8 Chapter 8 Solution Set

Problems

8.1 Compute the coherent point spread function (CPSF) of the low pass filtered approximation generated via Eq.(8.3).

We find from Eq.(8.3) that

$$V_{LP}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{|\mathbf{K}|\leq 2k_0} d^3K\,\tilde{V}(\mathbf{K})e^{i\mathbf{K}\cdot\mathbf{r}}$$

$$= \int d^3r'\,V(\mathbf{r}')\overbrace{\frac{1}{(2\pi)^3}\int_{|\mathbf{K}|\leq 2k_0} d^3K\,e^{i\mathbf{K}\cdot(\mathbf{r}-\mathbf{r}')}}^{H(\mathbf{r},\mathbf{r}')},$$

where $H(\mathbf{r},\mathbf{r}')$ is the coherent point spread function associated with the low pass image of the scattering potential. We can express the CPSF using spherical polar integration variables to find that

$$H(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\int_0^{2k_0} K^2dK\int_{4\pi} d\Omega_K\,e^{iK\hat{\mathbf{K}}\cdot(\mathbf{r}-\mathbf{r}')} = \frac{1}{2\pi^2}\int_0^{2k_0} K^2dK\,j_0(K|\mathbf{r}-\mathbf{r}'|),$$

where we have used the result (cf., Example 3.4 of Chapter 3)

$$\int_{4\pi} d\Omega_K\,e^{iK\hat{\mathbf{K}}\cdot\mathbf{R}} = 4\pi j_0(KR)$$

where j_0 is the zero order spherical Bessel function of the first kind. If we now use the recurrence relationship

$$x^2j_0(x) = \frac{d}{dx}[x^2j_1(x)]$$

and make a simple change of integration variables we find that

$$H(\mathbf{r},\mathbf{r}') = \frac{1}{2\pi^2R^3}\int_0^{2k_0R} x^2dx\,j_0(x) = \frac{2k_0^2}{\pi^2R}j_1(2k_0R),$$

where $R = |\mathbf{r}-\mathbf{r}'|$ and j_1 is the spherical Bessel function of the first kind and order one.

8.2 Complete the derivation of Eq.(8.6) using contour integration techniques.

We begin with the second line from Eq.(8.6):

$$\tilde{H}(\mathbf{K}) = 4\pi \int_0^\infty R^2 dR\, j_0^2(k_0 R) j_0(KR) = \frac{2\pi}{k_0^2 K} \int_{-\infty}^\infty \frac{dR}{R} \sin^2(k_0 R) \sin(KR),$$

where the contour of integration can be arbitrarily deformed since $R = 0$ is a removable singularity. We now use the Euler identity to expand the product of sin functions to obtain

$$\tilde{H}(\mathbf{K}) = i\frac{\pi}{4k_0^2 K} \int_{-\infty}^\infty \frac{dR}{R} [e^{i(2k_0+K)R} - e^{i(2k_0-K)R} + e^{-i(2k_0-K)R} \\ -e^{-i(2k_0+K)R} - 2e^{iKR} + 2e^{-iKR}].$$

Since the contour of integration can be arbitrarily deformed we will select it to lie below the removable singularity at $R = 0$. Note that although $R = 0$ is removable for the entire integrand it is a pole for each component in the above equation. We can thus use residue calculus to evaluate each component and then sum them to obtain the final expression for $\tilde{H}(\mathbf{K})$.

We first consider $K > 2k_0$. In this case we can close the contour for the second term, fourth term and sixth term in the l.h.p. to obtain zero. The remaining terms all must be closed in the u.h.p. and each yields a residue of unity so that we find that

$$\tilde{H}(\mathbf{K}) = i\frac{\pi}{4k_0^2 K} 2\pi i[1 + 1 - 2] = 0, \quad 2k_0 < K < \infty.$$

Now consider $0 < K < 2k_0$. In this case we close the first, second, and fifth terms in the u.h.p. and the remaining terms in the l.h.p. to obtain

$$\tilde{H}(\mathbf{K}) = i\frac{\pi}{4k_0^2 K} 2\pi i[1 - 1 - 2] = \frac{\pi^2}{k_0^2 K}, \quad 0 < K < 2k_0,$$

which then yield Eq.(8.6).

8.3 Fill in the steps in the derivation of Eq.(8.18) from Eq.(8.17).

The only missing step is that $|\mathbf{s} - \mathbf{s}_0| = \sqrt{2 - 2\mathbf{s} \cdot \mathbf{s}_0} = \sqrt{2}\sqrt{1 - \cos\alpha}$. The rest of the steps are clearly explained in the text.

8.4 Verify that the FBP algorithm with $\mathbf{s}_0$ and $\mathbf{s}$ restricted to the solid angles Ω_{s_0} and Ω_s satisfies the integral equation Eq.(8.19).

The integral equation to be satisfied is

$$f_B(\mathbf{s}, \mathbf{s}_0) = -\frac{1}{4\pi} \int d^3r\, V(\mathbf{r}) e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$

where f_B is the Born approximation to the scattering amplitude and $\mathbf{s}_0$ and $\mathbf{s}$ restricted to the solid angles Ω_{s_0} and Ω_s. If we substitute the FBP algorithm for the scattering potential in the above equation we obtain

$$f_B(\mathbf{s},\mathbf{s}_0) = -\frac{1}{4\pi}\int d^3r \overbrace{-\frac{k_0^3}{4\pi^3}\int d\Omega_{s_0'}\int d\Omega_{s'}\,|\mathbf{s}'-\mathbf{s}_0'|f_B(\mathbf{s}',\mathbf{s}_0')e^{ik_0(\mathbf{s}'-\mathbf{s}_0')\cdot\mathbf{r}}}^{V(\mathbf{r})}\, e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$
$$= \frac{k_0^3}{(2\pi)^4}\int d\Omega_{s_0'}\int d\Omega_{s'}\,|\mathbf{s}'-\mathbf{s}_0'|f_B(\mathbf{s}',\mathbf{s}_0')\int d^3r\, e^{ik_0[(\mathbf{s}'-\mathbf{s}_0')-(\mathbf{s}-\mathbf{s}_0)]\cdot\mathbf{r}}$$
$$= \frac{k_0^3}{2\pi}\int d\Omega_{s_0'}\int d\Omega_{s'}\,|\mathbf{s}'-\mathbf{s}_0'|f_B(\mathbf{s}',\mathbf{s}_0')\delta\left[k_0[(\mathbf{s}'-\mathbf{s}_0')-(\mathbf{s}-\mathbf{s}_0)]\right]$$

Now $f_B(\mathbf{s}',\mathbf{s}_0')$ in the above integral must be equal to $-\tilde{V}[k_0(\mathbf{s}'-\mathbf{s}_0')]/4\pi$ so that we can use the inverse scattering identity given in Eq.(8.10b) to evaluate the integral. Thus, we define

$$\tilde{F}(\mathbf{K}) = -\frac{1}{4\pi}\tilde{V}(\mathbf{K})\delta[\mathbf{K}-k_0(\mathbf{s}-\mathbf{s}_0)]$$

with $\mathbf{K}=k_0(\mathbf{s}'-\mathbf{s}_0')$ restricted to that region of $\mathbf{K}$ space that corresponds to $\mathbf{s}'\in\Omega_{s'}$ and $\mathbf{s}_0'\in\Omega_{s_0'}$, respectively. We then find that the r.h.s. of the above equation becomes

$$\frac{k_0^3}{2\pi}\int d\Omega_{s_0'}\int d\Omega_{s'}\,|\mathbf{s}'-\mathbf{s}_0'|f_B(\mathbf{s}',\mathbf{s}_0')\delta\left[k_0[(\mathbf{s}'-\mathbf{s}_0')-(\mathbf{s}-\mathbf{s}_0)]\right]$$
$$= \int_{K\le 2k_0} d^3K\,\tilde{F}(\mathbf{K}) = -\frac{1}{4\pi}\tilde{V}[k_0(\mathbf{s}-\mathbf{s}_0)] = f_B(\mathbf{s},\mathbf{s}_0), \quad \mathbf{s}\in\Omega_s,\ \mathbf{s}_0\in\Omega_{s_0},$$

where we have used the inverse scattering identity.

8.5 Complete the derivation of Eq.(8.21).

This follows immediately upon using the definition of $H(\mathbf{R})$ given in Eq.(8.4c).

8.6 Derive the expressions for $\hat{T}\hat{T}^\dagger$ given in Eq.(8.31) and for $\hat{T}^\dagger\hat{T}$ given in Eq.(8.30).

Consider first $\hat{T}^\dagger\hat{T}V$ with $V\in\mathcal{H}_V$

$$\hat{T}^\dagger\hat{T}V = \frac{1}{(4\pi)^2}\int d\Omega_{s_0}\int d\Omega_s\, e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}\int d^3r'\, e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}V(\mathbf{r}')$$
$$= \int d^3r' \overbrace{\frac{1}{(4\pi)^2}\int d\Omega_{s_0}\int d\Omega_s\, e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot(\mathbf{r}-\mathbf{r}')}}^{H(\mathbf{r}-\mathbf{r}')}\, V(\mathbf{r}').$$

We can express $H(\mathbf{R})$ in the form

$$H(\mathbf{R}) = \left[\frac{1}{4\pi}\int d\Omega_{s_0}e^{-ik_0\mathbf{s}_0\cdot\mathbf{R}}\right]\left[\frac{1}{4\pi}\int d\Omega_s e^{ik_0\mathbf{s}\cdot\mathbf{R}}\right] = j_0^2(k_0R)$$

which completes the proof.

Now consider $\hat{T}\hat{T}^\dagger f$ with $f \in \mathcal{H}_f$

$$\hat{T}\hat{T}^\dagger f = \frac{1}{(4\pi)^2}\int d^3r\, e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}\int d\Omega_{s_0'}\int d\Omega_{s'}\, e^{ik_0(\mathbf{s}'-\mathbf{s}_0')\cdot\mathbf{r}} f(\mathbf{s}',\mathbf{s}_0')$$
$$= \frac{1}{(4\pi)^2}\int d\Omega_{s_0'}\int d\Omega_{s'}\left[\int d^3r\, e^{-ik_0[(\mathbf{s}-\mathbf{s}_0)-(\mathbf{s}'-\mathbf{s}_0')]\cdot\mathbf{r}}\right] f(\mathbf{s}',\mathbf{s}_0')$$
$$= \frac{\pi}{2}\int d\Omega_{s_0'}\int d\Omega_{s'}\,\delta(k_0[(\mathbf{s}-\mathbf{s}_0)-(\mathbf{s}'-\mathbf{s}_0')])f(\mathbf{s}',\mathbf{s}_0')$$

which completes the proof.

8.7 Complete the derivation of Eq.(8.42).

On substituting Eq.(8.41) into Eq.(8.40) we obtain

$$U^{(s)}(\mathbf{r};\nu) \sim -\frac{1}{4\pi}\frac{e^{ik_0 r}}{r}\int_{\partial\tau} dS'\,[e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{\partial}{\partial n'}U^{(s)}(\mathbf{r}';\nu) - U^{(s)}(\mathbf{r}';\nu)(-ik_0\hat{\mathbf{n}}'\cdot\mathbf{s})e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}]$$
$$= \overbrace{-\frac{1}{4\pi}\int_{\partial\tau} dS'\,[\frac{\partial}{\partial n'}U^{(s)}(\mathbf{r}';\nu) + ik_0(\mathbf{s}\cdot\hat{\mathbf{n}}')U^{(s)}(\mathbf{r}';\nu)]e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}}^{f(\mathbf{s},\nu)}\,\frac{e^{ik_0 r}}{r}.$$

8.8 Express the generalized scattering amplitude given in Eq.(8.42) in terms of Dirichlet data over a sphere that completely encloses the scattering volume τ_0. Use this result to express the generalized scattering amplitude in terms of the multipole moments of the scattered field.

The key to this problem is that the Dirichlet and Neumann boundary conditions of the scattered field over a sphere surrounding the scattering potential are related to each other as was established in our treatment of the so-called "Dirichlet to Neumann map" in Example 4.10. In that example we showed that the generalized Fourier coefficients of the Dirichlet and Neumann boundary conditions over any sphere lying outside a source volume are related via the equation

$$v_l^m = \frac{k_0 h_l^{+\prime}(k_0 a)}{h_l^+(k_0 a)}u_l^m \tag{8.1}$$

where

$$u_l^m = \int d\Omega\, U_+(\mathbf{r})|_{r=a}Y_l^{m*}(\hat{\mathbf{r}}), \quad v_l^m = \int d\Omega\,\frac{\partial}{\partial r}U_+(\mathbf{r})|_{r=a}Y_l^{m*}(\hat{\mathbf{r}})$$

are the generalized Fourier coefficients of the Dirichlet and Neumann boundary conditions over a measurement sphere of radius a that exceeds the support radius of the source (scattering potential in this problem).

To make use of Eq.(8.1) we use the multipole expansion of the plane wave that was developed in Example 3.4:

$$e^{-ik_0\mathbf{s}\cdot\mathbf{r}'} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l j_l(k_0 r')Y^{*m}_l(\hat{\mathbf{r}}')Y_l^m(\mathbf{s}).$$

Using this expansion and the derived expansion

$$ik_0(\mathbf{s}\cdot\hat{\mathbf{n}}')e^{-ik_0\mathbf{s}\cdot\mathbf{r}} = -\frac{\partial}{\partial r'}e^{-ik_0\mathbf{s}\cdot\mathbf{r}'} = -4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l k_0 j_l'(k_0 r')Y^{*m}_l(\hat{\mathbf{r}}')Y_l^m(\mathbf{s})$$

in Eq.(8.42) we obtain

$$f(\mathbf{s},\nu) = -\frac{1}{4\pi}\int_{\partial\tau} dS' \frac{\partial}{\partial n'}U^{(s)}(\mathbf{r}';\nu)[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l j_l(k_0 r')Y^{*m}_l(\hat{\mathbf{r}}')Y_l^m(\mathbf{s})]$$

$$+\frac{1}{4\pi}\int_{\partial\tau} dS'\, U^{(s)}(\mathbf{r}';\nu)[4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l k_0 j_l'(k_0 r')Y^{*m}_l(\hat{\mathbf{r}}')Y_l^m(\mathbf{s})]$$

$$= -a^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l[j_l(k_0 a)v_l^m - k_0 j_l'(k_0 a)u_l^m]Y_l^m(\mathbf{s})]$$

$$= -a^2\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^l[j_l(k_0 a)\frac{k_0 h_l^{+\prime}(k_0 a)}{h_l^+(k_0 a)} - k_0 j_l'(k_0 a)]u_l^m Y_l^m(\mathbf{s})].$$

As a final step we use the Wronskian

$$j_l(x)h_l'(x) - j_l'(x)h_l(x) = \frac{i}{x^2}$$

to simplify the above result to

$$f(\mathbf{s},\nu) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^{l+1}\frac{u_l^m}{k_0 h_l^+(k_0 a)}Y_l^m(\mathbf{s}).$$

It is easy to verify that the above result is correct by using the multipole expansion of the scattered field (see Section 6.5.3). Using the notation from Section 6.5.3 we have that

$$U_+^{(s)}(\mathbf{r},\nu) = -ik_0\sum_{l=0}^{\infty}\sum_{m=-l}^{l}q_l^m(\nu)h_l^+(k_0 r)Y_l^m(\hat{\mathbf{r}})$$

where the multipole moments $q_l^m(\nu)$ can be determined from Dirichlet conditions over any sphere surrounding the scattering potential. In particular, we find that

$$q_l^m(\nu) = \frac{i}{k_0 h_l^+(k_0 a)}\int_{4\pi} d\Omega\, U_+^{(s)}(\mathbf{r},\nu)Y_l^{m*}(\hat{\mathbf{r}}) = \frac{i}{k_0 h_l^+(k_0 a)}u_l^m \qquad (8.2)$$

where $\mathbf{r}$ denotes position on the measurement sphere having radius a larger than the support radius of the scatterer. On substituting this expression for the multipole moments in the multipole expansion and letting $r\to\infty$ we obtain

$$U_+^{(s)}(\mathbf{r},\nu) \sim -ik_0 \sum_{l=0}^{\infty}\sum_{m=-l}^{l}[\frac{i}{k_0 h_l^+(k_0a)}u_l^m](-i)^{(l+1)}\frac{e^{ik_0r}}{k_0r}Y_l^m(\hat{\mathbf{r}})$$

$$= \overbrace{\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(-i)^{l+1}\frac{u_l^m}{k_0h_l^+(k_0a)}Y_l^m(\mathbf{s})}^{f(\mathbf{s},\nu)}\frac{e^{ik_0r}}{r}$$

which is the same result obtained above for the scattering amplitude.

Finally we note that Eq.(8.2) allows us to express the scattering amplitude directly in terms of the multipole moments of the scattered field.

8.9 Derive Eqs.(8.44) directly from Eqs.(8.42). Hint: Use the relationship between the spatial Fourier transforms of the scattered field and its normal derivative over a plane surface derived in Example 4.4 of Chapter 4.

Specializing Eqs.(8.42) to the case of two parallel measurement planes located at $z = z_<$ and $z = z_>$ outside and bounding the support of the scattering potential we obtain

$$f(\mathbf{s},\nu) = -\frac{1}{4\pi}\int_{z_<} dS'\,[-\frac{\partial}{\partial z'}U^{(s)}(\mathbf{r}';\nu) - ik_0(\mathbf{s}\cdot\hat{\mathbf{z}})U^{(s)}(\mathbf{r}';\nu)]e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}$$
$$-\frac{1}{4\pi}\int_{z_>} dS'\,[\frac{\partial}{\partial z'}U^{(s)}(\mathbf{r}';\nu) + ik_0(\mathbf{s}\cdot\hat{\mathbf{z}}')U^{(s)}(\mathbf{r}';\nu)]e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}$$
$$= -\frac{1}{4\pi}[-\widetilde{\frac{\partial}{\partial z'}U_{z_<}^{(s)}}(k_0\mathbf{s}_{||};\nu) - ik_0s_z\widetilde{U_{z_<}^{(s)}}(k_0\mathbf{s}_{||};\nu)]e^{-ik_0s_zz_<}$$
$$-\frac{1}{4\pi}[\widetilde{\frac{\partial}{\partial z'}U_{z_>}^{(s)}}(k_0\mathbf{s}_{||};\nu) + ik_0s_z\widetilde{U_{z_>}^{(s)}}(k_0\mathbf{s}_{||};\nu)]e^{-ik_0s_zz_>}$$

where $\widetilde{U_z^{(s)}}$ and $\widetilde{\frac{\partial}{\partial z}U_z^{(s)}}$ denote spatial Fourier transforms of the field and its normal derivative over the plane z and $\mathbf{s}_{||} = (s_x, s_y)$ is the transverse component of the unit propagation vector and s_z its longitudinal component.

We now make use of the relationship between the spatial Fourier transforms of the scattered field and its normal derivative over a plane surface derived in Example 4.4 of Chapter 4. In that example it was shown that

$$\widetilde{\frac{\partial}{\partial z}U_z^{(s)}}(k_0\mathbf{s}_{||};\nu) = \pm ik_0|s_z|\widetilde{U_z^{(s)}}(k_0\mathbf{s}_{||};\nu)$$

with the plus sign holding if z lies to the right of the source of the field (in this case the scattering potential) and the minus sign if z lies to the left of the source. On making use of these relationships we then find that

$$f(\mathbf{s},\nu) = -\frac{1}{4\pi}[\overbrace{-\,-ik_0|s_z|\widetilde{U^{(s)}_{z<}}(k_0\mathbf{s}_{||};\nu)}^{\frac{\partial}{\partial z'}\widetilde{U^{(s)}_{z<}}(k_0\mathbf{s}_{||};\nu)} - ik_0 s_z \widetilde{U^{(s)}_{z<}}(k_0\mathbf{s}_{||};\nu)]e^{-ik_0 s_z z_<}$$

$$-\frac{1}{4\pi}[\overbrace{ik_0|s_z|\widetilde{U^{(s)}_{z>}}(k_0\mathbf{s}_{||};\nu)}^{\frac{\partial}{\partial z'}\widetilde{U^{(s)}_{z>}}(k_0\mathbf{s}_{||};\nu)} + ik_0 s_z \widetilde{U^{(s)}_{z>}}(k_0\mathbf{s}_{||};\nu)]e^{-ik_0 s_z z_>}$$

We now have to evaluate the above for the two cases where $s_z > 0$ (scattering into the r.h.s.) and $s_z < 0$ (scattering into the l.h.s.) We then obtain

$$f(\mathbf{s},\nu) = -\frac{ik_0}{2\pi} s_z \widetilde{U^{(s)}_{z>}}(k_0\mathbf{s}_{||};\nu)e^{-ik_0 s_z z_>}, \quad s_z > 0$$

$$f(\mathbf{s},\nu) = \frac{ik_0}{2\pi} s_z \widetilde{U^{(s)}_{z<}}(k_0\mathbf{s}_{||};\nu)]e^{-ik_0 s_z z_<} \quad s_z < 0$$

which are precisely Eqs.(8.44).

8.10 Derive the second line in Eq.(8.55) from the first line.

We start with the first line in Eq.(8.55) which we write in the form

$$\delta n(\boldsymbol{\rho}) = \frac{1}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_0^{\infty} KdK\,\tilde{\delta n}(K\hat{\mathbf{K}})e^{iK\hat{\mathbf{K}}\cdot\boldsymbol{\rho}}$$

$$+\frac{1}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_0^{\infty} KdK\,\tilde{\delta n}(K\hat{\mathbf{K}})e^{iK\hat{\mathbf{K}}\cdot\boldsymbol{\rho}}.$$

In the second integral we make the change of variables $K \to -K$ and $\alpha_0 \to \alpha_0 + \pi$. Under this change of integration variables we have that

$$\hat{\mathbf{K}} = \cos\alpha\hat{\mathbf{x}} + \sin\alpha\hat{\mathbf{y}} \to \cos(\alpha+\pi)\hat{\mathbf{x}} + \sin(\alpha+\pi)\hat{\mathbf{y}} = -\hat{\mathbf{K}}.$$

The second integral then becomes

$$\frac{1}{2(2\pi)^2}\int_0^{2\pi} d\alpha_0 \int_{-\infty}^{0} -KdK\,\tilde{\delta n}(K\hat{\mathbf{K}})e^{iK\hat{\mathbf{K}}\cdot\boldsymbol{\rho}} = \frac{1}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-\infty}^{0} |K|dK\,\tilde{\delta n}(K\hat{\mathbf{K}})e^{iK\hat{\mathbf{K}}\cdot\boldsymbol{\rho}}$$

which when added to the first integral yield the second line in Eq.(8.55).

8.11 Complete the derivation of Eq.(8.59).

We wish to show that for circularly symmetric objects the filtered back projection algorithm

$$\delta n(\boldsymbol{\rho}) = \frac{1}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-\infty}^{\infty} |K|dK\,\widetilde{P\delta n}(K)e^{iK\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\rho}}$$

can be transformed into

$$\delta n(\boldsymbol{\rho}) = \frac{1}{2\pi}\int_0^{\infty} KdK\,\widetilde{P\delta n}(K)J_0(K\rho).$$

To make this transformation we set $K\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\rho} = K\rho\cos(\phi-\alpha_0)$ where ϕ is the angle of ρ relative to the positive x axis and, as usual, α_0 is the angle of the ξ axis relative to the positive x axis. We then find that

$$\int_{-\pi}^{\pi} d\alpha_0 e^{iK\hat{\boldsymbol{\xi}}\cdot\boldsymbol{\rho}} = \int_{-\pi}^{\pi} d\alpha_0 e^{iK\rho\cos(\phi-\alpha_0)} = \int_{-\pi}^{\pi} d\alpha_0 e^{iK\rho\cos\alpha_0} = 2\pi J_0(K\rho),$$

since the integral has to be independent of ϕ. Using this result we then obtain

$$\delta n(\boldsymbol{\rho}) = \frac{1}{4\pi}\int_{-\infty}^{\infty} |K| dK\, \widetilde{P\delta n}(K) J_0(K\rho) = \frac{1}{2\pi}\int_0^{\infty} K dK\, \widetilde{P\delta n}(K) J_0(K\rho)$$

where we have used the fact that for circularly symmetric objects $\widetilde{P\delta n}(-K) = \widetilde{P\delta n}(K)$.

8.12 Derive Eq.(8.71a).

This is obtained by using the same procedure as employed in Example 8.1 with the exception that the angle α formed by the unit vector $\mathbf{s}$ with the positive ξ axis now varies only from $\alpha = 0$ to $\alpha = \pi$ corresponding to forward scattering. Because of this the Ewald circles are now only semi-circles and so the limiting Ewald circle has a radius of $\sqrt{2}k_0$ so that the algorithm returns the low pass filtered approximation given in Eq.(8.71b).

8.13 Derive Eq.(8.72).

This form follows from the change of integration variable $\alpha \to \kappa$ specified in the text. In particular, we set

$$\mathbf{s} = \hat{\boldsymbol{\xi}}\cos\alpha + \hat{\boldsymbol{\eta}}\sin\alpha, \quad \mathbf{s}_0 = \hat{\boldsymbol{\eta}}, \quad \mathbf{s}_0\cdot\mathbf{s} = \sin\alpha$$

so that

$$k_0(\mathbf{s}-\mathbf{s}_0) = \overbrace{k_0\cos\alpha}^{\kappa}\,\hat{\boldsymbol{\xi}} + [\overbrace{k_0\sin\alpha}^{\gamma} - k_0]\hat{\boldsymbol{\eta}}, \quad \mathbf{s}_0\cdot\mathbf{s} = \frac{\gamma}{k_0}, \quad d\kappa = -\gamma d\alpha.$$

On making these change of integration variables we find that Eq.(8.71a) becomes

$$\begin{aligned}\delta n_{LP}(\boldsymbol{\rho}) &= \frac{k_0^2}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_0^{\pi} d\alpha\, \sqrt{1-(\mathbf{s}\cdot\mathbf{s}_0)^2}\,\tilde{\delta n}[k_0(\mathbf{s}-\mathbf{s}_0)]e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\boldsymbol{\rho}} \\ &= \frac{k_0^2}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-k_0}^{k_0} \frac{d\kappa}{\gamma}\sqrt{1-(\frac{\gamma}{k_0})^2}\,\tilde{\delta n}[k_0(\mathbf{s}-\mathbf{s}_0)]e^{i[\kappa\xi+(\gamma-k_0)\eta]} \\ &= \frac{k_0}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-k_0}^{k_0} \frac{d\kappa}{\gamma}\overbrace{\sqrt{k_0^2-\gamma^2}}^{|\kappa|}\,\tilde{\delta n}[k_0(\mathbf{s}-\mathbf{s}_0)]e^{i[\kappa\xi+(\gamma-k_0)\eta]}\end{aligned}$$

We now make use of Eq.(8.70) and make the replacement

$$\tilde{\delta n}[k_0(\mathbf{s}-\mathbf{s}_0)] = \frac{\gamma}{k_0 e^{i(\gamma-k_0)l_0}}\widetilde{\delta W}_R(\kappa,\alpha_0)$$

to obtain

$$\delta n_{LP}(\boldsymbol{\rho}) = \frac{1}{2(2\pi)^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-k_0}^{k_0} d\kappa\, |\kappa| \widetilde{\delta W}_R(\kappa,\alpha_0) e^{i[\kappa\xi+(\gamma-k_0)(\eta-l_0)]}.$$

8.14 Complete the derivation of Eq.(8.73a); i.e, show that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_0 e^{i[\kappa\xi+(\gamma-k_0)\eta]} = J_0(\sqrt{\kappa^2+(\gamma-k_0)^2}\rho),$$

where α_0 is the angle formed by the η coordinate axis with the fixed x axis. [Hint: write $\kappa\xi+(\gamma-k_0)\eta$ as the dot product of two vectors.]

Following the hint we set

$$\kappa\hat{\boldsymbol{\xi}}+(\gamma-k_0)\hat{\boldsymbol{\eta}}=\mathbf{v}, \quad \xi\hat{\boldsymbol{\xi}}+\eta\hat{\boldsymbol{\eta}}=\boldsymbol{\rho}$$

to find that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_0 e^{i[\kappa\xi+(\gamma-k_0)\eta]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_0 e^{i\mathbf{v}\cdot\boldsymbol{\rho}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\alpha_0 e^{i\sqrt{\kappa^2+(\gamma-k_0)^2}\rho\cos\theta}$$

where θ is the angle formed between the two vectors $\mathbf{v}$ and $\boldsymbol{\rho}$. As α_0 varies between $-\pi$ and $+\pi$ the angle θ varies over a full 2π radians and the integral yields the result

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta e^{i\sqrt{\kappa^2+(\gamma-k_0)^2}\rho\cos\theta} = J_0(\sqrt{\kappa^2+(\gamma-k_0)^2}\rho).$$

8.15 Derive Eq.(8.75b).

We have

$$\tilde{\delta n}(\mathbf{K}) = \sum_{j=1}^{N}\delta n_j \int d^2\rho\, \mathrm{circ}[a_j(\boldsymbol{\rho}-\boldsymbol{\rho}_j)]e^{-i\mathbf{K}\cdot\boldsymbol{\rho}} = \sum_{j=1}^{N}\delta n_j e^{-i\mathbf{K}\cdot\boldsymbol{\rho}_j}\int d^2\rho\, \mathrm{circ}(a_j\boldsymbol{\rho})e^{-i\mathbf{K}\cdot\boldsymbol{\rho}}$$

$$= 2\pi\sum_{j=1}^{N}\delta n_j e^{-i\mathbf{K}\cdot\boldsymbol{\rho}_j}\int_0^{a_j}\rho d\rho\, J_0(K\rho) = 2\pi\sum_{j=1}^{N}\delta n_j \frac{J_1(Ka_j)}{K}e^{-i\mathbf{K}\cdot\boldsymbol{\rho}_j}.$$