ 1214 b is just over one-third as dense as Earth.

(b) The magnitude of the acceleration due to gravity on the surface of a planet is

$$g_\mathrm{P} = \frac{GM_\mathrm{P}}{R_\mathrm{P}^2},\tag{Eqn 4.48}$$

so on the surface of GJ 1214 b we have

$$\begin{aligned}g_\mathrm{P} &= \frac{6.673\times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}\times 6.55\times 5.97\times 10^{24}\,\mathrm{kg}}{(2.68\times 6.37\times 10^{6}\,\mathrm{m})^2}\\ &= 8.95\,\mathrm{m\,s^{-2}}.\end{aligned}$$

This is very similar to the value of g on Earth, which is $9.81\,\mathrm{m\,s^{-2}}$.

(c) Humans would be rather comfortable on a planet with this value of gravity. Golf would be perfectly feasible, and it would appear to be possible to hit a ball somewhat farther than on Earth (assuming comparable values of air resistance) because the downwards acceleration during flight would be slightly less than on Earth, so the ball would travel further before landing.

Exercise 8.3 (a) The appropriate equation to use is

$$\frac{A_\mathrm{S}}{A_\mathrm{RV}} \approx 4.7\left(\frac{V_\mathrm{S}\sin i_\mathrm{S}}{5\,\mathrm{km\,s^{-1}}}\right)\left(\frac{R_\mathrm{P}}{\mathrm{R}_\oplus}\right)^2\left(\frac{M_\mathrm{P}\sin i}{\mathrm{M}_\oplus}\right)^{-1}\left(\frac{P}{\mathrm{yr}}\right)^{1/3}\left(\frac{\rho_*}{\rho_\odot}\right)^{2/3},\tag{Eqn 5.31}$$

and everything to the right of the first term in parentheses is 1, because each of the terms is normalized to the Earth or the Sun. We are told that the Sun's spin period is 25 days, so we have

$$\begin{aligned}V_\mathrm{S} &= \frac{2\pi\,\mathrm{R}_\odot}{P_\mathrm{rot}}\\ &= \frac{2\pi\times 6.96\times 10^{8}\,\mathrm{m}}{25\times 24\times 3600\,\mathrm{s}}\\ &= 2.0\times 10^{3}\,\mathrm{m\,s^{-1}}\quad\text{(to 2 s.f.)}.\end{aligned}$$

The only thing left to evaluate is $\sin i_\mathrm{S}$. We are viewing so that the Earth transits with an impact parameter of $b = 0$, so we know $i = 90^\circ$. The ecliptic is the plane of the Earth's orbit, so $i_\mathrm{S} = (90 - 7.15)^\circ$. Putting these values into Equation 5.31, we have

$$\begin{aligned}\frac{A_\mathrm{S}}{A_\mathrm{RV}} &\approx 4.7 \left(\frac{2.0 \times 10^3\,\mathrm{m\,s^{-1}} \sin 82.85^\circ}{5\,\mathrm{km\,s^{-1}}}\right) \\ &\approx 1.9 \quad \text{(to 2 s.f.)}.\end{aligned}$$

The amplitude of the Rossiter–McLaughlin effect is almost twice the reflex radial velocity amplitude.

(b) With 8 planets, the Sun's reflex radial velocity curve will be a superposition of the effects of all of them. Since the Earth is a low-mass component of the Solar System, it will be difficult to unambiguously isolate the Sun's reflex radial velocity due to the Earth. Measurements would need to be made for longer than the orbital period of Jupiter. The Rossiter–McLaughlin effect could be measured in a few hours, and would verify that a terrestrial-planet-sized body was transiting the host star. The reflex radial velocity amplitude allows the mass of the planet to be deduced; without a measurement of this amplitude, the mass of the planet would remain unknown.