à Question 9

```
In[76]:= Needs["Graphics`PlotField`"]

In[77]:= Clear[x, y, a, arrows1, arrows2, arrows3]
```

(i)

```
In[78]:= a = {{-3, 1}, {1, -3}}; MatrixForm[a]
```

Out[78]//MatrixForm=

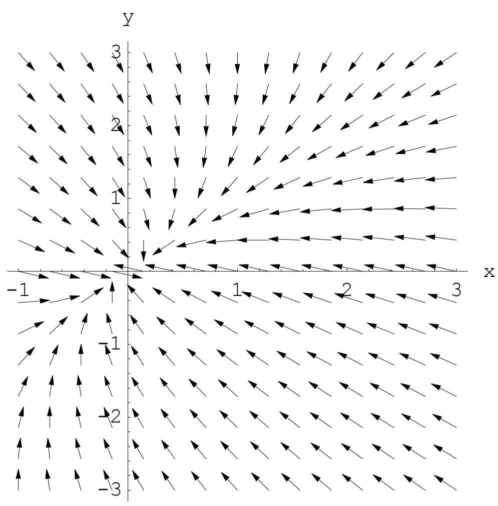
$$\begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

(a)

```
In[79]:= Eigenvalues[a]

Out[79]= {-4, -2}

In[80]:= Eigenvectors[a]

Out[80]= {{-1, 1}, {1, 1}}
```

(b)

```
In[81]:= arrows1 =
           PlotVectorField[ {-3 x + y, x - 3 y}, {x, -1, 3}, {y, -3, 3}, Axes -> True,
            AxesLabel -> {"x", "y"}, ScaleFunction -> (1 &), AspectRatio -> 1];
```




```
In[82]:= sol = DSolve[{x'[t] == -3 x[t] + y[t], y'[t] == x[t] - 3 y[t]}, {x[t], y[t]}, t]
```

Out[82]= $\left\{\left\{x[t] \to \frac{1}{2}\, e^{-4t}\,(C[1] + e^{2t}\,C[1] - C[2] + e^{2t}\,C[2]),\right.\right.$

$\left.\left. y[t] \to \frac{1}{2}\, e^{-4t}\,(-C[1] + e^{2t}\,C[1] + C[2] + e^{2t}\,C[2])\right\}\right\}$

```
In[83]:= sol1 = DSolve[{x'[t] == -3 x[t] + y[t],
             y'[t] == x[t] - 3 y[t], x[0] == x0, y[0] == y0}, {x[t], y[t]}, t]
```

Out[83]= $\left\{\left\{x[t] \to \frac{1}{2} e^{-4t} (x0 + e^{2t} x0 - y0 + e^{2t} y0),\ y[t] \to \frac{1}{2} e^{-4t} (-x0 + e^{2t} x0 + y0 + e^{2t} y0)\right\}\right\}$

(c)

```
In[84]:= x1 = sol1[[1, 1, 2]] /. {x0 -> 1, y0 -> 1}
```

Out[84]= e^{-2t}

```
In[85]:= y1 = sol1[[1, 2, 2]] /. {x0 -> 1, y0 -> 1}
```

Out[85]= e^{-2t}

```
In[86]:= tr1 = ParametricPlot[Evaluate[ {x1, y1}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[87]:= x2 = sol1[[1, 1, 2]] /. {x0 -> -1, y0 -> 1}
```

Out[87]= $-e^{-4t}$

```
In[88]:= y2 = sol1[[1, 2, 2]] /. {x0 -> -1, y0 -> 1}
```

Out[88]= e^{-4t}

```
In[89]:= tr2 = ParametricPlot[ Evaluate[ {x2, y2}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[90]:= x3 = sol1[[1, 1, 2]] /. {x0 -> -1, y0 -> -1}
```

Out[90]= $-e^{-2t}$

```
In[91]:= y3 = sol1[[1, 2, 2]] /. {x0 -> -1, y0 -> -1}
```

Out[91]= $-e^{-2t}$

```
In[92]:= tr3 = ParametricPlot[Evaluate[ {x3, y3}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[93]:= x4 = sol1[[1, 1, 2]] /. {x0 -> 1, y0 -> -1}
```

Out[93]= e^{-4t}

```
In[94]:= y4 = sol1[[1, 2, 2]] /. {x0 -> 1, y0 -> -1}
```

Out[94]= $-e^{-4t}$

```
In[95]:= tr4 = ParametricPlot[Evaluate[ {x4, y4}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[96]:= x5 = sol1[[1, 1, 2]] /. {x0 -> 2, y0 -> 0}
```

Out[96]= $\frac{1}{2} e^{-4t} (2 + 2 e^{2t})$

```
In[97]:= y5 = sol1[[1, 2, 2]] /. {x0 -> 2, y0 -> 0}
```

Out[97]= $\frac{1}{2} e^{-4t} (-2 + 2 e^{2t})$

```
In[98]:= tr5 = ParametricPlot[ Evaluate[ {x5, y5}], {t, 0, 1},
           DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];
```

```
In[99]:= x6 = sol1[[1, 1, 2]] /. {x0 -> 3, y0 -> 1}
```

Out[99]= $\frac{1}{2} e^{-4t} (2 + 4 e^{2t})$

```
In[100]:= y6 = sol1[[1, 2, 2]] /. {x0 -> 3, y0 -> 1}
```

Out[100]= $\frac{1}{2} e^{-4t} (-2 + 4 e^{2t})$

```
In[101]:= tr6 = ParametricPlot[ Evaluate[ {x6, y6}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];
```

```
In[102]:= x7 = sol1[[1, 1, 2]] /. {x0 -> 1, y0 -> 3}
```

Out[102]= $\frac{1}{2} e^{-4t} (-2 + 4 e^{2t})$

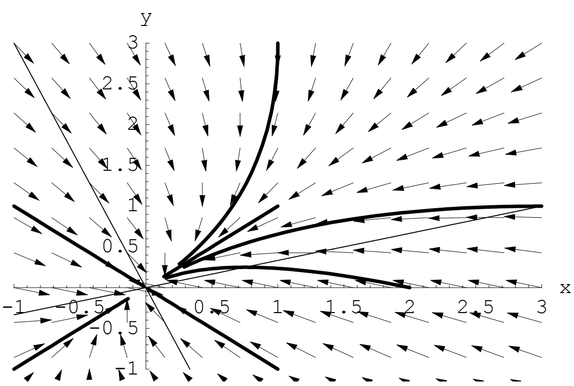
```
In[103]:= y7 = sol1[[1, 2, 2]] /. {x0 -> 1, y0 -> 3}
```

Out[103]= $\frac{1}{2} e^{-4t} (2 + 4 e^{2t})$

```
In[104]:= tr7 = ParametricPlot[Evaluate[ {x7, y7}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];
```

```
In[105]:= lines1 = Plot[{-3 x, (1 / 3) x}, {x, -1, 3}, DisplayFunction -> Identity];
```

```
In[106]:= Show[{tr1, tr2, tr3, tr4, tr5, tr6, tr7, arrows1, lines1},
           DisplayFunction -> $DisplayFunction,
           AxesLabel -> {"x", "y"}, PlotRange -> {{-1, 3}, {-1, 3}}];
```



(d)

All trajectories converge on the fixed point and the system shows an improper node.

(ii)

```
In[107]:= Clear[x, y]
```

In[108]:= $b = \begin{pmatrix} 2 & -4 \\ 1 & -3 \end{pmatrix}$

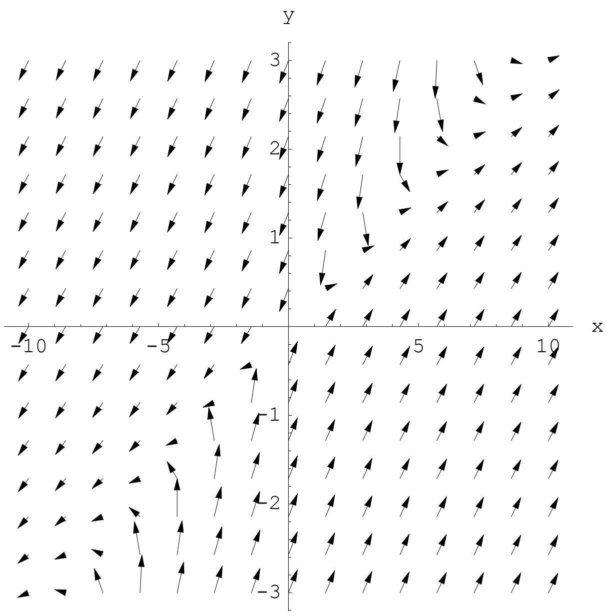
Out[108]= {{2, -4}, {1, -3}}

(a)

```
In[109]:= Eigenvalues[b]
Out[109]= {-2, 1}
In[110]:= Eigenvectors[b]
Out[110]= {{1, 1}, {4, 1}}
```

(b)

```
In[111]:= arrows2 =
           PlotVectorField[ {2 x - 4 y, x - 3 y}, {x, -10, 10}, {y, -3, 3}, Axes -> True,
            AxesLabel -> {"x", "y"}, ScaleFunction -> (1 &), AspectRatio -> 1];
```




```
In[112]:= sol20 =
          DSolve[ {x'[t] == 2 x[t] - 4 y[t], y'[t] == x[t] - 3 y[t]}, {x[t], y[t]}, t]
```

Out[112]= $\left\{\left\{x[t] \to \frac{1}{3} e^{-2t} (-C[1] + 4 e^{3t} C[1] + 4 C[2] - 4 e^{3t} C[2]),\ y[t] \to \frac{1}{3} e^{-2t} (-C[1] + e^{3t} C[1] + 4 C[2] - e^{3t} C[2])\right\}\right\}$

```
In[113]:= sol2 = DSolve[ {x'[t] == 2 x[t] - 4 y[t],
            y'[t] == x[t] - 3 y[t], x[0] == x0, y[0] == y0}, {x[t], y[t]}, t]
```

Out[113]= $\left\{\left\{x[t] \to \frac{1}{3} e^{-2t} (-x0 + 4 e^{3t} x0 + 4 y0 - 4 e^{3t} y0),\ y[t] \to \frac{1}{3} e^{-2t} (-x0 + e^{3t} x0 + 4 y0 - e^{3t} y0)\right\}\right\}$

(c)

```
In[114]:= x21 = sol2[[1, 1, 2]] /. {x0 -> 1, y0 -> 1}
```

Out[114]= e^{-2t}

```
In[115]:= y21 = sol2[[1, 2, 2]] /. {x0 -> 1, y0 -> 1}
```

Out[115]= e^{-2t}

```
In[116]:= tr21 = ParametricPlot[Evaluate[ {x21, y21}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

          General::spell1 :
           Possible spelling error: new symbol name "tr21" is similar to existing symbol "tr1".
```

```
In[117]:= x22 = sol2[[1, 1, 2]] /. {x0 -> -1, y0 -> -1}
```

Out[117]= $-e^{-2t}$

```
In[118]:= y22 = sol2[[1, 2, 2]] /. {x0 -> -1, y0 -> -1}
```

Out[118]= $-e^{-2t}$

```
In[119]:= tr22 = ParametricPlot[Evaluate[ {x22, y22}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];
```

```
In[120]:= x23 = sol2[[1, 1, 2]] /. {x0 -> 4, y0 -> 1}
```

Out[120]= $4\,e^{t}$

```
In[121]:= y23 = sol2[[1, 2, 2]] /. {x0 -> 4, y0 -> 1}
```

Out[121]= e^{t}

```
In[122]:= tr23 = ParametricPlot[Evaluate[ {x23, y23}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

          General::spell1 :
           Possible spelling error: new symbol name "tr23" is similar to existing symbol "tr3".
```

```
In[123]:= x24 = sol2[[1, 1, 2]] /. {x0 -> -4, y0 -> -1}
```

Out[123]= $-4\,e^{t}$

```
In[124]:= y24 = sol2[[1, 2, 2]] /. {x0 -> -4, y0 -> -1}
```

Out[124]= $-e^{t}$

```
In[125]:= tr24 = ParametricPlot[Evaluate[ {x24, y24}], {t, 0, 1},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

          General::spell1 :
           Possible spelling error: new symbol name "tr24" is similar to existing symbol "tr4".
```

```
In[126]:= x25 = sol2[[1, 1, 2]] /. {x0 -> 0, y0 -> 1}
```

Out[126]= $\frac{1}{3}\,e^{-2t}\,(4 - 4\,e^{3t})$

```
In[127]:= y25 = sol2[[1, 2, 2]] /. {x0 -> 0, y0 -> 1}
```

Out[127]= $\frac{1}{3}\,e^{-2t}\,(4 - e^{3t})$

```
In[128]:= tr25 = ParametricPlot[Evaluate[ {x25, y25}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

      General::spell1 :
       Possible spelling error: new symbol name "tr25" is similar to existing symbol "tr5".

In[129]:= x26 = sol2[[1, 1, 2]] /. {x0 -> 0, y0 -> -1}
```

Out[129]= $\frac{1}{3} e^{-2t} (-4 + 4 e^{3t})$

```
In[130]:= y26 = sol2[[1, 2, 2]] /. {x0 -> 0, y0 -> -1}
```

Out[130]= $\frac{1}{3} e^{-2t} (-4 + e^{3t})$

```
In[131]:= tr26 = ParametricPlot[Evaluate[ {x26, y26}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

      General::spell1 :
       Possible spelling error: new symbol name "tr26" is similar to existing symbol "tr6".

In[132]:= x27 = sol2[[1, 1, 2]] /. {x0 -> 3, y0 -> 2}
```

Out[132]= $\frac{1}{3} e^{-2t} (5 + 4 e^{3t})$

```
In[133]:= y27 = sol2[[1, 2, 2]] /. {x0 -> 3, y0 -> 2}
```

Out[133]= $\frac{1}{3} e^{-2t} (5 + e^{3t})$

```
In[134]:= tr27 = ParametricPlot[Evaluate[ {x27, y27}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

      General::spell1 :
       Possible spelling error: new symbol name "tr27" is similar to existing symbol "tr7".

In[135]:= x28 = sol2[[1, 1, 2]] /. {x0 -> -3, y0 -> -2}
```

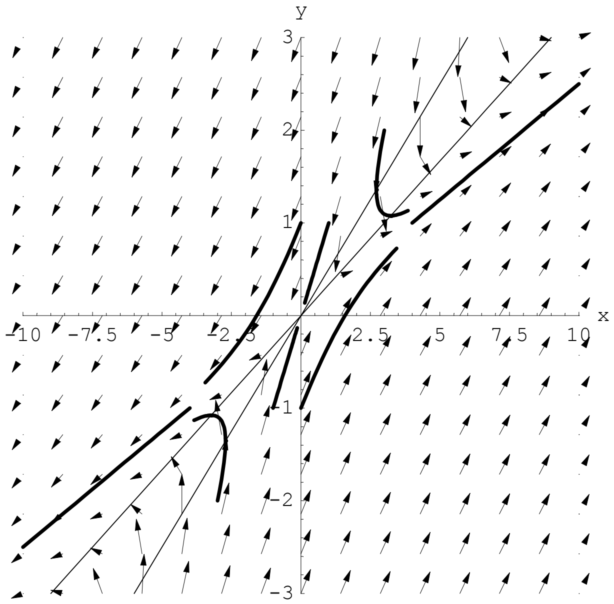
Out[135]= $\frac{1}{3} e^{-2t} (-5 - 4 e^{3t})$

```
In[136]:= y28 = sol2[[1, 2, 2]] /. {x0 -> -3, y0 -> -2}
```

Out[136]= $\frac{1}{3} e^{-2t} (-5 - e^{3t})$

```
In[137]:= tr28 = ParametricPlot[Evaluate[ {x28, y28}], {t, 0, 1},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[138]:= lines2 = Plot[ {(1 / 2) x, (1 / 3) x}, {x, -10, 10}, DisplayFunction -> Identity];
```

```
In[139]:= Show[{tr21, tr22, tr23, tr24, tr25, tr26, tr27, tr28, arrows2, lines2},
            DisplayFunction -> $DisplayFunction, AxesLabel -> {"x", "y"},
            AspectRatio -> 1, PlotRange -> {{-10, 10}, {-3, 3}}];
```



(d)

The diagram shows a saddle-point solution.

(iii)

```
In[140]:= Clear[x, y]
```

In[141]:= **c =** $\begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$

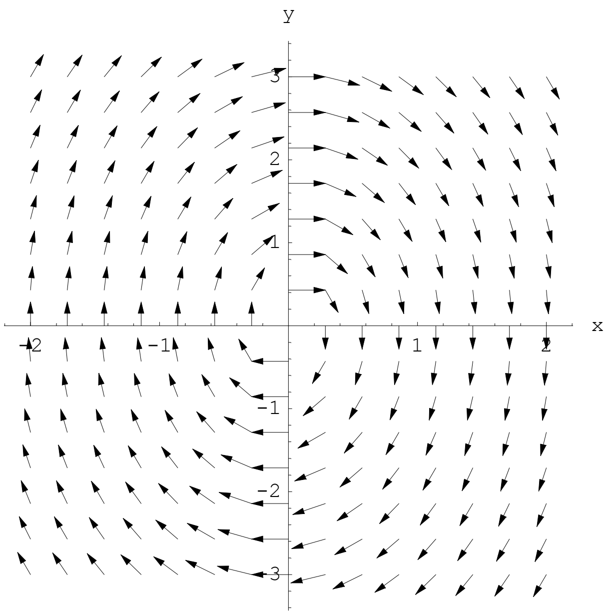
```
Out[141]= {{0, 1}, {-4, 0}}
```

(a)

```
In[142]:= Eigenvalues[c]

Out[142]= {-2 i, 2 i}

In[143]:= Eigenvectors[c]

Out[143]= {{i, 2}, {-i, 2}}
```

(b)

```
In[144]:= arrows3 = PlotVectorField[ {y, -4 x}, {x, -2, 2}, {y, -3, 3}, Axes -> True,
            AxesLabel -> {"x", "y"}, ScaleFunction -> (1 &), AspectRatio -> 1];
```




```
In[145]:= sol30 = DSolve[ {x'[t] == y[t], y'[t] == -4 x[t]}, {x[t], y[t]}, t]
```

Out[145]= $\{\{x[t] \to C[1]\, Cos[2\,t] + \frac{1}{2} C[2]\, Sin[2\,t],\ y[t] \to C[2]\, Cos[2\,t] - 2\, C[1]\, Sin[2\,t]\}\}$

```
In[146]:= sol3 = DSolve[
           {x'[t] == y[t], y'[t] == -4 x[t], x[0] == x0, y[0] == y0}, {x[t], y[t]}, t]
```

Out[146]= $\{\{x[t] \to \frac{1}{2} (2\, x0\, Cos[2\,t] + y0\, Sin[2\,t]),\ y[t] \to y0\, Cos[2\,t] - 2\, x0\, Sin[2\,t]\}\}$

(c)

```
In[147]:= x31 = sol3[[1, 1, 2]] /. {x0 -> 0, y0 -> 1}
```

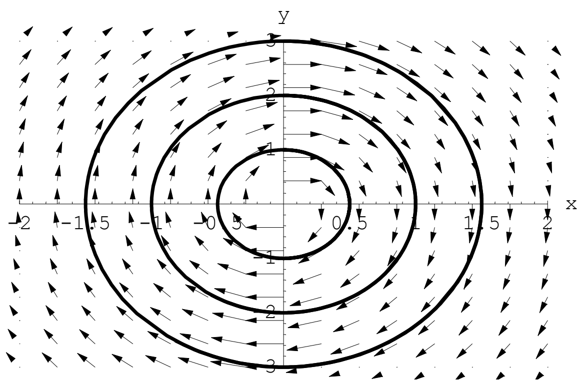
Out[147]= $\frac{1}{2} Sin[2\,t]$

```
In[148]:= y31 = sol3[[1, 2, 2]] /. {x0 -> 0, y0 -> 1}

Out[148]= Cos[2 t]

In[149]:= tr31 = ParametricPlot[Evaluate[ {x31, y31}], {t, 0, 5},
            DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

      General::spell1 :
       Possible spelling error: new symbol name "tr31" is similar to existing symbol "tr1".

In[150]:= x32 = sol3[[1, 1, 2]] /. {x0 -> 0, y0 -> 2}

Out[150]= Sin[2 t]

In[151]:= y32 = sol3[[1, 2, 2]] /. {x0 -> 0, y0 -> 2}

Out[151]= 2 Cos[2 t]
```

```
In[152]:= tr32 = ParametricPlot[Evaluate[ {x32, y32}], {t, 0, 5},
              DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

          General::spell :
           Possible spelling error: new symbol name "tr32" is similar to existing symbols {tr2, tr23}.

In[153]:= x33 = sol3[[1, 1, 2]] /. {x0 -> 0, y0 -> 3}
```

Out[153]= $\frac{3}{2}$ Sin[2 t]

```
In[154]:= y33 = sol3[[1, 2, 2]] /. {x0 -> 0, y0 -> 3}

Out[154]= 3 Cos[2 t]

In[155]:= tr33 = ParametricPlot[Evaluate[ {x33, y33}], {t, 0, 5},
              DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[156]:= Show[{tr31, tr32, tr33, arrows3}, DisplayFunction -> $DisplayFunction,
             AxesLabel -> {"x", "y"}, PlotRange -> {{-2, 2}, {-3, 3}}];
```



(d)

The diagram shows a centre node, with a clear cyclical pattern.

(iv)

```
In[157]:= Clear[x, y]
```

In[158]:= d = $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$

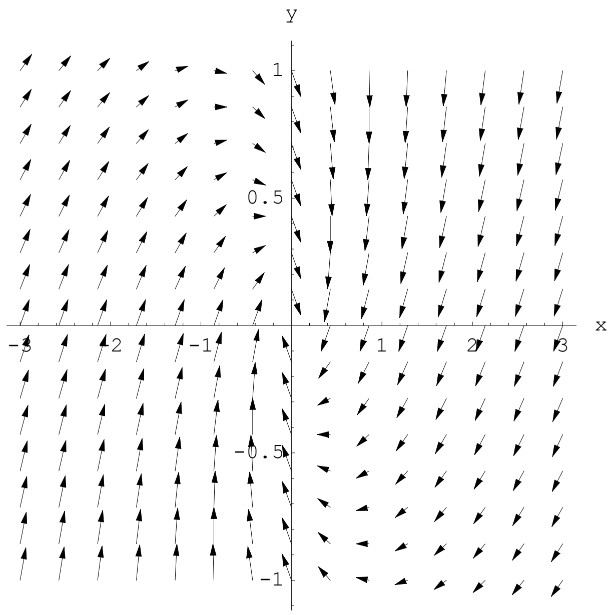
```
Out[158]= {{-1, 1}, {-1, -1}}
```

(a)

```
In[159]:= Eigenvalues[d]

Out[159]= {-1 - ⅈ, -1 + ⅈ}

In[160]:= Eigenvectors[d]

Out[160]= {{ⅈ, 1}, {-ⅈ, 1}}
```

(b)

```
In[161]:= arrows4 =
            PlotVectorField[ {-x + y, -x - y}, {x, -3, 3}, {y, -1, 1}, Axes -> True,
             AxesLabel -> {"x", "y"}, ScaleFunction -> (1 &), AspectRatio -> 1];
```



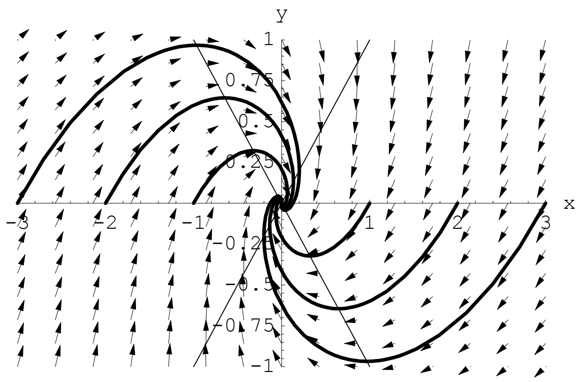

```
In[162]:= sol40 = DSolve[{x'[t] == -x[t] + y[t], y'[t] == -x[t] - y[t]}, {x[t], y[t]}, t]

Out[162]= {{x[t] → e^-t (C[1] Cos[t] + C[2] Sin[t]), y[t] → e^-t (C[2] Cos[t] - C[1] Sin[t])}}

In[163]:= sol4 = DSolve[{x'[t] == -x[t] + y[t],
             y'[t] == -x[t] - y[t], x[0] == x0, y[0] == y0}, {x[t], y[t]}, t]

Out[163]= {{x[t] → e^-t (x0 Cos[t] + y0 Sin[t]), y[t] → e^-t (y0 Cos[t] - x0 Sin[t])}}
```

(c)

```
In[164]:= x41 = sol4[[1, 1, 2]] /. {x0 -> 1, y0 -> 0}

Out[164]= e^-t Cos[t]

In[165]:= y41 = sol4[[1, 2, 2]] /. {x0 -> 1, y0 -> 0}

Out[165]= -e^-t Sin[t]

In[166]:= tr41 = ParametricPlot[ Evaluate[ {x41, y41}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

          General::spell1 :
           Possible spelling error: new symbol name "tr41" is similar to existing symbol "tr1".

In[167]:= x42 = sol4[[1, 1, 2]] /. {x0 -> 2, y0 -> 0}

Out[167]= 2 e^-t Cos[t]

In[168]:= y42 = sol4[[1, 2, 2]] /. {x0 -> 2, y0 -> 0}

Out[168]= -2 e^-t Sin[t]
```

```
In[169]:= tr42 = ParametricPlot[Evaluate[ {x42, y42}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

        General::spell :
         Possible spelling error: new symbol name "tr42" is similar to existing symbols {tr2, tr24}.

In[170]:= x43 = sol4[[1, 1, 2]] /. {x0 -> 3, y0 -> 0}

Out[170]= 3 e^-t Cos[t]

In[171]:= y43 = sol4[[1, 2, 2]] /. {x0 -> 3, y0 -> 0}

Out[171]= -3 e^-t Sin[t]

In[172]:= tr43 = ParametricPlot[ Evaluate[ {x43, y43}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

        General::spell1 :
         Possible spelling error: new symbol name "tr43" is similar to existing symbol "tr3".

In[173]:= x44 = sol4[[1, 1, 2]] /. {x0 -> -1, y0 -> 0}

Out[173]= -e^-t Cos[t]

In[174]:= y44 = sol4[[1, 2, 2]] /. {x0 -> -1, y0 -> 0}

Out[174]= e^-t Sin[t]

In[175]:= tr44 = ParametricPlot[ Evaluate[ {x44, y44}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

In[176]:= x45 = sol4[[1, 1, 2]] /. {x0 -> -2, y0 -> 0}

Out[176]= -2 e^-t Cos[t]

In[177]:= y45 = sol4[[1, 2, 2]] /. {x0 -> -2, y0 -> 0}

Out[177]= 2 e^-t Sin[t]

In[178]:= tr45 = ParametricPlot[ Evaluate[ {x45, y45}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

        General::spell1 :
         Possible spelling error: new symbol name "tr45" is similar to existing symbol "tr5".

In[179]:= x46 = sol4[[1, 1, 2]] /. {x0 -> -3, y0 -> 0}

Out[179]= -3 e^-t Cos[t]

In[180]:= y46 = sol4[[1, 2, 2]] /. {x0 -> -3, y0 -> 0}

Out[180]= 3 e^-t Sin[t]

In[181]:= tr46 = ParametricPlot[ Evaluate[ {x46, y46}], {t, 0, 5},
             DisplayFunction -> Identity, PlotStyle -> {Thickness[0.007]}];

        General::spell1 :
         Possible spelling error: new symbol name "tr46" is similar to existing symbol "tr6".

In[182]:= lines4 = Plot[{x, -x}, {x, -3, 3}, DisplayFunction -> Identity];
```

```
In[183]:= Show[{tr41, tr42, tr43, tr44, tr45, tr46, arrows4, lines4},
            DisplayFunction -> $DisplayFunction,
            AxesLabel -> {"x", "y"}, PlotRange -> {{-3, 3}, {-1, 1}}];
```



(d)

The diagram shows a spiral node, with a clockwise trajectory.

à Question 10

■ (i)

Fixed points are

In[184]:= $\text{Solve}\left[\left\{0 == x\left(1-\frac{x}{6}\right)-\frac{6\,x\,y}{(8+8\,x)},\ 0 == 0.2\,y\left(1-\frac{0.4\,y}{x}\right)\right\},\ \{x, y\}\right]$

Out[184]= $\{\{y \to -17.739,\ x \to -7.0956\},\ \{y \to 0.,\ x \to 6.\},\ \{y \to 2.11399,\ x \to 0.845595\}\}$

We rule out the negative values of *x* and *y*, so that there are two fixed points to consider
P1 = (0,6) and P2 = (0.8456,2.1140).

■ (ii)

In[185]:= $\text{fx} = \text{D}\left[x\left(1-\frac{x}{6}\right)-\frac{6\,x\,y}{(8+8\,x)},\ x\right]$

Out[185]= $1-\frac{x}{3}+\frac{48\,x\,y}{(8+8\,x)^2}-\frac{6\,y}{8+8\,x}$

In[186]:= $\text{fy} = \text{D}\left[x\left(1-\frac{x}{6}\right)-\frac{6\,x\,y}{(8+8\,x)},\ y\right]$

Out[186]= $-\frac{6\,x}{8+8\,x}$

In[187]:= $\text{gx} = \text{D}\left[0.2\,y\left(1-\frac{0.4\,y}{x}\right),\ x\right]$

Out[187]= $\frac{0.08\,y^2}{x^2}$

```
In[188]:= gy = D[0.2 y (1 - 0.4 y/x), y]
```

$$Out[188]= -\frac{0.08\,y}{x} + 0.2\left(1 - \frac{0.4\,y}{x}\right)$$

```
In[189]:= fx1 == fx /. {x → 6, y → 0}

Out[189]= fx1 == -1

In[190]:= fy1 == fy /. {x → 6, y → 0}
```

$$Out[190]= fy1 == -\frac{9}{14}$$

```
In[191]:= gx1 == gx /. {x → 6, y → 0}

Out[191]= gx1 == 0

In[192]:= gy1 = gy /. {x → 6, y → 0}

Out[192]= 0.2
```

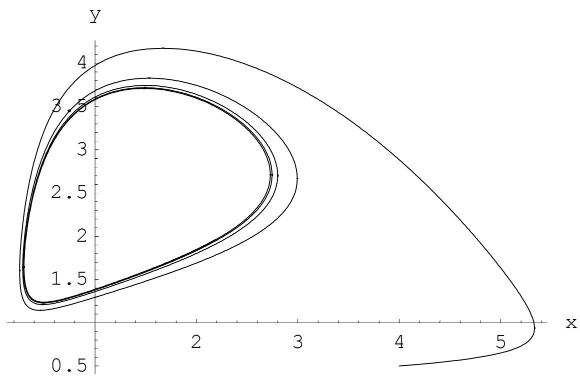
$$In[193]:= mA := \left\{\left\{-1, -\frac{9}{14}\right\}, \{0, 0.2\}\right\}$$

```
In[194]:= Eigenvalues[mA]

Out[194]= {-1., 0.2}

In[195]:= fx2 == fx /. {x → 0.8456, y → 2.1140}

Out[195]= fx2 == 0.252664

In[196]:= fy2 == fy /. {x → 0.8456, y → 2.1140}

Out[196]= fy2 == -0.343628

In[197]:= gx2 == gx /. {x → 0.8456, y → 2.1140}

Out[197]= gx2 == 0.5

In[198]:= gy2 = gy /. {x → 0.8456, y → 2.1140}

Out[198]= -0.2

In[199]:= mB := {{0.2527, -0.3436}, {0.5, -0.2}}

In[200]:= Eigenvalues[mB]

Out[200]= {0.02635 + 0.347226 i, 0.02635 - 0.347226 i}
```

Since only the eigenvalues of matrix mB are complex conjugate, then point P2 is a limit cycle. We can verify this by plotting the phase portrait. We take point (4, 0.5) as an initial point.

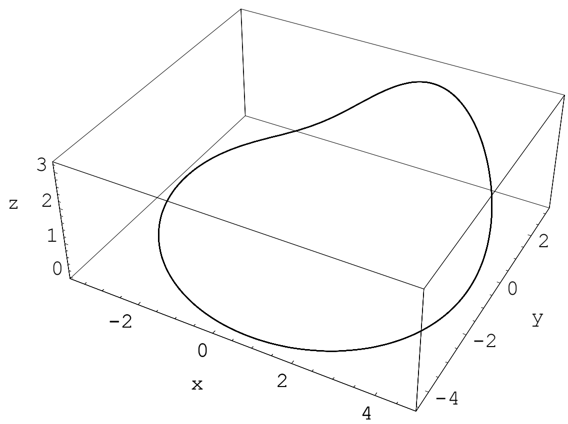
```
In[201]:= sol = NDSolve[
    {x'[t] == x[t] (1 - x[t]/6) - 6 x[t] y[t]/(8 + 8 x[t]), y'[t] == 0.2 y[t] (1 - 0.4 y[t]/x[t]),
     x[0] == 4, y[0] == 0.5}, {x, y}, {t, 0, 100}, MaxSteps → 1000]

Out[201]= {{x → InterpolatingFunction[{{0., 100.}}, <>],
    y → InterpolatingFunction[{{0., 100.}}, <>]}}
```

```
In[202]:= plotxy = ParametricPlot[Evaluate[{x[t], y[t]} /. sol], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"x", "y"}];
```



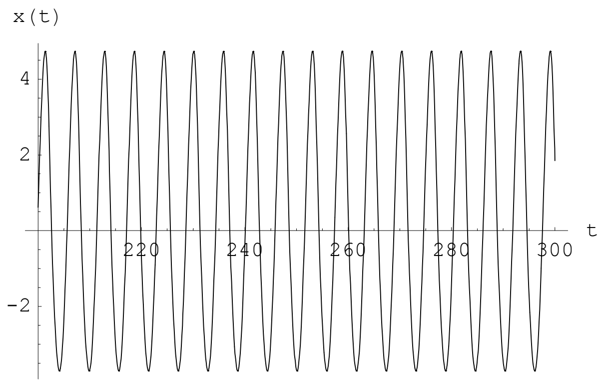
à Question 11

```
In[203]:= sol11 = NDSolve[{x'[t] == -y[t] - z[t], y'[t] == x[t] + 0.2 y[t],
             z'[t] == 0.2 + z[t] (x[t] - 2.5), x[0] == 1, y[0] == 1, z[0] == 1},
            {x, y, z}, {t, 200, 300}, MaxSteps → 5000]

Out[203]= {{x → InterpolatingFunction[{{200., 300.}}, <>],
            y → InterpolatingFunction[{{200., 300.}}, <>],
            z → InterpolatingFunction[{{200., 300.}}, <>]}}

In[204]:= plot11a =
            ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. sol11], {t, 200, 300},
             PlotPoints → 3000, PlotRange → All, AxesLabel → {"x", "y", "z"}];
```



```
In[205]:= plot11b = ParametricPlot[Evaluate[{t, x[t]} /. sol11], {t, 200, 300},
            PlotPoints → 1000, PlotRange → All, AxesLabel → {"t", "x(t)"}];
```

```
General::spell1 :
 Possible spelling error: new symbol name "plot11b" is similar to existing symbol "plot11a".
```



à Question 12

■ (i)

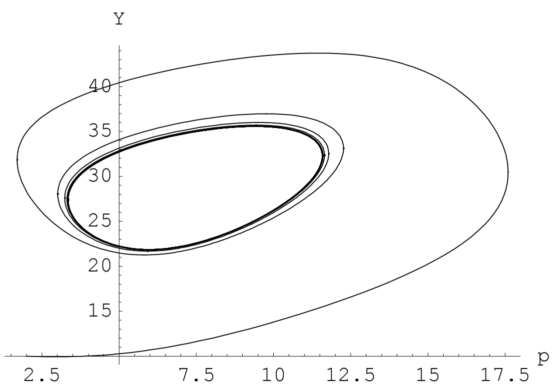
```
In[206]:= Solve[{p == 0.5 + 0.25 Y, Y == -0.025 p^3 + 0.75 p^2 - 6 p + 40}, {p, Y}]

Out[206]= {{Y → 26.2213, p → 7.05533}, {Y → 43.8893 - 41.2802 ⅈ, p → 11.4723 - 10.32 ⅈ},
            {Y → 43.8893 + 41.2802 ⅈ, p → 11.4723 + 10.32 ⅈ}}
```

So the economically meaninful fixed point is (p^*, Y^*) = (7.0553, 26.2213).

■ (ii)

```
In[207]:= sol12 = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 2, Y[0] == 10},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

Out[207]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}
```

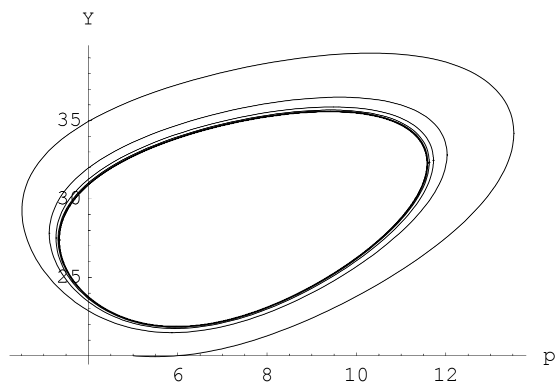
```
In[208]:= plot12 = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol12], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];
```



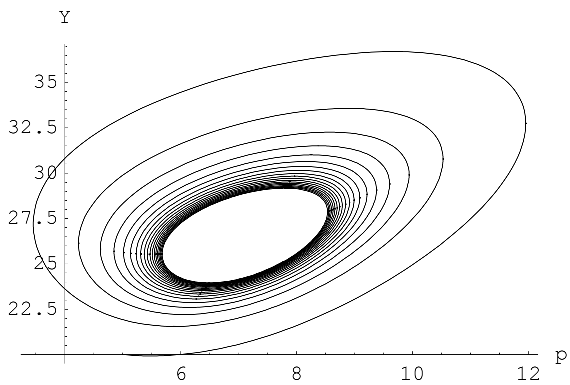
It can be seen that the system has a stable limit cycle.

à Question 13

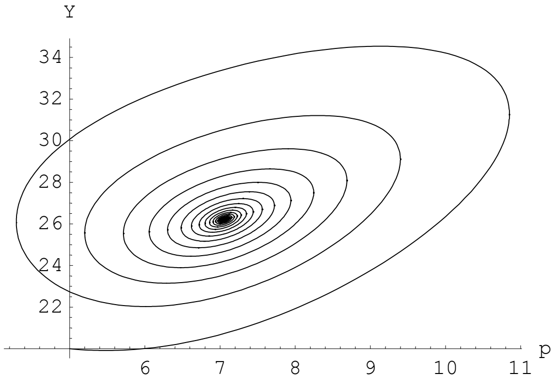
```
In[209]:= sol13a = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

Out[209]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[210]:= plot13a = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol13a], {t, 0, 100},
              PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];
```



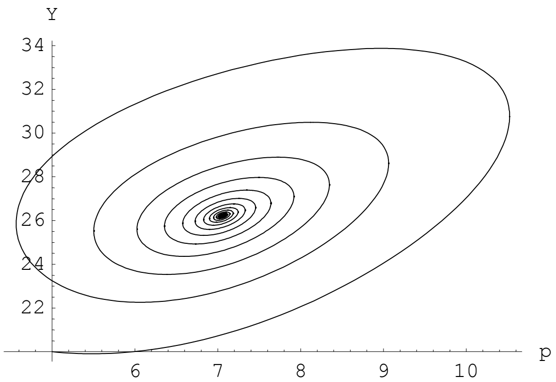

```
In[211]:= sol13b = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2.5 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

          General::spell1 :
           Possible spelling error: new symbol name "sol13b" is similar to existing symbol "sol13a".

Out[211]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[212]:= plot13a = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol13b], {t, 0, 100},
              PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];
```



```
In[213]:= sol13c = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
            Y'[t] == 3 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
           {p, Y}, {t, 0, 100}, MaxSteps → 3000]

        General::spell : Possible spelling error: new
           symbol name "sol13c" is similar to existing symbols {sol13a, sol13b}.

Out[213]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[214]:= plot13a = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol13c], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];
```




```
In[215]:= sol13d = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
            Y'[t] == 3.2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
           {p, Y}, {t, 0, 100}, MaxSteps → 3000]

        General::spell : Possible spelling error: new symbol
           name "sol13d" is similar to existing symbols {sol13a, sol13b, sol13c}.

Out[215]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[216]:= plot13d = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol13d], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

        General::spell1 :
         Possible spelling error: new symbol name "plot13d" is similar to existing symbol "plot13a".
```



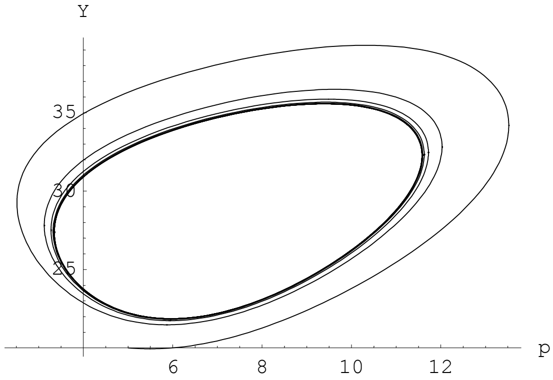
à Question 14

```
In[217]:= Solve[{p == 0.5 + 0.25 Y, Y == -0.025 p^3 + 0.75 p^2 - 8 p + 40}, {p, Y}]

Out[217]= {{Y → 16.5418, p → 4.63546}, {Y → 48.7291 - 56.7921 ⅈ, p → 12.6823 - 14.198 ⅈ},
           {Y → 48.7291 + 56.7921 ⅈ, p → 12.6823 + 14.198 ⅈ}}
```

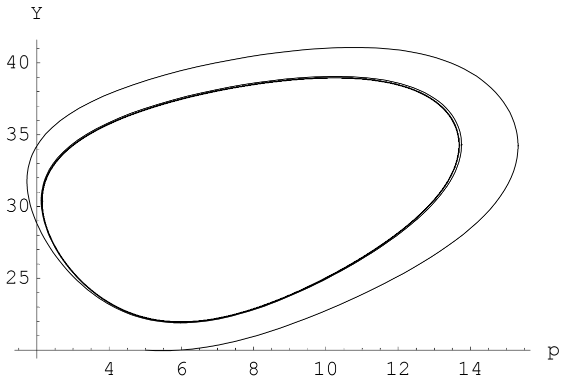
Economically meaningful fixed point is then (p^*, Y^*) = (4.6355, 16.5418).

```
In[218]:= sol14a = NDSolve[{p'[t] == 0.5 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

Out[218]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[219]:= plot14a = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol14a], {t, 0, 100},
             PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

In[222]:= sol14c = NDSolve[{p'[t] == (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

        General::spell : Possible spelling error: new
           symbol name "sol14c" is similar to existing symbols {sol14a, sol14b}.

Out[222]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[223]:= plot14c = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol14c], {t, 0, 100},
             PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

        General::spell : Possible spelling error: new
           symbol name "plot14c" is similar to existing symbols {plot14a, plot14b}.

In[220]:= sol14b = NDSolve[{p'[t] == 0.75 (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
             Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
            {p, Y}, {t, 0, 100}, MaxSteps → 3000]

        General::spell1 :
         Possible spelling error: new symbol name "sol14b" is similar to existing symbol "sol14a".

Out[220]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[221]:= plot14b = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol14b], {t, 0, 100},
             PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

        General::spell1 :
         Possible spelling error: new symbol name "plot14b" is similar to existing symbol "plot14a".
```

```
In[221]:= plot14b = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol14b], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

        General::spell1 :
         Possible spelling error: new symbol name "plot14b" is similar to existing symbol "plot14a".
```




```
In[222]:= sol14c = NDSolve[{p'[t] == (-0.025 p[t]^3 + 0.75 p[t]^2 - 6 p[t] + 40 - Y[t]),
            Y'[t] == 2 (p[t] - 0.5 - 0.25 Y[t]), p[0] == 5, Y[0] == 20},
           {p, Y}, {t, 0, 100}, MaxSteps → 3000]

        General::spell : Possible spelling error: new
           symbol name "sol14c" is similar to existing symbols {sol14a, sol14b}.

Out[222]= {{p → InterpolatingFunction[{{0., 100.}}, <>],
            Y → InterpolatingFunction[{{0., 100.}}, <>]}}

In[223]:= plot14c = ParametricPlot[Evaluate[{p[t], Y[t]} /. sol14c], {t, 0, 100},
            PlotPoints → 3000, PlotRange → All, AxesLabel → {"p", "Y"}];

        General::spell : Possible spelling error: new
           symbol name "plot14c" is similar to existing symbols {plot14a, plot14b}.
```



à Question 15

Although requested to perform this on a spreadsheet, we shall display the result here in *Mathematica*.

```
In[224]:= sol15a = NDSolve[{x'[t] == -y[t] - z[t], y'[t] == x[t] + 0.4 y[t],
            z'[t] == 2 + z[t] (x[t] - 4), x[0] == 0.1, y[0] == 0.1, z[0] == 0.1},
           {x, y, z}, {t, 200, 300}, MaxSteps → 5000]

Out[224]= {{x → InterpolatingFunction[{{200., 300.}}, <>],
            y → InterpolatingFunction[{{200., 300.}}, <>],
            z → InterpolatingFunction[{{200., 300.}}, <>]}}
```