Appendix B

Matrices

This appendix provides an overview of several of the concepts and properties relating to matrices.

B.1 Definitions and basic operations

An $m \times n$ *real matrix* $\mathbf{A}$ is a two-dimensional *rectangular array* of mn real numbers arranged in m rows and n columns as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

It is often denoted as $\mathbf{A} = [a_{ij}]$ with $1 \leq i \leq m$ called the *row index* and $1 \leq j \leq n$ called the *column index*. The set of all real $m \times n$ matrices is denoted by $\mathbb{R}^{m\times n}$. When $m = n$, it is called a *square* matrix of *size* or *order* m. When all the elements of $\mathbf{A}$ are zero, it is called the *null* or *zero* matrix. The ith row of $\mathbf{A}$ is denoted by a_{i*} and the jth column as a_{*j}. Hence, the matrix $\mathbf{A}$ can be represented as

$$\mathbf{A} = [a_{ij}] = [a_{*1} \ a_{*2} \ \cdots \ a_{*n}] = \begin{bmatrix} a_{1*} \\ a_{2*} \\ \vdots \\ a_{m*} \end{bmatrix}.$$

The set of elements $(a_{11}, a_{22}, \ldots, a_{mm})$ is called the *principal or main diagonal* of $\mathbf{A}$. Diagonals parallel to this principal diagonal and *above*(*below*) the main diagonal are called *super*(*sub*) diagonals.

Operations on matrices Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be matrices in $\mathbb{R}^{m\times n}$, p, q be scalars, and $\mathbf{y} \in \mathbb{R}^m$ and $\mathbf{x} \in \mathbb{R}^n$.

(a) **Sum/Difference** $\mathbf{C} = \mathbf{A} \pm \mathbf{B}$, where $c_{ij} = a_{ij} \pm b_{ij}$, is called the element-wise sum/difference of matrices $\mathbf{A}$ and $\mathbf{B}$.
(b) **Scalar multiplication** $\mathbf{C} = p\mathbf{A}$, where $c_{ij} = pa_{ij}$, is called the multiplication of $\mathbf{A}$ by the scalar p.
(c) **Matrix-vector product** $\mathbf{y} = \mathbf{A}\mathbf{x}$, where the ith element of $\mathbf{y}$ is defined by the *inner product* of the ith row of $\mathbf{A}$ with $\mathbf{x}$ as

$$y_i = \sum_{j=1}^{n} a_{ij} x_j \text{ for } 1 \leq i \leq m.$$

Alternatively, $\mathbf{y}$ can also be expressed as a *linear combination of the columns* of $\mathbf{A}$ as

$$\mathbf{y} = \sum_{j=1}^{m} a_{*j} x_j$$

where the elements of the vector $\mathbf{x}$ are used as the coefficients of the linear combination.
(d) **Matrix-matrix product** Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times r}$. Then, the product $\mathbf{C} = \mathbf{AB} \in \mathbb{R}^{m \times r}$ can be defined in three equivalent ways:
  (a) **Inner product** The element c_{ij} is the *inner product* of the ith row of $\mathbf{A}$, and jth column of $\mathbf{B}$:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}, \quad 1 \leq i \leq m, 1 \leq j \leq r.$$

  (b) **Scalar times a vector** The jth column c_{*j} of $\mathbf{C}$ is the *linear combination of the columns of* $\mathbf{A}$, using the elements of the jth column of $\mathbf{B}$ as the coefficients, that is,

$$c_{*j} = \sum_{i=1}^{n} a_{*j} b_{ij}, \; 1 \leq j \leq r.$$

  (c) **Outer product** The product matrix $\mathbf{C}$ can also be expressed as the sum of n *outer product* matrices obtained by the jth column of $\mathbf{A}$ and jth row of $\mathbf{B}$ as

$$\mathbf{C} = \sum_{j=1}^{n} a_{*j} b_{j*}.$$

  The following properties of these operations are easily verified:
  (i) $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
  (ii) $p\mathbf{A} = \mathbf{A}p$
  (iii) $\mathbf{AB} \neq \mathbf{BA}$
  (iv) $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}$
  (v) $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
  (vi) $(p + q)\mathbf{A} = p\mathbf{A} + q\mathbf{A}$

(vii) $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$
(viii) $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$
(ix) $p(\mathbf{A} + \mathbf{B}) = p\mathbf{A} + p\mathbf{B}$.

Special matrices, other operations and properties
Let $\mathbf{A} \in \mathbb{R}^{m\times m}$, a square matrix.

(a) **Diagonal matrix** The matrix $\mathbf{A}$ is said to be a *diagonal* matrix, if $a_{ij} = 0$ for $i \neq j$. Diagonal matrix $\mathbf{A}$ is denoted by

$$\mathbf{A} = \mathrm{Diag}(a_{11}, a_{22}, \ldots, a_{mm})$$

(b) **Identity matrix $\mathbf{I}_m$** is a diagonal matrix of order m with all its m diagonal elements equal to 1, and all other elements zero. Thus,

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) **Upper/lower triangular matrix** $\mathbf{A}$ is an *upper triangular* matrix, if $a_{ij} = 0$, for $i > j$. Similarly, $\mathbf{A}$ is a *lower triangular* matrix, if $a_{ij} = 0$, for $i < j$.
(d) **Tri-diagonal matrix** $\mathbf{A}$ is *tri-diagonal* if $a_{ij} = 0$ for $|i - j| > 1$.
(e) **Transpose of a matrix** If $\mathbf{A} \in \mathbb{R}^{m\times n}$, then $\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{n\times m}$ is called the *transpose* of $\mathbf{A}$ obtained by interchanging the columns and rows of $\mathbf{A}$. It can be verified that

$$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}$$
$$(\mathbf{A} + \mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$$
$$(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}.$$

(f) **Trace of a matrix** If $\mathbf{A} \in \mathbb{R}^{m\times m}$, *trace* of $\mathbf{A}$ denoted by $\mathrm{tr}(\mathbf{A})$ is (a scalar) defined by the sum of its diagonal elements:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{m} a_{ii}.$$

It can be verified that

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^{\mathrm{T}})$$
$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$
$$\mathrm{tr}(\alpha\mathbf{A}) = \alpha\mathrm{tr}(\mathbf{A})$$
$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$
$$\mathrm{tr}(\mathbf{ABC}) = \mathrm{tr}(\mathbf{BCA}) = \mathrm{tr}(\mathbf{CAB})$$
$$\mathrm{tr}(\mathbf{A}^{-1}\mathbf{BA}) = \mathrm{tr}(\mathbf{B})(\mathbf{A}^{-1}\text{is defined below as item j}).$$

(g) **Determinant of a matrix** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. Determinant of $\mathbf{A}$, denoted by $\det(\mathbf{A})$ is (a scalar) defined recursively as follows: For any row index i, fixed,

$$\det(\mathbf{A}) = \sum_{j=1}^{m} a_{ij} A_{ij}$$

where A_{ij} is called the *cofactor* of the element a_{ij} given by

$$A_{ij} = (-1)^{i+j} M_{ij}$$

with M_{ij}, called the *minor* of a_{ij} which is the *determinant* of the $n-1 \times n-1$ matrix obtained by deleting the ith row and jth column of $\mathbf{A}$. It can be verified that

$$\begin{aligned}
&\det(\mathbf{A}^{\mathrm{T}}) = \det(\mathbf{A})\\
&\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})\\
&\det(\mathbf{A}^{-1}) = \tfrac{1}{\det(\mathbf{A})} (\mathbf{A}^{-1}\text{defined below as item j}).
\end{aligned}$$

(h) **Rank of a matrix** Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. The number of *linearly independent columns(rows)* of $\mathbf{A}$ (Appendix A) is called the *column(row) rank* of $\mathbf{A}$. Thus, the column (row) rank of $\mathbf{A}$ is less than or equal to $n(m)$. It turns out that for any given matrix $\mathbf{A}$, its column and row ranks are always equal and this common integer value is called the *rank* of $\mathbf{A}$, and is denoted by $\mathrm{Rank}(\mathbf{A})$, that is

$$0 \leq \mathrm{Rank}(\mathbf{A}) \leq \min\{m, n\}.$$

If the $\mathrm{Rank}(\mathbf{A}) = \min\{m, n\}$, then it is said to be a matrix of *full rank*, otherwise it is *rank-deficient*. We now list several important properties of rank.

(a) $\mathrm{Rank}(\mathbf{A}^{\mathrm{T}}) = \mathrm{Rank}(\mathbf{A})$
(b) $\mathrm{Rank}(\mathbf{A}+\mathbf{B}) \leq \mathrm{Rank}(\mathbf{A}) + \mathrm{Rank}(\mathbf{B})$
(c) $\mathrm{Rank}(\mathbf{A}-\mathbf{B}) \geq |\mathrm{Rank}(\mathbf{A}) - \mathrm{Rank}(\mathbf{B})|$
(d) If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times r}$, then

$$\mathrm{Rank}(\mathbf{AB}) \leq \min\{\mathrm{Rank}(\mathbf{A}), \mathrm{Rank}(\mathbf{B})\}$$

(e) If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{m}$, then the rank of the outer product matrix $\mathbf{xy}^{\mathrm{T}}$ is unity, i.e. $\mathrm{Rank}(\mathbf{xy}^{\mathrm{T}}) = 1$.

(i) **Non-singularity of a matrix** A square matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is said to be *non-singular*, if any one of the following conditions holds:

(a) $\det(\mathbf{A}) \neq 0$.
(b) $\mathrm{Rank}(\mathbf{A}) = m$.
(c) All columns (rows) of $\mathbf{A}$ are linearly independent.

(j) **Inverse of a matrix** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be non-singular. Then, there exists a (unique) multiplicative inverse denoted by $\mathbf{A}^{-1} \in \mathbb{R}^{m\times m}$, such that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_m.$$

It can be verified that

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$
$$(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}} = \mathbf{A}^{-\mathrm{T}}.$$

(k) **Sherman–Morrison–Woodbury formula** Let $\mathbf{c}, \mathbf{d} \in \mathbb{R}^m$. Then $\mathbf{cd}^{\mathrm{T}}$ is a rank-one outer-product matrix. Adding an outer-product matrix to a non-singular matrix is called the *rank-one perturbation*. There is an interesting relation between the inverse of $\mathbf{A}$ and that of $(\mathbf{A} + \mathbf{cd}^{\mathrm{T}})$.

(a) **An identity** The following identity is easily verified:

$$(\mathbf{I}_m + \mathbf{cd}^{\mathrm{T}})^{-1} = \mathbf{I}_m - \frac{\mathbf{cd}^{\mathrm{T}}}{1 + \mathbf{d}^{\mathrm{T}}\mathbf{c}}.$$

(b) **Sherman–Morrison formula**

$$(\mathbf{A} + \mathbf{cd}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{cd}^{\mathrm{T}}\mathbf{A}^{-1}}{1 + \mathbf{d}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{c}}.$$

(c) **Woodbury's Extension** Let $\mathbf{C} \in \mathbb{R}^{m\times k}$ and $\mathbf{D} \in \mathbb{R}^{m\times k}$ be matrices of rank k. Then

$$(\mathbf{A} + \mathbf{CD}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}[\mathbf{I}_k + \mathbf{D}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}]^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{A}^{-1}.$$

Similarly, if $\mathbf{A}$ and $\mathbf{B}$ are invertible, then

$$(\mathbf{A} + \mathbf{CBD}^{\mathrm{T}})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{C}[\mathbf{B}^{-1} + \mathbf{D}^{\mathrm{T}}\mathbf{A}^{-1}\mathbf{C}]^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{A}^{-1}.$$

(l) **Another useful matrix identity**

$$[\mathbf{A}^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{A} + \mathbf{D}^{-1}]\mathbf{A}^{\mathrm{T}}\mathbf{B}^{-1} = \mathbf{DA}^{\mathrm{T}}[\mathbf{B} + \mathbf{ADA}^{\mathrm{T}}]^{-1}.$$

(m) **Moore–Penrose generalized inverse** Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Then, $\mathbf{A}^{+} \in \mathbb{R}^{n\times m}$ is called the *Moore–Penrose inverse* of $\mathbf{A}$ if the following four conditions hold:

(a) $\mathbf{AA}^{+}\mathbf{A} = \mathbf{A}$
(b) $\mathbf{A}^{+}\mathbf{AA}^{+} = \mathbf{A}^{+}$
(c) $(\mathbf{A}^{+}\mathbf{A})^{\mathrm{T}} = \mathbf{A}^{+}\mathbf{A}$, ($\mathbf{A}^{+}\mathbf{A}$ is a symmetric matrix)
(d) $(\mathbf{AA}^{+})^{\mathrm{T}} = \mathbf{AA}^{+}$, ($\mathbf{AA}^{+}$ is a symmetric matrix).

Remark B.1.1 $\mathbf{AA}^{+} \in \mathbb{R}^{m\times m}$ as a symmetric matrix represents the *orthogonal projection* on to the subspace spanned by the columns of $\mathbf{A}$, also called the *range space* of $\mathbf{A}$, $R(\mathbf{A})$. Similarly, $\mathbf{AA}^{+} \in \mathbb{R}^{n\times n}$ as a symmetric matrix represents the orthogonal projection on to the subspace spanned by the rows of $\mathbf{A}$, called the range space of $\mathbf{A}^{\mathrm{T}}$, $R(\mathbf{A}^{\mathrm{T}})$. When $\mathbf{A} \in \mathbb{R}^{m\times m}$ is non-singular, then $\mathbf{A}^{+} = \mathbf{A}^{-1}$ and $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_m$.

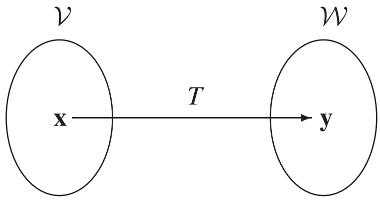


Fig. B.2.1 Concept of a transformation of vector space $\mathcal{V}$ into $\mathcal{W}$.

B.2 Linear transformation and operators

Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces, and let $T : \mathcal{V} \longrightarrow \mathcal{W}$ be a mapping or a transformation that associates a vector $\mathbf{x} \in \mathcal{V}$ to a vector $\mathbf{y} \in \mathcal{W}$, i.e. $\mathbf{y} = T(\mathbf{x})$. Refer to Figure B.2.1. The vector space $\mathcal{V}$ is called the *domain* of T and the set $\{\mathbf{y}|\mathbf{y} = T(\mathbf{x}),\ \text{for } \mathbf{x} \in \mathcal{V}\}$ is called the *range* of T. When $\mathcal{V} = \mathcal{W}$, then T is called an *operator*. Thus, an operator is a transformation that maps a vector space into itself.

The *null space* of T is the set of all $\mathbf{x} \in \mathcal{V}$ such that $T(\mathbf{x}) = 0$. The transformation T is said to be one-to-one, if

$$T(\mathbf{x}_1) = T(\mathbf{x}_2) \quad \text{exactly when } \mathbf{x}_1 = \mathbf{x}_2,$$

and T is called *linear* if

$$T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2) - \text{ additivity}$$

and for any scalar $a \in \mathbb{R}$ and every $\mathbf{x} \in \mathcal{V}$

$$T(a\mathbf{x}) = aT(\mathbf{x}) - \text{ homogeneity}$$

hold.

Given a vector space, we can find a basis and a system of coordinates for it. It is a fundamental fact that with respect to the chosen basis for $\mathcal{V}$ and $\mathcal{W}$, every linear transformation between finite dimensional vector spaces $\mathcal{V}$ and $\mathcal{W}$ can be represented by a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, where n and m are respectively the dimensions of $\mathcal{V}$ and $\mathcal{W}$. Henceforth, we will *not* distinguish between the linear transformation and the matrix representing it. Thus, $\mathbf{y} = T(\mathbf{x})$ is replaced with $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A} \in \mathbb{R}^{m\times n}$. In view of this, we can draw from the properties of matrices to draw conclusions about linear transformations.

Remark B.2.1 It is obvious that there are more than one linear transformations from the vector space $\mathcal{V}$ into the vector space $\mathcal{W}$. The set of all linear transformations from $\mathcal{V}$ to $\mathcal{W}$ is denoted by $B(\mathcal{V}, \mathcal{W})$. When $\mathcal{V} = \mathcal{W}$, the $B(\mathcal{V})$ denotes the set of all *linear operators* on $\mathcal{V}$. When $\mathcal{V}$ is a finite-dimensional vector space, we can analyze the properties of linear operators *via* the properties of square matrices.

Examples of transformation/operator

(a) **Translation** The operator $T : \mathcal{V} \longrightarrow \mathcal{V}$ is called a *translation* if

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{a} \text{ for every } \mathbf{x} \in \mathcal{V},$$

for some fixed $\mathbf{a} \in \mathcal{V}$. Since

$$T(\mathbf{x}+\mathbf{y}) = \mathbf{a} + \mathbf{x} + \mathbf{y} \neq T(\mathbf{x}) + T(\mathbf{y}) = 2\mathbf{a} + x + y,$$

it follows that translation is *not* a linear operator.
A well-known translation operation in $\mathbb{R}^m$ is called *centering* defined by

$$T(\mathbf{x}) = \mathbf{x} - \bar{\mathbf{x}}\mathbf{1}$$

where $\bar{\mathbf{x}} = \frac{1}{m}\sum_{i=1}^{m} x_i$ and $\mathbf{1} = (1\ 1\ 1\ \cdots\ 1)^{\mathrm{T}}$, a vector with all its elements as 1. Translation leaves the distance between vectors unaltered, that is $\|\mathbf{x}-\mathbf{y}\|_2 = \|T(\mathbf{x}) - T(\mathbf{y})\|_2$.

(b) **Rotation** Every $m \times m$ orthogonal matrix (recall $\mathbf{Q}$ is orthogonal if $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$, that is, $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$) represents a rigid-body rotation in $\mathbb{R}^m$. As an example,

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is a 2×2 orthogonal matrix that represents a clockwise rotation by an angle θ. Since $\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}^{-1}$,

$$\mathbf{Q}^{\mathrm{T}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

gives the anti-clockwise rotation by an angle θ. Orthogonal rotation leaves the lengths of the vectors unaltered, that is,

$$\|\mathbf{Q}(\mathbf{x})\|_2^2 = (\mathbf{Q}(\mathbf{x}))^{\mathrm{T}}(\mathbf{Q}(\mathbf{x})) = \mathbf{x}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{x} = \|\mathbf{x}\|_2^2 .$$

Remark B.2.2 Translation and rotation can also be viewed as mechanisms for transformation of coordinate systems.

(c) **General coordinate transformation** Let $\mathcal{B}_1 = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ be the standard basis for $\mathbb{R}^m$. Let $\mathcal{B}_2 = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m\}$ be a new basis for $\mathbb{R}^m$. First express each member of the new basis in terms of the standard basis as follows:

$$\mathbf{g}_i = t_{1i}\mathbf{e}_1 + t_{2i}\mathbf{e}_2 + \cdots + t_{mi}\mathbf{e}_m, \ 1 \le i \le m, \tag{B.2.1}$$

where $(t_{1i}, t_{2i}, \ldots, t_{mi})^{\mathrm{T}}$ is the representation of the vector $\mathbf{g}_i$ in the standard basis. Rewriting these m relations in the matrix notation, we obtain:

$$[\mathbf{g}_1\, \mathbf{g}_2\, \cdots\, \mathbf{g}_m] = [\mathbf{e}_1\, \mathbf{e}_2\, \cdots\, \mathbf{e}_m] \begin{bmatrix} t_{11} & t_{12} \cdots & t_{1m} & \\ t_{21} & t_{22} \cdots & t_{2m} & \\ \vdots & \vdots & \vdots & \vdots \\ t_{m1} & t_{m2} \cdots & t_{mm} & \end{bmatrix}$$

or

$$\mathbf{G} = \mathbf{ET}$$

where $\mathbf{G} = [\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_m] \in \mathbb{R}^{m\times m}$, the matrix formed by the members of the new basis $\mathcal{B}_2$, $\mathbf{E} = [\mathbf{e}_1\, \mathbf{e}_2\, \cdots\, \mathbf{e}_m] \in \mathbb{R}^{m\times m}$, the matrix formed by the members of the standard basis $\mathcal{B}_1$ and $\mathbf{T} \in \mathbb{R}^{m\times m}$ is the matrix whose elements are the coordinates of the new basis with respect to the standard basis. It can be readily verified that $\mathbf{T}$ is a *non-singular* matrix.
Now consider a vector $\mathbf{x} \in \mathbb{R}^m$. Let

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_m\mathbf{e}_m \tag{B.2.2}$$

be the representation of $\mathbf{x}$ in the standard basis $\mathcal{B}_1$ and let

$$\mathbf{x} = x_1^*\mathbf{g}_1 + x_2^*\mathbf{g}_2 + \cdots + x_m^*\mathbf{g}_m \tag{B.2.3}$$

be the representation of the same $\mathbf{x}$ in the new basis $\mathcal{B}_2$. Now combining (B.2.1) with (B.2.3), it follows that

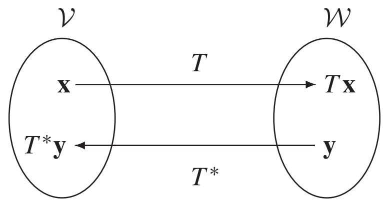
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1m} \\ t_{21} & t_{22} & \cdots & t_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ t_{m1} & t_{m2} & \cdots & t_{mm} \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_m^* \end{bmatrix}$$

or

$$\mathbf{x} = \mathbf{T}\mathbf{x}^*. \tag{B.2.4}$$

Thus, $\mathbf{T} : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a matrix that provides a bridge between the representation of the vectors in the standard and new bases. This matrix $\mathbf{T}$ is non-singular but is *not* orthogonal and, hence, sometimes called *oblique* rotation.

(d) **Similarity transformation** Let $\mathbf{x}$ and $\mathbf{y}$ denote two vectors in $\mathbb{R}^m$ in the standard basis and let $\mathbf{A}$ be the matrix representation of a linear operator in the standard basis such that $\mathbf{y} = \mathbf{Ax}$. Let $\mathbf{T}$ be the non-singular matrix that represents the transformation of the standard basis $\mathcal{B}_1$ to the new basis $\mathcal{B}_2$. Let $\mathbf{x}^*$ and $\mathbf{y}^*$ be the representation of $\mathbf{x}$ and $\mathbf{y}$ in the new basis $\mathcal{B}_2$, where $\mathbf{x} = \mathbf{T}\mathbf{x}^*$ and $\mathbf{y} = \mathbf{T}\mathbf{y}^*$.



$$< T\mathbf{x}, \mathbf{y} > = < \mathbf{x}, T^*\mathbf{y} >$$

Fig. B.2.2 Definition of the adjoint operator.

Since $\mathbf{y} = \mathbf{A}\mathbf{x}$, by substituting, we obtain

$$\mathbf{y} = \mathbf{T}\mathbf{y}^* = \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{T}\mathbf{x}^*$$

or

$$\mathbf{y}^* = (\mathbf{T}^{-1}\mathbf{A}\mathbf{T})\mathbf{x}^*. \tag{B.2.5}$$

In other words, $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is the representation of the matrix $\mathbf{A}$ in the new basis $\mathcal{B}_2$. This transformation of the matrix $\mathbf{A}$ to the matrix $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is called the *similarity transformation*. Similarity transformation preserves the eigenvalues (Section B.5). That is, $\mathbf{A}$ and $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ have the same set of eigenvalues.

(e) **Congruence transformation** Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. If $\mathbf{B} \in \mathbb{R}^{m\times m}$ is a non-singular matrix, then $\mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{B}$ is called the *congruent transformation* of $\mathbf{A}$. That is, $\mathbf{A}$ and $\mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{B}$ are said to be *congruent* matrices. Congruent transformation preserves several properties of $\mathbf{A}$. Thus, $\mathbf{B}^{\mathrm{T}}\mathbf{A}\mathbf{B}$ is symmetric, skew-symmetric, positive (semi)definite, whenever $\mathbf{A}$ is symmetric, skew-symmetric or positive (semi)definite.

(f) **Adjoint Operator** Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces, each endowed with its own inner product. Let $< \mathbf{x}, \mathbf{y} >_{\mathcal{V}}$ denote the inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{V}$. Likewise, $< \mathbf{x}, \mathbf{y} >_{\mathcal{W}}$ is defined. Let $T : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation such that $\mathbf{y} = T\mathbf{x}$, where $\mathbf{x} \in \mathcal{V}$ and $\mathbf{y} \in \mathcal{W}$. If there exists a linear transformation $T^* : \mathcal{W} \longrightarrow \mathcal{V}$, such that

$$< T\mathbf{x}, \mathbf{y} >_{\mathcal{W}} = < \mathbf{x}, T^*\mathbf{y} >_{\mathcal{V}}$$

then T^* is called the *adjoint* of T. Refer to Figure B.2.2 for an illustration.

We now specialize this definition. Let $\mathcal{V} \subseteq \mathbb{R}^n$ and $\mathcal{W} \subseteq \mathbb{R}^m$. Then, both T and T^* as linear operators have a matrix representation in the chosen basis for $\mathcal{V}$ and $\mathcal{W}$. Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{A}^* \in \mathbb{R}^{n\times m}$ be the matrices corresponding to T and T^*, where $\mathbf{A} : \mathcal{V} \longrightarrow \mathcal{W}$ and $\mathbf{A}^* : \mathcal{W} \longrightarrow \mathcal{V}$. Then, from the above definition, we immediately have

$$< \mathbf{A}\mathbf{x}, \mathbf{y} > = (\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{y} = < \mathbf{x}, \mathbf{A}^*\mathbf{y} > = \mathbf{x}^{\mathrm{T}}\mathbf{A}^*\mathbf{y}$$

from which it follows that $\mathbf{A}^* = \mathbf{A}^{\mathrm{T}}$. That is, when $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional vector spaces, transpose of $\mathbf{A}$ is, in fact, the *adjoint of* $\mathbf{A}$.

When $\mathcal{V} = \mathcal{W} = \mathbb{R}^m$, then $\mathbf{A} \in \mathbb{R}^{m\times m}$ is a linear *operator* and $\mathbf{A}^* = \mathbf{A}^{\mathrm{T}}$ is called the adjoint operator. In the special case, when $\mathbf{A}$ is a *symmetric matrix*, then $\mathbf{A}^* = \mathbf{A}^{\mathrm{T}} = \mathbf{A}$, and $\mathbf{A}$ is called a *self-adjoint operator*.

The following properties can be readily established.

(a) Adjoint $\mathbf{A}^*$ of a linear transformation $\mathbf{A}$ is also linear.
(b) $(\mathbf{A}^*)^* = \mathbf{A}$.
(c) $(a\mathbf{A})^* = a\mathbf{A}^*$.
(d) $(\mathbf{A} + \mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$.
(e) $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$.
(f) If $\mathbf{A}$ is invertible, then $(\mathbf{A}^{-1})^* = (\mathbf{A}^*)^{-1}$.

Range and null space of a matrix Let $\mathcal{V} \subseteq \mathbb{R}^n$ and $\mathcal{W} \subseteq \mathbb{R}^m$ and $\mathbf{A} : \mathcal{V} \longrightarrow \mathcal{W}$ be such that $\mathbf{A} \in \mathbb{R}^{m\times n}$. The *range* of $\mathbf{A}$, also known as *column space* of $\mathbf{A}$, denoted by $\mathcal{R}(\mathbf{A})$ is defined as (refer to Figure B.2.1)

$$\mathcal{R}(\mathbf{A}) = \left\{\mathbf{y} \in \mathbb{R}^m \,|\, \mathbf{y} = \mathbf{Ax} \text{ for all } \mathbf{x} \in \mathcal{V}\right\} \subseteq \mathcal{W}$$

is the set of linear combinations of the columns of $\mathbf{A}$. That is, $\mathcal{R}(\mathbf{A})$ is the linear vector space generated by the columns of $\mathbf{A}$ and, hence, is called the *image* of $\mathcal{V}$ under $\mathbf{A}$ or simply as *image space* of $\mathbf{A}$. Likewise, we can define

$$\mathcal{R}(\mathbf{A}^{\mathrm{T}}) = \left\{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{x} = \mathbf{A}^{\mathrm{T}}\mathbf{y} \text{ for all } \mathbf{y} \in \mathbb{R}^m\right\} \subseteq \mathcal{V}$$

called the *range* of $\mathbf{A}^{\mathrm{T}}$ which is also called the *row space* of $\mathbf{A}$.

The *null space* of $\mathbf{A}$ denoted by $\mathcal{N}(\mathbf{A})$ is defined as

$$\mathcal{N}(\mathbf{A}) = \left\{\mathbf{x} \in \mathbb{R}^n \,|\, \mathbf{Ax} = 0\right\} \subseteq \mathcal{V}.$$

Thus, $\mathcal{N}(\mathbf{A})$ consists of all solutions to the homogeneous systems $\mathbf{Ax} = 0$. It can be verified that $\mathcal{N}(\mathbf{A})$ is a vector space and $\mathcal{N}(\mathbf{A})$ is also called the *kernel* of $\mathbf{A}$. Similarly, we can define

$$\mathcal{N}(\mathbf{A}^{\mathrm{T}}) = \left\{\mathbf{y} \in \mathbb{R}^m \,|\, \mathbf{A}^{\mathrm{T}}\mathbf{y} = 0\right\} \subseteq \mathcal{W}$$

the null space of $\mathbf{A}^{\mathrm{T}}$.

Indeed, given a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, there are four vector spaces – $\mathcal{R}(\mathbf{A})$, $\mathcal{R}(\mathbf{A}^{\mathrm{T}})$, $\mathcal{N}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^{\mathrm{T}})$ associated with $\mathbf{A}$. In the following, we enlist several key properties and the relations between these four vector spaces.

(a) $\mathcal{N}(\mathbf{A}) = \mathcal{R}^{\perp}(\mathbf{A}^{\mathrm{T}})$, that is, the null space of $\mathbf{A}$ is orthogonal to the range space of $\mathbf{A}^{\mathrm{T}}$. Similarly, $\mathcal{N}(\mathbf{A}^{\mathrm{T}}) = \mathcal{R}^{\perp}(\mathbf{A}^{\mathrm{T}})$.
(b) When $\mathbf{A}$ is a square symmetric matrix, then $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}^{\mathrm{T}})$ and $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}^{\mathrm{T}})$, and $\mathcal{N}(\mathbf{A}) = \mathcal{R}^{\perp}(\mathbf{A}^{\mathrm{T}})$.
(c) $\mathrm{Dim}[\mathcal{R}(\mathbf{A})] = \mathrm{Rank}(\mathbf{A}) = \mathrm{Dim}[\mathcal{R}(\mathbf{A}^{\mathrm{T}})]$.
(d) $\mathrm{Dim}[\mathcal{N}(\mathbf{A})] = n - r$, where $r = \mathrm{Rank}(\mathbf{A})$, is called the *nullity* of $\mathbf{A}$.

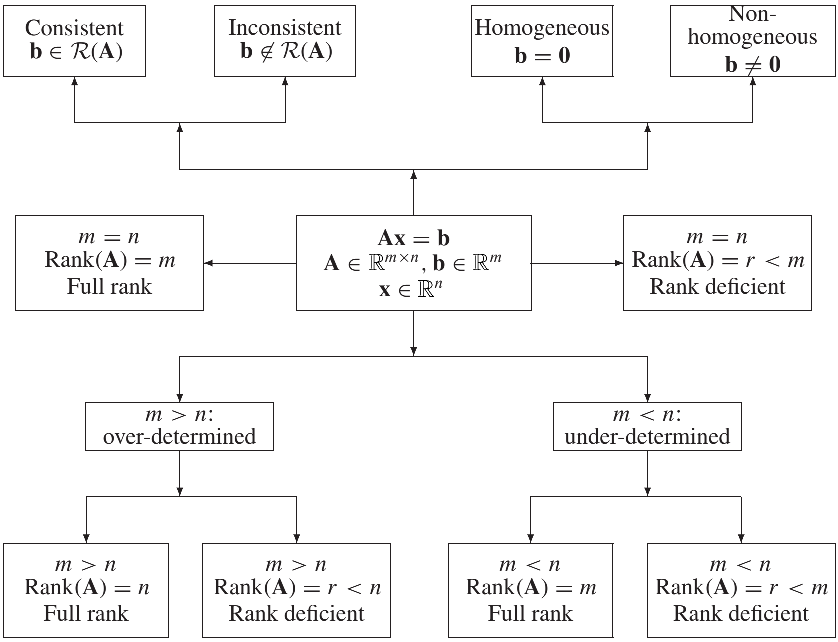


Fig. B.3.1 A classification of linear systems.

(e) Dim[$\mathcal{R}(\mathbf{A})$] + Dim[$\mathcal{N}(\mathbf{A})$] $= n$ for all $\mathbf{A} \in \mathbb{R}^{m \times n}$.
(f) Dim[$\mathcal{N}(\mathbf{A})$] $= 0$ exactly when $\mathbf{A}$ is square and non-singular.
(g) Dim[$\mathcal{N}(\mathbf{A}^{\mathrm{T}})$] $= m - r$.

B.3 Solution of linear systems

Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Given $\mathbf{A}$ and $\mathbf{b}$, the problem of finding the vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ is known as the problem of solving a set of m equations in n unknowns that are the components of $\mathbf{x}$. If the vector $\mathbf{b} \in \mathcal{R}(\mathbf{A})$, the range of $\mathbf{A}$ (also known as the column space of $\mathbf{A}$), then $\mathbf{b}$ must be expressible as a linear combination of the columns of $\mathbf{A}$ and the coefficients of this linear combination constitutes the components of the vector $\mathbf{x}$ we are seeking. In this case, the linear system $\mathbf{Ax} = \mathbf{b}$ is called a *consistent* system. If $\mathbf{b}$ does *not* belong to $\mathcal{R}(\mathbf{A})$, then there does *not* exist a vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ax} = \mathbf{b}$ and in this case it is termed as an *inconsistent* system. When the vector $\mathbf{b} = 0$, the linear system $\mathbf{Ax} = \mathbf{0}$ is called a *homogeneous system*, otherwise, it is known as a *non-homogeneous system*. Recall that Rank($\mathbf{A}$) $\leq \min(m, n)$. If Rank($\mathbf{A}$) $= \min(m, n)$, then the linear system is said to be of *full rank*, otherwise, it is called *rank deficient*. If m, the *number of equations*, is equal to n, the *number of variables*, it is called a *determined* system; if $m > n$, it is known as an *over-determined* system and if $m < n$,

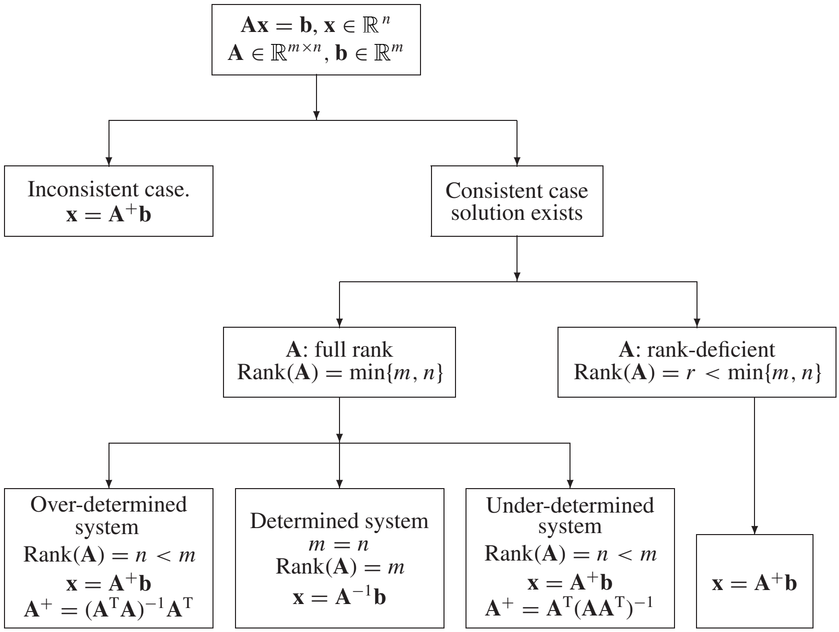


Fig. B.3.2 Solution of linear systems in various cases where $\mathbf{A}^+$ is the Moore–Penrose generalized inverse of $\mathbf{A}$.

it is known as an *under-determined* system. By combining various attributes – consistent/inconsistent, homogeneous/non-homogeneous, full-rank/rank-deficient, and determined/over or under-determined, we get a variety of combinations that are pictorially represented in Figure B.3.1.

(a) If $m = n$, and $\mathbf{A}$ is of full rank, it is always consistent and the solution exists and is unique. If $\mathbf{b} \neq 0$, then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ and if $\mathbf{b} = 0$, $\mathbf{x} = 0$ is the only solution.
(b) If $m > n$, it becomes an over-determined system. The unique solution is given by $\mathbf{x} = \mathbf{A}^+\mathbf{b}$ where $\mathbf{A}^+$ is the Moore–Penrose inverse of $\mathbf{A}$. In the special case when $\mathbf{A}$ is of full rank, this generalized inverse is given by $\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$.
(c) If $m < n$, it becomes an under-determined system. The unique solution is given by $\mathbf{x} = \mathbf{A}^+\mathbf{b}$, where $\mathbf{A}^+$ is the Moore–Penrose inverse. In the special case when $\mathbf{A}$ is of full rank, this generalized inverse is given by $\mathbf{A}^+ = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$.

A summary of these solutions is given in Figure B.3.2.

B.4 Special matrices

Symmetric Matrix The matrix $\mathbf{A} \in \mathbb{R}^{m \times m}$ is *symmetric* if $\mathbf{A}^T = \mathbf{A}$. The following properties are easily verified:

(a) $\mathbf{A}^{-1}$ is symmetric if $\mathbf{A}$ is.
(b) If $\mathbf{A}$ and $\mathbf{B}$ are symmetric, then $\mathbf{AB}$ is symmetric exactly when $\mathbf{A}$ and $\mathbf{B}$ *commute*, that is $\mathbf{AB} = \mathbf{BA}$.
(c) If $\mathbf{A} \in \mathbb{R}^{m\times n}$, then $\mathbf{AA}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ and $\mathbf{A}^{\mathrm{T}}\mathbf{A} \in \mathbb{R}^{n\times n}$ are both symmetric matrices called *Grammian*.
(d) The eigenvalues of a real-symmetric matrix are real.

Skew symmetric matrix The matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is skew symmetric if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. Thus, the diagonal elements of a skew-symmetric matrix are zeros. Given any matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ –

(a) $\mathbf{A}_{\mathrm{s}} = (\mathbf{A} + \mathbf{A}^{\mathrm{T}})/2$ is a symmetric matrix, and
(b) $\mathbf{A}_{\mathrm{ss}} = (\mathbf{A} - \mathbf{A}^{\mathrm{T}})/2$ is a skew-symmetric matrix.

$\mathbf{A}_{\mathrm{s}}$ and $\mathbf{A}_{\mathrm{ss}}$ are called the symmetric and skew-symmetric parts of $\mathbf{A}$, respectively.

Bilinear form Let $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Then $f_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) : \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (scalar-valued function of two vectors) defined by:

$$f_{\mathbf{A}} = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y}$$

is called the *bilinear form*, since the components of the vectors $\mathbf{x}$ and $\mathbf{y}$ appear in their first degree in $f_{\mathbf{A}}(\mathbf{x}, \mathbf{y})$.

Quadratic form Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{x} \in \mathbb{R}^m$. Then $\mathbf{Q}_{\mathbf{A}} : \mathbb{R}^m \longrightarrow \mathbb{R}$ (scalar-valued function of a vector) is defined as

$$\mathbf{Q}_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

which is a *second-degree multivariate polynomial* in the elements of $\mathbf{x}$. As an example, when $n = 2$,

$$\mathbf{Q}_{\mathbf{A}}(\mathbf{x}) = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2.$$

Since $\mathbf{Q}_{\mathbf{A}}(\mathbf{x})$ is a scalar, we immediately have

$$[\mathbf{Q}_{\mathbf{A}}(\mathbf{x})]^{\mathrm{T}} = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{x} = \mathbf{Q}_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}.$$

Thus,

$$\mathbf{Q}_{\mathbf{A}}(\mathbf{x}) = 1/2[\mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{x} + \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}] = \mathbf{x}^{\mathrm{T}}\left(\frac{\mathbf{A} + \mathbf{A}^{\mathrm{T}}}{2}\right)\mathbf{x} = \mathbf{x}^{\mathrm{T}}\mathbf{A}_{\mathrm{s}}\mathbf{x}$$

where $\mathbf{A}_{\mathrm{s}}$ is the symmetric part of $\mathbf{A}$. Henceforth, without loss of generality, we assume that the matrix $\mathbf{A}$ in $\mathbf{Q}_{\mathbf{A}}(\mathbf{x})$ is a symmetric matrix, and

$$\mathbf{Q}_{\mathbf{A}}(\mathbf{x}) = a_{11}\mathbf{x}_1^2 + 2a_{12}\mathbf{x}_1\mathbf{x}_2 + a_{22}\mathbf{x}_2^2.$$

Positive/negative definite matrix A real symmetric matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is said to be *positive definite* if

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \begin{cases} > 0 \text{ if } \mathbf{x} \neq 0 \\ = 0 \text{ exactly when } \mathbf{x} = 0. \end{cases}$$

A rea-symmetric matrix $\mathbf{A}$ is said to be *positive semi-definite* if

$$\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^m.$$

Thus, $\mathbf{A}$ is positive definite when the quadratic form $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ vanishes only at the origin and positive semi-definite if, in addition to the origin, $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ vanishes at least for one $\mathbf{x} \neq \mathbf{0}$. A real symmetric matrix $\mathbf{A}$ is said to be *negative definite* and *negative semi-definite*, if the above properties hold with the inequalities reversed. The matrix $\mathbf{A}$ is said to be *indefinite* if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \geq 0$ for some $\mathbf{x} \in \mathbb{R}^m$ and < 0 for some other values of $\mathbf{x}$.

We now list several important properties of positive and positive semi-definite matrices.

(a) The diagonal elements, the principal minors of all orders and the determinant of a positive definite matrix are all positive.

(b) Positive definiteness does *not* imply that the elements of the matrix are positive. For example,

$$\mathbf{A} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$

is positive definite, but

$$\mathbf{B} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

is not positive definite.

(c) $\mathbf{A}^{-1}$ is positive definite if $\mathbf{A}$ is.

(d) If $\mathbf{A}$ is positive definite and $\mathbf{B}$ is non-singular, then $(\mathbf{B}^{-1})^{\mathrm{T}}\mathbf{A}\mathbf{B}^{-1}$ is positive definite.

(e) Eigenvalues of a real-symmetric and positive definite matrix are all real and positive, and those of a positive semi-definite matrix are real and non-negative.

(f) If $\mathbf{A}$ is a symmetric and positive definite, then there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{\Lambda} = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$ where $\lambda_i > 0$ are the eigenvalues of $\mathbf{A}$ and the columns of $\mathbf{Q}$ are the corresponding orthonormal set of eigenvectors.

(g) From $\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q} = \mathbf{\Lambda}$, we immediately have

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{\Lambda}^{1/2}\mathbf{\Lambda}^{1/2}\mathbf{Q}^{\mathrm{T}} = (\mathbf{Q}\mathbf{\Lambda}^{1/2})(\mathbf{Q}\mathbf{\Lambda}^{1/2})^{\mathrm{T}} = \mathbf{C}\mathbf{C}^{\mathrm{T}}.$$

Thus, if $\mathbf{A}$ is positive definite, we have an important decomposition of $\mathbf{A} = \mathbf{C}\mathbf{C}^{\mathrm{T}}$.

Diagonally dominant matrix A square matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is said to be *diagonally dominant* if

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$$

and **strictly diagonally dominant** if strict inequality holds for all $i = 1, 2, \ldots, m$. It can be shown that a diagonally dominant matrix is non-singular.

Orthogonal matrix A matrix $\mathbf{Q} \in \mathbb{R}^{m\times m}$ is said to be an *orthogonal* matrix, if $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$, that is $\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}_m$. The following are properties of an orthogonal matrix.

(a) If $\mathbf{Q}$ is orthogonal, it is non-singular.
(b) The columns (rows) of $\mathbf{Q} \in \mathbb{R}^{m\times m}$ form a complete orthonormal basis for $\mathbb{R}^m$
(c) If $\mathbf{x} \in \mathbb{R}^m$, then $\|\mathbf{x}\|_2 = \|\mathbf{Q}\mathbf{x}\|_2$, that is the Euclidean norm is *invariant* under an orthogonal transformation; hence, orthogonal transformations are called *isometric* transformations.
(d) $\det(\mathbf{Q}^{\mathrm{T}}\mathbf{A}\mathbf{Q}) = \det(\mathbf{A})$.
(e) If $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are orthogonal, then so is their product $\mathbf{Q}_1\mathbf{Q}_2$.
(f) As a linear transformation, orthogonal transformation represents a rigid body rotation. Thus,

$$\mathbf{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is an orthogonal matrix representing a clockwise rotation by an angle θ.

Permutation matrix $\mathbf{P} = [\mathbf{p}_{ij}] \in \mathbb{R}^{m\times m}$ is a *permutation* matrix if $\mathbf{p}_{ij} \in \{0, 1\}$ and there is only one 1 in each row and in each column. For example,
$\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is a permutation matrix. Some other properties of permutation matrices are as follows.

(a) $\mathbf{I}$, the identity matrix, is a permutation matrix.
(b) If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is any matrix and $\mathbf{P} \in \mathbb{R}^{m\times m}$ is a permutation matrix, then $\mathbf{PA}$ permutes the rows of $\mathbf{A}$ and $\mathbf{AP}$ permutes the columns of $\mathbf{A}$.
(c) Every permutation matrix is orthogonal.
(d) Product of two permutation matrices is also a permutation matrix.
(e) Since $\mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$, we get $\det(\mathbf{P}) = \pm 1$, and hence every permutation matrix is non-singular.

Idempotent matrix A matrix $\mathbf{P} \in \mathbb{R}^{m\times m}$ is said to be *idempotent* if $\mathbf{P}^2 = \mathbf{P}$.

(a) With the exception of the identity matrix, every idempotent matrix is singular.
(b) If $\mathbf{P}$ is idempotent, then so is $(\mathbf{I} - \mathbf{P})$.

(c) If $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent, then $\mathbf{P}_1\mathbf{P}_2$ is idempotent only when they commute, and $\mathbf{P}_1 + \mathbf{P}_2$ is idempotent when $\mathbf{P}_1\mathbf{P}_2 = \mathbf{P}_2\mathbf{P}_1 = \mathbf{0}$.

(d) If $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{m\times n}$, and $\text{Rank}(\mathbf{A}) = n = \text{Rank}(\mathbf{B})$, then

$$\mathbf{P} = \mathbf{B}(\mathbf{A}^{\mathrm{T}}\mathbf{B})^{-1}\mathbf{A}^{\mathrm{T}}$$

is idempotent. When $\mathbf{A} = \mathbf{B}$, then

$$\mathbf{P} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$$

is clearly idempotent.

Remark B.4.1 Idempotent matrices play a very basic role in the sense that every projection matrix is idempotent. Further, every symmetric, idempotent matrix defines an orthogonal projection that plays a critical role in the solution of linear least squares problems.

Nilpotent matrix A matrix $\mathbf{A} \in \mathbb{R}^{m\times m}$ is said to be a *nilpotent* matrix of index k if $\mathbf{A}^k = \mathbf{0}$ and $\mathbf{A}^{k-1} \neq \mathbf{0}$. Strictly lower triangular matrices, for example, are nilpotent. For example,

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

is a nilpotent matrix of index 3.

Grammian matrix If $\mathbf{A} \in \mathbb{R}^{m\times n}$, then $\mathbf{A}^{\mathrm{T}}\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{A}\mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$ are called *Grammian* matrices.

(a) Grammian matrices are symmetric.

(b) If $\mathbf{A}$ is of full rank, then its Grammian is a symmetric positive definite, since $\mathbf{x}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{x} = (\mathbf{A}\mathbf{x})^{\mathrm{T}}\mathbf{A}\mathbf{x} = \|\mathbf{A}\mathbf{x}\| > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and likewise for $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

(c) If $\mathbf{A}$ is *not* of full rank, then the Grammian matrices are symmetric positive semi-definite matrices.

(d)

$$\begin{aligned} &\text{Rank}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \text{Rank}(\mathbf{A}) = \text{Rank}(\mathbf{A}\mathbf{A}^{\mathrm{T}}) \\ &\mathcal{R}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \mathcal{R}(\mathbf{A}^{\mathrm{T}}),\ \mathcal{N}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \mathcal{N}(\mathbf{A}) \\ &\mathcal{R}(\mathbf{A}\mathbf{A}^{\mathrm{T}}) = \mathcal{R}(\mathbf{A}),\ \mathcal{N}(\mathbf{A}\mathbf{A}^{\mathrm{T}}) = \mathcal{N}(\mathbf{A}^{\mathrm{T}}). \end{aligned}$$

Projection matrix Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^m$ such that $\mathbf{u}^{\mathrm{T}}\mathbf{v} \neq 0$. Then, it can be verified that

$$\text{Span}(\mathbf{v}) \oplus \text{Span}(\mathbf{u}^{\perp}) = \mathbb{R}^m$$

where $\mathbf{u}^{\perp}$ is the set of all vectors orthogonal to $\mathbf{u}$. Thus, $\mathbb{R}^m$ is a *direct sum* of these two subspaces, since

$$\text{Span}(\mathbf{v}) \cap \text{Span}(\mathbf{u}^{\perp}) = \{0\}\,.$$

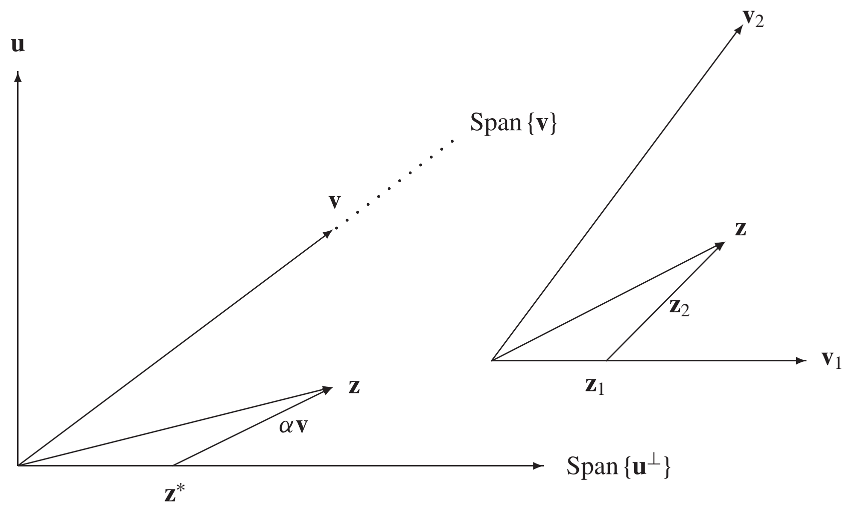


Fig. B.4.1 Oblique projections.

(Notice that the subspaces Span($\mathbf{v}$) and Span($\mathbf{u}^{\perp}$) are required to be orthogonal to each other.) Refer to Figure B.4.1 for an illustration.

The intuitive notion of projection of a vector $\mathbf{z}$ is related to the concept of "shining light" on $\mathbf{z}$ in the direction parallel to $\mathbf{v}$ and let $\mathbf{z}^*$ be the "shadow" cast by $\mathbf{z}$ on Span($\mathbf{u}^{\perp}$). Then, $\mathbf{z}^*$ is called the *oblique* projection of $\mathbf{z}$ onto Span($\mathbf{u}^{\perp}$) along Span($\mathbf{v}$). Referring to Figure B.4.1, by construction, for some real constant a, we have

$$\mathbf{u} \perp \mathbf{z}^* = \mathbf{z} - a\mathbf{v}$$

from which, we obtain

$$\mathbf{u}^{\mathrm{T}}(\mathbf{z} - a\mathbf{v}) = 0 \quad \text{or} \quad a = \frac{\mathbf{u}^{\mathrm{T}}\mathbf{z}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}.$$

Hence,

$$\mathbf{z}^* = \mathbf{z} - \frac{\mathbf{u}^{\mathrm{T}}\mathbf{z}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}\mathbf{v} = \mathbf{z} - \frac{\mathbf{v}\mathbf{u}^{\mathrm{T}}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}\mathbf{z} = \left[\mathbf{I} - \frac{\mathbf{v}\mathbf{u}^{\mathrm{T}}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}\right]\mathbf{z} = \mathbf{P}\mathbf{z}$$

where

$$\mathbf{P} = \left[\mathbf{I} - \frac{\mathbf{v}\mathbf{u}^{\mathrm{T}}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}\right]$$

is called the (oblique) projection matrix, projecting vectors $\mathbf{z} \in \mathbb{R}^m$ on to the Span$\{\mathbf{u}^{\perp}\}$ along the Span$\{\mathbf{v}\}$.

(a) This matrix $\mathbf{P}$ is idempotent.

(b) $(\mathbf{I}-\mathbf{P}) = \frac{1}{\mathbf{u}^{\mathrm{T}}\mathbf{v}}\mathbf{v}\mathbf{u}^{\mathrm{T}}$ is a rank-one outer-product matrix and denotes the projection matrix projecting vectors $\mathbf{z} \in \mathbb{R}^m$ on to the Span$\{\mathbf{v}\}$ along Span$\{\mathbf{u}^{\perp}\}$.
(c) $(\mathbf{I}-\mathbf{P})$ is an idempotent matrix.
(d) $\mathbf{P}$ and hence $(\mathbf{I}-\mathbf{P})$ are *not* symmetric.

We now generalize the above development. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of linearly independent vectors in $\mathbb{R}^m$. Let $\mathcal{V}_1 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathcal{V}_2 = \text{Span}\{\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m\}$. Clearly,

$$\mathbb{R}^m = \mathcal{V}_1 + \mathcal{V}_2 \quad \text{and} \quad \mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$$

that is, $\mathbb{R}^m$ is the *direct sum* of $\mathcal{V}_1$ and $\mathcal{V}_2$. Any $\mathbf{z} \in \mathbb{R}^m$ can be expressed as $\mathbf{z} = \mathbf{z}_1 + \mathbf{z}_2$, where $\mathbf{z}_1 \in \mathcal{V}_1$ and $\mathbf{z}_2 \in \mathcal{V}_2$. $\mathbf{z}_1(\mathbf{z}_2)$ is called the *oblique projection* of $\mathbf{z}$ on to $\mathcal{V}_1(\mathcal{V}_2)$ along $\mathcal{V}_2(\mathcal{V}_1)$. Refer to Figure B.4.1. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be the respective projection matrices, that is,

$$\mathbf{z}_1 = \mathbf{P}_1\mathbf{z} \quad \text{and} \quad \mathbf{z}_2 = \mathbf{P}_2\mathbf{z}$$

then $\mathbf{P}_1(\mathbf{P}_2)$ is called the *oblique projection matrix* on $\mathcal{V}_1(\mathcal{V}_2)$ along $\mathcal{V}_2(\mathcal{V}_1)$.

Let $\mathbf{V}_1 = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] \in \mathbb{R}^{m\times n}$, and $\mathbf{V}_2 = [\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m] \in \mathbb{R}^{m\times(m-n)}$ be two matrices. We have $\mathbf{V} = [\mathbf{V}_1, \mathbf{V}_2] \in \mathbb{R}^{m\times m}$ is non-singular, and

$$\begin{array}{llll} \mathbf{P}_1\mathbf{v}_i = \mathbf{v}_i & \text{for } i = 1 \text{ to } n, & \mathbf{P}_1\mathbf{v}_j = 0 & \text{for } j = n+1 \text{ to } m \\ \mathbf{P}_2\mathbf{v}_j = \mathbf{v}_j & \text{for } j = n+1 \text{ to } m, & \mathbf{P}_2\mathbf{v}_i = 0 & \text{for } i = 1 \text{ to } n. \end{array}$$

Then,

$$\mathbf{P}_1\mathbf{V} = \mathbf{P}_1[\mathbf{V}_1, \mathbf{V}_2] = [\mathbf{P}_1\mathbf{V}_1, \mathbf{P}_1\mathbf{V}_2] = [\mathbf{V}_1, \mathbf{0}]$$

and

$$\mathbf{P}_1 = [\mathbf{V}_1, \mathbf{0}]\mathbf{V}^{-1} = [\mathbf{V}_1, \mathbf{V}_2]\begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{V}^{-1} = \mathbf{V}\Sigma_1\mathbf{V}^{-1}$$

where

$$\Sigma_1 = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Similarly,

$$\mathbf{P}_2\mathbf{V} = \mathbf{P}_2[\mathbf{V}_1, \mathbf{V}_2] = [\mathbf{P}_2\mathbf{V}_1, \mathbf{P}_2\mathbf{V}_2] = [\mathbf{0}, \mathbf{V}_2]$$

and

$$\mathbf{P}_2 = [\mathbf{0}, \mathbf{V}_2]\mathbf{V}^{-1} = [\mathbf{V}_1, \mathbf{V}_2]\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m-n} \end{bmatrix}\mathbf{V}^{-1} = \mathbf{V}\Sigma_2\mathbf{V}^{-1}$$

where

$$\Sigma_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{m-n} \end{bmatrix}.$$

The following properties of (oblique) projection matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ are readily established.

(a) $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent.
(b) $\mathbf{I} = \mathbf{P}_1 + \mathbf{P}_2$.
(c) $\mathbf{P}_1$ and $\mathbf{P}_2$ are *not* symmetric.

We now conclude this development with the following result without proof:

- *Every (oblique) projection matrix is idempotent and conversely every idempotent matrix defines an oblique projection.*

Orthogonal projection matrix When $\mathbf{v} = \mathbf{u}$ in Figure B.4.1, then the light shines parallel to $\mathbf{v} = \mathbf{u}$, and the shadow $\mathbf{z}^*$ is called *orthogonal projection* of $\mathbf{z}$ on to $\mathbf{u}^\perp$. In this case,

$$\mathbf{z}^* = \left[\mathbf{I} - \frac{\mathbf{u}\mathbf{u}^\mathrm{T}}{\mathbf{u}^\mathrm{T}\mathbf{u}}\right]\mathbf{z} = \mathbf{P}\mathbf{z}$$

with

$$\mathbf{P} = \left[\mathbf{I} - \frac{\mathbf{u}\mathbf{u}^\mathrm{T}}{\mathbf{u}^\mathrm{T}\mathbf{u}}\right].$$

(a) $\mathbf{P}$ is an orthogonal projection to the Span$\{\mathbf{u}^\perp\}$.
(b) $\mathbf{P}^2 = \mathbf{P}$.
(c) $(\mathbf{I} - \mathbf{P})\mathbf{z} = \frac{\mathbf{u}\mathbf{u}^\mathrm{T}}{\mathbf{u}^\mathrm{T}\mathbf{u}}\mathbf{z}$ is an orthogonal projection on to the Span$\{\mathbf{u}\}$.
(d) $\mathbf{P}$ is a symmetric matrix.

An important conclusion from this is that *every orthogonal projection matrix is idempotent and symmetric*. In fact, *the converse is also true.*

We now generalize the notion of orthogonal projection. Let $\mathbb{R}^m = \mathcal{V}_1 + \mathcal{V}_2$ with $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ and $\mathcal{V}_1 \perp \mathcal{V}_2$, that is, every vector in $\mathcal{V}_1$ is orthogonal to every vector in $\mathcal{V}_2$. In this case, $\mathcal{V}_1 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\mathcal{V}_2 = \mathrm{Span}\{\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m\}$ are called orthogonal complements of each other.

Again denoting $\mathbf{V}_1 = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] \in \mathbb{R}^{m\times n}$, and $\mathbf{V}_2 = [\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m] \in \mathbb{R}^{m\times(m-n)}$ are such that

$$\begin{aligned} &\mathbf{V}_1^\mathrm{T}\mathbf{V}_1 \in \mathbb{R}^{n\times n} \text{ is non-singular.}\\ &\mathbf{V}_2^\mathrm{T}\mathbf{V}_2 \in \mathbb{R}^{(m-n)\times(m-n)} \text{ is non-singular.}\\ &\mathbf{V}_1^\mathrm{T}\mathbf{V}_2 = 0 = \mathbf{V}_2^\mathrm{T}\mathbf{V}_1. \end{aligned}$$

Notice that we are not requiring $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ to be orthogonal; nor do we require $\{\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m\}$ to be orthogonal. All we need is their linear independence and that $\mathcal{V}_1 \perp \mathcal{V}_2$. As an example,

$$\mathbf{V}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \mathbf{V}_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

If $\mathbf{P}_1$ and $\mathbf{P}_2$ are the orthogonal projection matrices onto the subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, we have the following.

(a) $\mathbf{P}_1 = \mathbf{V}_1(\mathbf{V}_1^{\mathrm{T}}\mathbf{V}_1)^{-1}\mathbf{V}_1^{\mathrm{T}} \in \mathbb{R}^{m\times m}$.
(b) $\mathbf{P}_2 = \mathbf{V}_2(\mathbf{V}_2^{\mathrm{T}}\mathbf{V}_2)^{-1}\mathbf{V}_2^{\mathrm{T}} \in \mathbb{R}^{m\times m}$.
(c) $\mathbf{I} = \mathbf{P}_1 + \mathbf{P}_2$.
(d) $\mathbf{P}_1^2 = \mathbf{P}_1$ and $\mathbf{P}_2^2 = \mathbf{P}_2$.
(e) $\mathbf{P}_1$ and $\mathbf{P}_2$ are symmetric.
(f) In addition, if the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are orthonormal vectors, and likewise $\mathbf{v}_{n+1}, \mathbf{v}_{n+2}, \ldots, \mathbf{v}_m$ are also orthonormal, then $\mathbf{V}_1^{\mathrm{T}}\mathbf{V}_1 = \mathbf{I}_n$ and $\mathbf{V}_2^{\mathrm{T}}\mathbf{V}_2 = \mathbf{I}_{m-n}$ and $\mathbf{P}_1 = \mathbf{V}_1\mathbf{V}_1^{\mathrm{T}}$ and $\mathbf{P}_2 = \mathbf{V}_2\mathbf{V}_2^{\mathrm{T}}$.

We conclude this discussion with the following fact.

- *Every orthogonal projection matrix is symmetric and idempotent and conversely every symmetric and idempotent matrix defines an orthogonal projection.*

B.5 Eigenvalues and eigenvectors

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$. If $\mathbf{x} \in \mathbb{R}^m$, a non-null vector and λ is a scalar (real or complex), such that

$$\mathbf{Ax} = \lambda\mathbf{x} \quad \text{or} \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

then the pair $(\lambda, \mathbf{x})$ is called the *eigenvalue* and *eigenvector* (simply called the *eigen pair*) of the matrix $\mathbf{A}$. The linear homogeneous equation on the right above has a non-null solution only when $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which is an mth-degree polynomial, called the *characteristic polynomial* in λ. Since the mth-degree polynomial has m roots – real or complex with complex roots occurring in conjugate pairs, $\mathbf{A} \in \mathbb{R}^{m\times m}$ has m eigenvalues. If

$$\det(\mathbf{A} - \lambda\mathbf{I}) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

then there are k distinct eigenvalues λ_1 through λ_k with m_i being the multiplicity of λ_i and $\sum_{i=1}^{k} m_i = m$. Here m_i is called the *algebraic multiplicity* of λ_i. The set of all eigenvalues of $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$. The magnitude of the largest eigenvalue is called the *spectral radius* and is denoted by $\rho(\mathbf{A})$. We now list several properties.

(a) Distinct eigenvalues have distinct eigenvectors. That is, if $(\lambda_1, \mathbf{x}_1)$ and $(\lambda_2, \mathbf{x}_2)$ are such that $\lambda_1 \neq \lambda_2$, then $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

(b) There could be more than one eigenvector corresponding to the same eigenvalue – witness the identity matrix $\mathbf{I}_m$. For $\mathbf{I}_m$, 1 is the only eigenvalue, but it has m linearly independent eigenvectors.

(c) If $(\lambda, \mathbf{x})$ is an eigen pair of $\mathbf{A}$, then $\mathbf{x} \in \mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$, the *null space* of $(\mathbf{A} - \lambda\mathbf{I})$. The number of distinct, that is, linearly independent eigenvectors $\mathbf{x}$ corresponding to the eigenvalue m is known as the *geometric multiplicity*. Thus, geometric multiplicity denotes the dimension of $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I})$. Recall that the dimension of $\mathcal{N}(\mathbf{A} - \lambda\mathbf{I}) = \text{Rank}(\mathbf{A} - \lambda\mathbf{I})$ which is known as the *nullity* of $(\mathbf{A} - \lambda\mathbf{I})$. It is a fact that

$$\text{geometric multiplicty of } \lambda \leq \text{algebraic multiplicity of } \lambda$$

(d) If $\mathbf{B} \in \mathbb{R}^{m\times m}$ is non-singular, then $\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ is similar to $\mathbf{A}$. It can be verified that similar matrices have the same set of eigenvalues. Thus, if $(\lambda, \mathbf{x})$ is an eigen pair of $\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ then $(\lambda, \mathbf{B}\mathbf{x})$ is the corresponding pair for $\mathbf{A}$.

(e) $\text{tr}(\mathbf{A}) = m_1\lambda_1 + m_2\lambda_2 + \cdots + m_k\lambda_k$, where m_i is the *algebraic multiplicity* of λ_i.

(f) $\det(\mathbf{A}) = \lambda_1^{m_1}\lambda_2^{m_2}\ldots\lambda_k^{m_k}$.

(g) **Gerschgorin circles** Let $\mathbf{A} = [a_{ij}]$. Define a disk or circle D_i in the complex plane

$$D_i = \left\{ \mathbf{z} \,\middle|\, |a_{ii} - \mathbf{z}| \leq \sum_{j\neq i} |a_{ij}| \right\}, i = 1, 2, \ldots, m$$

which is centered at a_{ii} and radius $\sum_{j\neq i} |a_{ij}|$. Then, every eigenvalue of $\mathbf{A}$ lies in the union of the disks D_i denoted by $S = \cup_{i=1}^{m} D_i$.

(h) Let $\lambda = 0$ be an eigenvalue of $\mathbf{A} \in \mathbb{R}^{m\times m}$ with algebraic multiplicity p and geometric multiplicity q (with $q \leq p$), then

$$\text{Rank}(\mathbf{A}) = m - q \geq m - p$$

that is, the rank of a matrix may exceed the number of non-zero eigenvalues – witness a nilpotent matrix

$$\mathbf{A} = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & c_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

.

(i) If $(\lambda, \mathbf{x})$ is an eigen pair for $\mathbf{A}$, then $(\lambda^k, \mathbf{x})$ is the corresponding eigen pair for $\mathbf{A}^k$ for any integer $k \geq 1$.

(j) If $(\lambda, \mathbf{x})$ is an eigen pair of $\mathbf{A}$, then $(\lambda + a, \mathbf{x})$ is the corresponding eigen pair of $(\mathbf{A} + a\mathbf{I})$ for any real scalar a.

(k) Let $p(x) = a_0 + a_1 x + a_2 x^2$. Then, $p(\mathbf{A}) = a_0\mathbf{I} + a_1\mathbf{A} + a_2\mathbf{A}^2$ is called a matrix polynomial in **A** of degree 2. If $(\lambda, \mathbf{x})$ is an eigen pair of **A**, then $(p(\lambda), \mathbf{x})$ is an eigen pair of the matrix $p(\mathbf{A})$.

We now state the properties of the eigenvalues of special matrices.

Symmetric matrices Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a *symmetric matrix*

(a) Eigenvalues of **A** are real.
(b) The number of non-zero eigenvalues of $\mathbf{A} = \text{Rank}(\mathbf{A})$.
(c) Eigenvectors of a real symmetric matrix are orthogonal. Without loss of generality, we can assume that they are also normalized.
(d) Let $\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$ denote the orthonormal matrix of the m eigenvectors of **A**. Then $\mathbf{AP} = \mathbf{P\Lambda}$, where $\mathbf{\Lambda} = \text{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the diagonal matrix of the eigenvalues of **A**.
(e) **Spectral decomposition** Since **P** is orthogonal, we can rewrite **A** as

$$\begin{aligned}\mathbf{A} = \mathbf{P\Lambda P}^{\mathrm{T}} &= [\mathbf{x}_1\ \mathbf{x}_2\ \cdots\ \mathbf{x}_m]\begin{bmatrix}\lambda_1 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_m\end{bmatrix}\begin{bmatrix}\mathbf{x}_1^{\mathrm{T}} \\ \mathbf{x}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{x}_m^{\mathrm{T}}\end{bmatrix} \\ &= \textstyle\sum_{i=1}^m \lambda_i \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} = \sum_{i=1}^m \lambda_i \mathbf{P}_i\end{aligned} \tag{B.5.1}$$

where $\mathbf{P}_i = \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}$ are the *rank-one outer-product* matrices. Recall that $\mathbf{P}_i = \frac{\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}}{\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_i} = \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}}$ (since $\mathbf{x}_i$'s are normalized) denotes the *orthogonal projection matrix* onto the Span$\{\mathbf{x}_i\}$. This expansion of **A** as the linear combination of the orthogonal projection matrices is called the *spectral decomposition* of **A**.

Simultaneous Diagonalizability If $\mathbf{A} \in \mathbb{R}^{m\times m}$ and $\mathbf{B} \in \mathbb{R}^{m\times m}$ are two symmetric matrices, then there exists a common orthogonal matrix **P** such that

$$\mathbf{AP} = \mathbf{P}\Lambda \quad \text{and} \quad \mathbf{BP} = \mathbf{P}\Sigma$$

where $\mathbf{\Lambda}$ and Σ are diagonal matrices of eigenvalues of **A** and **B**, respectively, exactly when $\mathbf{AB} = \mathbf{BA}$.

Krylov Subspace Let $\mathbf{y} \in \mathbb{R}^m$ be any vector. Then, there exists an eigenvector **x** of **A** that belongs to the vector space

$$\mathcal{K}(\mathbf{A}, \mathbf{y}) = \left\{\mathbf{y}, \mathbf{Ay}, \mathbf{A}^2\mathbf{y}, \ldots, \mathbf{A}^{r-1}\mathbf{y}\right\}$$

for some integer $r \geq 0$, called the *Krylov subspace*

Rayleigh coefficient Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Then,

$$r(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{Ax}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} \tag{B.5.2}$$

is called the *Rayleigh coefficient*.

Properties of Rayleigh coefficient

(a) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times n}$ be two symmetric and positive definite matrices. Let $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ be such that

$$\mathbf{Ax} = \lambda\mathbf{Bx}. \tag{B.5.3}$$

Then, λ is called the **generalized eigenvalue** and $\mathbf{x}$ is called the corresponding **generalized eigenvector** of the pair $(\mathbf{A},\ \mathbf{B})$. Multiplying both sides of (B.5.3) by $\mathbf{B}^{-1}$, from $(\mathbf{B}^{-1}\mathbf{A})\mathbf{x} = \lambda\mathbf{x}$ it follows that λ and $\mathbf{x}$ are the eigenvalue and eigenvector of $(\mathbf{B}^{-1}\mathbf{A})$. Further, multiplying both sides of (B.5.3) by $\mathbf{x}^{\mathrm{T}}$ and rearranging we readily see that λ is given by Rayleigh coefficient

$$\lambda = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{Ax}}{\mathbf{x}^{\mathrm{T}}\mathbf{Bx}}. \tag{B.5.4}$$

(b) Let $\mathbf{A}$ and $\mathbf{B}$ be two symmetric positive matrices. Consider the function

$$\begin{aligned} g(\lambda) &= \| (\mathbf{A} - \lambda\mathbf{B})\mathbf{x} \|^2_{\mathbf{B}^{-1}} \\ &= \mathbf{x}^{\mathrm{T}}[(\mathbf{A} - \lambda\mathbf{B})\mathbf{B}^{-1}(\mathbf{A} - \lambda\mathbf{B})]\mathbf{x} \\ &= \mathbf{x}^{\mathrm{T}}(\mathbf{AB}^{-1}\mathbf{A})\mathbf{x} - 2\lambda(\mathbf{x}^{\mathrm{T}}\mathbf{Ax}) + \lambda^2(\mathbf{x}^{\mathrm{T}}\mathbf{Bx}) \end{aligned} \tag{B.5.5}$$

The **stationary point** of $g(\lambda)$ is obtained by solving

$$0 = \frac{\mathrm{d}g}{\mathrm{d}\lambda} = -2(\mathbf{x}^{\mathrm{T}}\mathbf{Ax}) + 2\lambda(\mathbf{x}^{\mathrm{T}}\mathbf{Bx}).$$

That is,

$$\lambda = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{Ax}}{\mathbf{x}^{\mathrm{T}}\mathbf{Bx}}. \tag{B.5.6}$$

Since

$$\frac{\mathrm{d}^2 g}{\mathrm{d}\lambda^2} = \mathbf{x}^{\mathrm{T}}\mathbf{Bx} > 0$$

it follows that λ in (B.5.6) is the minimizer of $g(\lambda)$. That is, the Rayleigh coefficient (B.5.6) minimizes the norm in (B.5.5).

(c) Let

$$r(\mathbf{x}) = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{Ax}}{\mathbf{x}^{\mathrm{T}}\mathbf{Bx}}. \tag{B.5.7}$$

Then, it can be verified that

$$\nabla r(\mathbf{x}) = \frac{2}{(\mathbf{x}^{\mathrm{T}}\mathbf{Bx})}[\mathbf{Ax} - r(\mathbf{x})\mathbf{Bx}]. \tag{B.5.8}$$

This gradient vanished exactly when $\mathbf{x}$ is such that $r(\mathbf{x})$ is the generalized eigenvalue of the pair $(\mathbf{A}, \mathbf{B})$, that is

$$\mathbf{Ax} = r(\mathbf{x})\mathbf{Bx}. \tag{B.5.9}$$

(d) Let $r(\mathbf{x})$ be given by (B.5.7). Let $\mathbf{B} = \mathbf{G}\mathbf{G}^{\mathrm{T}}$ be the Cholesky factorization of $\mathbf{B}$ (Chapter 9). Then defining $\mathbf{y} = \mathbf{G}^{\mathrm{T}}\mathbf{x}$ and substituting into (B.5.7) we obtain

$$r(\mathbf{y}) = \frac{\mathbf{y}^{\mathrm{T}}\mathbf{A}_1\mathbf{y}}{\mathbf{y}^{\mathrm{T}}\mathbf{y}} \tag{B.5.10}$$

where

$$\mathbf{A}_1 = \mathbf{G}^{-1}\mathbf{A}\mathbf{G}^{-\mathrm{T}} \tag{B.5.11}$$

is a symmetric and positive definite matrix. Given this equivalence between $r(\mathbf{x})$ in (B.5.7) and $r(\mathbf{y})$ in (B.5.10), without loss of generality we can analyze the simpler form in (B.5.10).

(e) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric and positive definite matrix. Then

$$\begin{aligned} r(\mathbf{x}) &= \frac{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}} = \frac{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}}{\|\ \mathbf{x}\ \|^2} = \left(\frac{\mathbf{x}}{\|\ \mathbf{x}\ \|}\right)^{\mathrm{T}} \mathbf{A} \left(\frac{\mathbf{x}}{\|\ \mathbf{x}\ \|}\right) \\ &= (\widehat{\mathbf{x}})^{\mathrm{T}}\mathbf{A}(\widehat{\mathbf{x}}) \end{aligned}$$

where $\widehat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{x}$. Hence without loss of generality consider

$$r(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} \qquad \text{when } \ \|\ \mathbf{x}\ \|= 1. \tag{B.5.12}$$

(f) Let $(\lambda_i, \mathbf{p}_i)$, $i = 1, 2, \ldots, N$ be the eigenvalue/vector pair of the matrix $\mathbf{A}$ in (B.5.12), where it is assumed that

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n > 0. \tag{B.5.13}$$

If $\mathbf{P} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n]$ and $\mathbf{\Lambda} = \mathrm{Diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]$, then we get

$$\mathbf{A}\mathbf{P} = \mathbf{P}\mathbf{\Lambda}.$$

It can be verified that $\mathbf{P}$ is orthogonal and $\mathbf{P}^{\mathrm{T}}\mathbf{P} = \mathbf{P}\mathbf{P}^{\mathrm{T}} = \mathbf{I}$. Using this fact it follows that

$$\mathbf{A} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{\mathrm{T}} \quad \text{or} \quad \mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P} = \mathbf{\Lambda}. \tag{B.5.14}$$

Hence, any vector $\mathbf{x} \in \mathbb{R}^n$ can be expressed as a linear combination of the columns of $\mathbf{P}$, that is,

$$\mathbf{x} = \mathbf{P}\mathbf{y}. \tag{B.5.15}$$

Substituting (B.5.15) and using (B.5.14) we get

$$r(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} = \mathbf{y}^{\mathrm{T}}(\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P})\mathbf{y} = \mathbf{y}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{y} = r(\mathbf{y}). \tag{B.5.16}$$

That is,

$$r(\mathbf{y}) = \sum_{i=1}^{n} \lambda_i y_i^2 \tag{B.5.17}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ is such that

$$\sum_{i=1}^{n} y_i^2 = 1. \tag{B.5.18}$$

Combining this with (B.5.13), we get

$$\lambda_n \leq \sum_{i=1}^{n} \lambda_i y_i^2 \leq \lambda_1. \tag{B.5.19}$$

More precisely, we get

$$\lambda_1 = \max_{\|\mathbf{y}\|=1} \{\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y}\} = \max_{\|\mathbf{x}\|=1} \{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\} \tag{B.5.20}$$

and

$$\lambda_n = \min_{\|\mathbf{y}\|=1} \{\mathbf{y}^{\mathrm{T}}\mathbf{A}\mathbf{y}\} = \min_{\|\mathbf{x}\|=1} \{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\}. \tag{B.5.21}$$

To understand the relation between $r(\mathbf{x})$ and other eigenvalues, let $S_1 = \mathrm{Span}\{\mathbf{p}_1\}$ and let $S_1^{\perp}$ denote the set of all vectors orthogonal to S_1. Then, for $\mathbf{x} \in S_1^{\perp}$ we get

$$0 = \mathbf{p}_1^{\mathrm{T}}\mathbf{x} = \mathbf{p}_1\mathbf{P}\mathbf{y} = \mathbf{e}_1^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}\mathbf{P}\mathbf{y} = \mathbf{e}_1^{\mathrm{T}}\mathbf{y} \tag{B.5.22}$$

where $\mathbf{e}_1 = (1, 0, 0, \ldots, 0)^{\mathrm{T}}$. That is, $\mathbf{y}$ is a vector whose first component $y_1 = 0$. Thus,

$$\begin{aligned} \lambda_2 &= \max_{\substack{\|\mathbf{y}\|=1 \\ y_1=0}} \left\{\sum_{i=2}^{n} \lambda_i y_i^2\right\} \\ &= \max_{\substack{\|\mathbf{y}\|=1 \\ y_1=0}} \left\{\mathbf{y}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{y}\right\} \\ &= \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\in S_1^{\perp}}} \left\{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\right\}. \end{aligned} \tag{B.5.23}$$

Similarly, if

$$S_j = \mathrm{Span}\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$$

then

$$\lambda_j = \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\in S_j^{\perp}}} \left\{\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}\right\}. \tag{B.5.24}$$

These properties play a crucial role in the numerical computation of the eigenvalues of $\mathbf{A}$.

Eigenvalues of orthogonal matrices

(a) If $\mathbf{Q} \in \mathbb{R}^{m\times m}$ is orthogonal, then $\mathbf{Q}$ can have only two values $\{+1, -1\}$.
(b) The eigenvectors of $\mathbf{Q}$ are real and orthogonal.

Eigenvalues of projection matrices

(a) Eigenvalues of a projection matrix can only take two values $\{0, 1\}$.
(b) $\mathrm{tr}(\mathbf{P}) = \mathrm{Rank}(\mathbf{P})$.

Symmetric and positive definite matrix Let $\mathbf{A} \in \mathbb{R}^{m\times m}$.

(a) If $\mathbf{A}$ is symmetric and positive (negative) definite, then all the eigenvalues are positive (negative).
(b) If $\mathbf{A}$ is symmetric and positive (negative) semi-definite then the eigenvalues are non-negative (non-positive);
(c) The matrix $\mathbf{A}$ is said to be *indefinite* if the eigenvalues simultaneously take positive/zero/negative values.
(d) Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a symmetric and positive definite matrix. Then

$$\mathbf{Q}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x}$$

denotes a *hyper ellipsoid* in $\mathbb{R}^m$. If

$$\boldsymbol{\Lambda} = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

is the diagonal matrix of the eigenvalues and $\mathbf{P} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m]$. The orthogonal matrix of the corresponding eigenvectors of $\mathbf{A}$, then

$$\mathbf{AP} = \mathbf{P}\boldsymbol{\Lambda} \text{ and } \mathbf{A} = \mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{\mathrm{T}}.$$

Hence,

$$\mathbf{Q}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}(\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{\mathrm{T}})\mathbf{x} = (\mathbf{Px})^{\mathrm{T}}\boldsymbol{\Lambda}(\mathbf{Px}).$$

If $\mathbf{x} = \mathbf{P}^{\mathrm{T}}\mathbf{y}$ or $\mathbf{Px} = \mathbf{y}$ (a rigid-body rotation of the standard coordinate system using the orthogonal matrix $\mathbf{P}$), then

$$\begin{aligned}\mathbf{Q}(\mathbf{x}) = \mathbf{Q}(\mathbf{P}^{\mathrm{T}}\mathbf{y}) = &= \mathbf{y}^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{y}\\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_m y_m^2.\end{aligned}$$

In other words, $\mathbf{Q}(\mathbf{x})$ in the hyper-ellipsoid whose m principal axes coincide with the m orthogonal eigenvectors of $\mathbf{A}$ and $1/\sqrt{\lambda_i}$ denotes the length of the semi-axis in the ith principal direction.

B.6 Matrix norms

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ and denote $\mathbf{A} = [a_{ij}]$. Conceptually, there are two ways in which one can approach the definition of the matrix norm. When viewed as a mathematical *entity* or *object*, we can define the norm to denote its *size*. The *Frobenius norm* of a matrix is defined as follows:

$$\|\mathbf{A}\|_{\mathrm{F}} = \left[\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}^2\right]^{1/2} .$$

This is an extension of the concept of Euclidean norm for vectors. Dually, when viewed as an *operator/mapping* vectors to vectors we can define the matrix norm by its *action* using vector norms. Let p denote $1, 2$, or ∞. Let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} = \mathbf{Ax}$. Then, $\mathbf{y}$ is called the *image* of $\mathbf{x}$ under $\mathbf{A}$. Define

$$\|\mathbf{A}\|_p = \sup_{\|\mathbf{x}\|\neq 0} \frac{\|\mathbf{Ax}\|_p}{\|\mathbf{x}\|_p} = \sup_{\mathbf{x}\neq\mathbf{0}} \left\|\mathbf{A}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_p}\right)\right\| = \max_{\|\mathbf{x}=1\|} \|\mathbf{Ax}\|_p .$$

Thus, the p-norm of $\mathbf{A}$ is the *maximum* of the ratio of the p-norm of $\mathbf{y} = \mathbf{Ax}$ to that of $\mathbf{x}$ where the maximum is taken over all unit vectors. For this reason, this is also known as the *induced* norm. If $\|\mathbf{A}\|_p > 1 (< 1)$, then $\mathbf{A}$ *magnifies* (*contracts*) the length of the vector $\mathbf{x}$. When $p = 1, 2$, or ∞, we get the *one*, *two* or the *infinity* norm of $\mathbf{A}$. From the above definition, we have the following useful inequalities:

$$\begin{aligned} \|\mathbf{Ax}\| &\leq \|\mathbf{A}\| \, \|\mathbf{x}\| \\ \|\mathbf{AB}\| &\leq \|\mathbf{A}\| \, \|\mathbf{B}\| , \end{aligned}$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times m}$. It can be shown that we can compute the matrix norms rather directly as follows:

(N1) $\|\mathbf{A}\|_1 = \max_j \left\{\sum_i |a_{ij}|\right\}$ – *Column norm.*
(N2) $\|\mathbf{A}\|_\infty = \max_i \left\{\sum_j |a_{ij}|\right\}$ – *Row norm.*
(N3) $\|\mathbf{A}\|_2 = \sigma_1$, where σ_1^2 is the maximum eigenvalue of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$. Recall that σ_1 is the largest *singular value* of $\mathbf{A}$.

When $\mathbf{A}$ is symmetric, from $\mathbf{A}^{\mathrm{T}}\mathbf{A} = \mathbf{A}^2$ and the fact that λ^2 is the eigenvalue of $\mathbf{A}^2$, where λ is an eigenvalue of $\mathbf{A}$, it follows that $\|\mathbf{A}\|_2 = |\lambda_{max}|$, where λ_{max} is the maximum eigenvalue of $\mathbf{A}$. Further, these matrix norms are equivalent and the following inequalities hold.

$$\begin{aligned} &\|\mathbf{A}\|_2 \leq \left[\|\mathbf{A}\|_1 \, \|\mathbf{A}\|_\infty\right]^{1/2} \\ &\tfrac{1}{\sqrt{m}} \|\mathbf{A}\|_\infty \leq \|\mathbf{A}\|_1 \leq \sqrt{m}\, \|\mathbf{A}\|_\infty \\ &\tfrac{1}{\sqrt{m}} \|\mathbf{A}\|_1 \leq \|\mathbf{A}\|_2 \leq \sqrt{m}\, \|\mathbf{A}\|_1 \\ &\|\mathbf{A}\|_2 \leq \|\mathbf{A}\|_F \leq \sqrt{m}\, \|\mathbf{A}\|_2 . \end{aligned}$$

Furthermore, the spectral radius $\rho(\mathbf{A})$ is less than or equal to any matrix norm of $\mathbf{A}$, that is, $\rho(\mathbf{A}) \leq \|\mathbf{A}\|$. This fact is very useful in proving convergence of an iterative process.

Example B.6.1

$$\mathbf{A} = \begin{bmatrix} 5.0 & -2.0 \\ -3.0 & 4.0 \end{bmatrix}.$$

Then $\|\mathbf{A}\|_1 = \max\{8.0, 6.0\} = 8.0$, $\|\mathbf{A}\|_\infty = \max\{7.0, 7.0\} = 7.0$, and $\|\mathbf{A}\|_F = 7.348$. Eigenvalues of $\mathbf{A}^T\mathbf{A}$ are given by the solution of

$$\det[\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}] = \lambda^2 - 54\lambda + 196 = 0.$$

Thus, $\sigma_1^2 = 50.087$ and $\sigma_2^2 = 3.913$, and, hence, $\sigma_1 = 7.07$ and $\sigma_2 = 1.978$. Hence, $\|\mathbf{A}\|_2 = 7.07$. Again, it can be verified that the eigenvalues of $\mathbf{A}$ are given by the roots of

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 9\lambda + 14 = 0$$

Thus, the eigenvalues are given by $\lambda_1 = 7$, and $\lambda_2 = 2.0$. All the inequalities given above are easily verified.

The reader can easily verify that the norm of an identity matrix is 1 under all matrix norms. Clearly, each of the norms, $\|\mathbf{A}\|_1$ and $\|\mathbf{A}\|_\infty$, require $(n^2 - n)$ additions and $(n - 1)$ comparisons. But, the spectral norm $\|\mathbf{A}\|_2$ is rather expensive to compute. This latter computation could require $O(n^3)$ operations. Spectral norms are, however, very useful in theoretical analysis of convergence of iterative methods.

Neumman series If x is a real number such that $|x| < 1$, then

$$(1 - x)^{-1} = \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots +$$

is a well-known identity. If $\mathbf{A} \in \mathbb{R}^{m\times m}$ is such that $\|\mathbf{A}\| < 1$. Then, it can likewise be verified that

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots) = \mathbf{I}$$

from which it follows that

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^k + \cdots$$

which is called the *Neumman series expansion* of the inverse of $(\mathbf{I} - \mathbf{A})$.

B.7 Condition number of a matrix

Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a matrix. The *condition number* of a matrix, denoted by $\kappa(\mathbf{A})$ is defined as

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \left\|\mathbf{A}^{-1}\right\|$$

where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$. Thus, the value of the condition number is norm dependent. Since $\mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$, using the property of matrix norms, we get

$$1 = \|\mathbf{I}\| = \left\|\mathbf{A}\mathbf{A}^{-1}\right\| \leq \|\mathbf{A}\| \left\|\mathbf{A}^{-1}\right\| = \kappa(\mathbf{A}).$$

Thus, for any matrix,

$$1 \leq \kappa(\mathbf{A}) \leq \infty.$$

Matrices with small (large) condition numbers are said to *well (ill) conditioned*. Since the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of those of $\mathbf{A}$, we have

$$\left\|\mathbf{A}^{-1}\right\|_2 = \max_i\{\frac{1}{|\lambda_i|}\} = \frac{1}{\min_i\{|\lambda_i|\}}$$

and *spectral condition number*

$$\kappa_2(\mathbf{A}) = \|\mathbf{A}\|_2 \left\|\mathbf{A}^{-1}\right\|_2 = \frac{\max_i\{|\lambda_i|\}}{\min_i\{|\lambda_i|\}}.$$

Using the relation between various norms, we immediately have

$$\begin{aligned}
&\tfrac{1}{m}\kappa_2(\mathbf{A}) \leq \kappa_1(\mathbf{A}) \leq m\kappa_2(\mathbf{A}) \\
&\tfrac{1}{m}\kappa_\infty(\mathbf{A}) \leq \kappa_2(\mathbf{A}) \leq m\kappa_\infty(\mathbf{A}) \\
&\tfrac{1}{m^2}\kappa_1(\mathbf{A}) \leq \kappa_\infty(\mathbf{A}) \leq m^2\kappa_1(\mathbf{A}).
\end{aligned}$$

Much like the matrix norms, the spectral condition number theoretically useful, is rather difficult to compute. However, $\kappa_1(\mathbf{A})$ and $\kappa_\infty(\mathbf{A})$ can be readily computed and we can estimate $\kappa_2(\mathbf{A})$ using these inequalities.

It turns out that we are often more interested in the order of magnitude of $\kappa(\mathbf{A})$ rather than its exact value. There are excellent subroutines for computing $\kappa_1(\mathbf{A})$ in the widely available **LAPACK** library.

It can be verified

$$\kappa(\mathbf{AB}) \leq \kappa(\mathbf{A})\kappa(\mathbf{B})$$

and

$$\kappa(\alpha\mathbf{A}) \leq \kappa(\mathbf{A})$$

where $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times m}$, and $\alpha \neq 0$. Thus, condition number is *independent* of the scaling of the matrix.

In conclusion, we wish to emphasize that there is no connection or correlation between det($\mathbf{A}$), the determinant of $\mathbf{A}$, and $\kappa(\mathbf{A})$, the condition number of $\mathbf{A}$. Let $\mathbf{A} \in \mathbb{R}^{m\times m}$ be a diagonal matrix where each diagonal element is $1/2$. Then

$$\det(\mathbf{A}) = \frac{1}{2^n} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

but

$$\kappa_p(\mathbf{A}) = 1 \text{ for } p = 1, 2, \text{ or } \infty.$$

Similarly, if $\mathbf{A} \in \mathbb{R}^{m \times m}$ is an upper triangular matrix such that

$$a_{ij} = \begin{cases} 1 & i = j \\ -1 & i > j \\ 0 & i < j \end{cases}$$

then, $\det(\mathbf{A}) = 1$ and $\kappa_\infty(\mathbf{A}) = n2^{n-1} \longrightarrow \infty$ as $n \longrightarrow \infty$.

B.8 Sensitivity analysis: solution of linear systems

Challenges facing the design of algorithms for scientific computing involves, among other things computation and/or control of accumulation of *round-off* errors resulting from *finite-precision arithmetic*. These are basically two approaches to this problem. In the *forward analysis*, we strive to compute the overall accumulated uncertainty resulting from a sequence of computations of an algorithm. The *backward analysis*, on the other hand, is a technique in which we throw back the round-off errors as perturbations on the input data to the algorithm. This latter technique pioneered by J. H. Wilkinson [1965] reduces the round-off error to *sensitivity analysis.*

We now quote without proof a basic result relating to the sensitivity analysis in linear systems. Let

$$\mathbf{Ax} = \mathbf{b}$$

be the given system and let $\kappa(\mathbf{A})$ be the condition number of $\mathbf{A}$. Let $f \in \mathbb{R}^m$ and $\mathbf{B} \in \mathbb{R}^{m \times m}$ and $\epsilon > 0$ be a real parameter. Let $\mathbf{y}$ be the solution of the perturbed system

$$(\mathbf{A} + \epsilon\mathbf{B})\mathbf{y} = \mathbf{b} + \epsilon f.$$

Notice that $\epsilon \, \|\mathbf{B}\| \,/\, \|\mathbf{A}\|$ and $\epsilon \, \|f\| \,/\, \|\mathbf{b}\|$ denote the relative errors induced by the perturbations $\epsilon\mathbf{B}$ and ϵf in $\mathbf{A}$ and $\mathbf{b}$ respectively. It can be shown that the relative error in the solution $\mathbf{y}$ is given by

$$\frac{\|\mathbf{y} - \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(\mathbf{A}) \left(\epsilon \frac{\|\mathbf{B}\|}{\|\mathbf{A}\|} + \epsilon \frac{f}{\|\mathbf{b}\|} \right).$$

Since $\kappa(\mathbf{A}) \geq 1$ *amplifies* the relative errors in $\mathbf{A}$ and $\mathbf{b}$. Thus, for given relative errors on $\mathbf{A}$ and $\mathbf{b}$, the relative errors in the solution are directly proportional to $\kappa(\mathbf{A})$. Herein lies one of the fundamental consequences of the conditioning of

A – the larger the value of $\kappa(\mathbf{A})$, the larger is the potential for errors in the computed solution, that is, *the larger is the sensitivity of the solution.*

Notes and references

The material covered in this Appendix is normally the basis for the standard first year graduate level course on matrices. There are virtually countless books dealing with these topics. We mention only a handful of classics – Bellman (1960), Basilevsky (1983), Golub and van Loan (1989), Trefethen and Bau (1997), and Meyer (2000).