```
Clear["Global`*"]
```

Thomas Precession in Special Relativity

In many elementary problems in special relativity it is assumed that the relative velocity of two frames is along the x-axis with no velocity along either the y or z axes. We have previously introduced the complete expression for a general relative velocity in 3D. It is not pretty and one can imagine that even a small frame velocity orthogonal to the x-axis will introduce immediate complication. In this notebook, we investigate such a situation in an attempt to derive what the essential character of the complication is.

In many elementary problems in special relativity it is assumed that the relative velocity of two frames is along the x-axis with no velocity along either the y or z axes. We have previously introduced the complete expression for a general relative velocity in 3D. It is not pretty and one can imagine that even a small frame velocity orthogonal to the x-axis will introduce immediate complication. In this notebook, we investigate such a situation in an attempt to derive what the essential character of the complication is. Here we have adopted the derivation used in Goldstein, Poole, and Safko (2002). It must be noted, however, that in following Goldstein, Poole, and Safko we have adopted their definition of the Minkowski metric. That is $d s^2 = c^2 (d \tau)^2 = c^2 (d t)^2 - (d x^2 + d y^2 + d z^2)$. This means that $ds^2>0$ for bodies moving slower than the speed of light (time-like) and $ds^2<0$ for objects that that participate in events that can be simultaneous in time dt =0, but separated in space (space-like). This convention is indicated if the metric tensor has a plus sign in the upper left of the matrix representation. Some authors define ds^2 such that the space part is positive and the time part has a negative sign. This then reverses the definitions of time-like and space-like given above. This is indicated if the metric tensor has a minus sign in the upper left of the matrix representation.

Preliminary Definitions

The x-axis Lorentz Transformation in list form

```
lorentzX = {{γ, -γ β, 0, 0}, {-γ β, γ, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}};
```

```
lorentzX // MatrixForm
```

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(Note here that the standard relativistic notation uses unprimed, primed, double primed etc. notation for different frames. But we cannot do that in *Mathematica* because it uses primes to indicate derivatives in DSolve[ ] and NDSolve[ ]. So our notation uses arabic numerals.)

The Generalized Lorentz Transformation in full list form

```
lorentzXYZ = {{γ1, -γ1 β1x, -γ1 β1y, -γ1 β1z},
   {-γ1 β1x, 1 + (γ1 - 1) β1x^2/β1^2, (γ1 - 1) β1x β1y/β1^2, (γ1 - 1) β1x β1z/β1^2},
   {-γ1 β1y, (γ1 - 1) β1x β1y/β1^2, 1 + (γ1 - 1) β1y^2/β1^2, (γ1 - 1) β1y β1z/β1^2},
   {-γ1 β1z, (γ1 - 1) β1x β1z/β1^2, (γ1 - 1) β1y β1z/β1^2, 1 + (γ1 - 1) β1z^2/β1^2}};
```

```
lorentzXYZ // MatrixForm
```

$$\begin{pmatrix} \gamma 1 & -\beta 1x\,\gamma 1 & -\beta 1y\,\gamma 1 & -\beta 1z\,\gamma 1 \\ -\beta 1x\,\gamma 1 & 1+\frac{\beta 1x^2\,(-1+\gamma 1)}{\beta 1^2} & \frac{\beta 1x\,\beta 1y\,(-1+\gamma 1)}{\beta 1^2} & \frac{\beta 1x\,\beta 1z\,(-1+\gamma 1)}{\beta 1^2} \\ -\beta 1y\,\gamma 1 & \frac{\beta 1x\,\beta 1y\,(-1+\gamma 1)}{\beta 1^2} & 1+\frac{\beta 1y^2\,(-1+\gamma 1)}{\beta 1^2} & \frac{\beta 1y\,\beta 1z\,(-1+\gamma 1)}{\beta 1^2} \\ -\beta 1z\,\gamma 1 & \frac{\beta 1x\,\beta 1z\,(-1+\gamma 1)}{\beta 1^2} & \frac{\beta 1y\,\beta 1z\,(-1+\gamma 1)}{\beta 1^2} & 1+\frac{\beta 1z^2\,(-1+\gamma 1)}{\beta 1^2} \end{pmatrix}$$

The 1 to 3 Transformation with no z-velocity of the frame

```
lorentzXY = lorentzXYZ /. β1z → 0
```

$$\left\{\{\gamma 1, -\beta 1x\,\gamma 1, -\beta 1y\,\gamma 1, 0\}, \left\{-\beta 1x\,\gamma 1, 1+\frac{\beta 1x^2\,(-1+\gamma 1)}{\beta 1^2}, \frac{\beta 1x\,\beta 1y\,(-1+\gamma 1)}{\beta 1^2}, 0\right\}, \left\{-\beta 1y\,\gamma 1, \frac{\beta 1x\,\beta 1y\,(-1+\gamma 1)}{\beta 1^2}, 1+\frac{\beta 1y^2\,(-1+\gamma 1)}{\beta 1^2}, 0\right\}, \{0, 0, 0, 1\}\right\}$$

Setting up the Successive Lorentz Transformations

We start the derivation by applying two successive Lorentz transformation matrix multiplications. It is asumed that the axes are all parallel to each other. The first matrix is a standard boost along the x-axis so the matrix is very simple. The second matrix is a matrix where there are two orthogonal boosts, both relatively small one again along x and another along y. direct matrix product is a mess and should be cleared of second order terms. Because the multiplication of two aligned Lorentz matrices produces another Lorentz matrix one might expect the same to happen here. But instead we find that the product is not symmetrical so that the order of the matrix multiplication matters.

```
lorentz2 = lorentzXY.lorentzX
```

$$\Big\{\{\gamma\,\gamma 1+\beta\,\beta 1x\,\gamma\,\gamma 1, -\beta\,\gamma\,\gamma 1-\beta 1x\,\gamma\,\gamma 1, -\beta 1y\,\gamma 1, 0\},$$
$$\left\{-\beta\,\gamma\left(1+\frac{\beta 1x^2\,(-1+\gamma 1)}{\beta 1^2}\right)-\beta 1x\,\gamma\,\gamma 1, \gamma\left(1+\frac{\beta 1x^2\,(-1+\gamma 1)}{\beta 1^2}\right)+\beta\,\beta 1x\,\gamma\,\gamma 1, \frac{\beta 1x\,\beta 1y\,(-1+\gamma 1)}{\beta 1^2}, 0\right\},$$
$$\left\{-\frac{\beta\,\beta 1x\,\beta 1y\,\gamma\,(-1+\gamma 1)}{\beta 1^2}-\beta 1y\,\gamma\,\gamma 1, \frac{\beta 1x\,\beta 1y\,\gamma\,(-1+\gamma 1)}{\beta 1^2}+\beta\,\beta 1y\,\gamma\,\gamma 1, 1+\frac{\beta 1y^2\,(-1+\gamma 1)}{\beta 1^2}, 0\right\}, \{0, 0, 0, 1\}\Big\}$$

```
lorentz2 // MatrixForm
```

$$\begin{pmatrix} \gamma\,\gamma1 + \beta\,\beta1x\,\gamma\,\gamma1 & -\beta\,\gamma\,\gamma1 - \beta1x\,\gamma\,\gamma1 & -\beta1y\,\gamma1 & 0 \\ -\beta\,\gamma\left(1 + \frac{\beta1x^2\,(-1+\gamma1)}{\beta1^2}\right) - \beta1x\,\gamma\,\gamma1 & \gamma\left(1 + \frac{\beta1x^2\,(-1+\gamma1)}{\beta1^2}\right) + \beta\,\beta1x\,\gamma\,\gamma1 & \frac{\beta1x\,\beta1y\,(-1+\gamma1)}{\beta1^2} & 0 \\ -\frac{\beta\,\beta1x\,\beta1y\,\gamma\,(-1+\gamma1)}{\beta1^2} - \beta1y\,\gamma\,\gamma1 & \frac{\beta1x\,\beta1y\,\gamma\,(-1+\gamma1)}{\beta1^2} + \beta\,\beta1y\,\gamma\,\gamma1 & 1 + \frac{\beta1y^2\,(-1+\gamma1)}{\beta1^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Reduction of the $\boldsymbol{L_{1\to3}}$ matrix to first order

It is assumed that the second Lorentz boost is small so that we can omit the second order terms. But even without doing that it can be seen that the result is no longer symmetrical so that a systematic effect in addition to the boost has been introduced. There is no really automatic way to do the reduction so we do it by hand. Various substitutions/approximations will be applied (following Goldstein, Poole, and Safko, 2000) and many will not be obvious as most represent working backwards from the final result. First we make a copy of the original matrix.

```
lorentz2
```

$$\Big\{\{\gamma\,\gamma1 + \beta\,\beta1x\,\gamma\,\gamma1,\ -\beta\,\gamma\,\gamma1 - \beta1x\,\gamma\,\gamma1,\ -\beta1y\,\gamma1,\ 0\},\ \Big\{-\beta\,\gamma\left(1 + \frac{\beta1x^2\,(-1+\gamma1)}{\beta1^2}\right) - \beta1x\,\gamma\,\gamma1,\ \gamma\left(1 + \frac{\beta1x^2\,(-1+\gamma1)}{\beta1^2}\right) + \beta\,\beta1x\,\gamma\,\gamma1,\ \frac{\beta1x\,\beta1y\,(-1+\gamma1)}{\beta1^2},\ 0\Big\},\ \Big\{-\frac{\beta\,\beta1x\,\beta1y\,\gamma\,(-1+\gamma1)}{\beta1^2} - \beta1y\,\gamma\,\gamma1,\ \frac{\beta1x\,\beta1y\,\gamma\,(-1+\gamma1)}{\beta1^2} + \beta\,\beta1y\,\gamma\,\gamma1,\ 1 + \frac{\beta1y^2\,(-1+\gamma1)}{\beta1^2},\ 0\Big\},\ \{0,\ 0,\ 0,\ 1\}\Big\}$$

In a given term we invoke that $\beta1 > \beta1x$ and $\beta1y \sim \beta1$ and are small with respect to β and hence reduce it to "first" order. This emphasizes the asymmetry, but even now it is not simple. It must consist of a boost plus some other operation. We need to figure out what that operation is.

```
lorentzP2R =
  {{γ γ1, -β γ γ1, -β1y γ1, 0}, {-β γ , γ , 0, 0}, {-β1y γ γ1, β β1y γ γ1, γ1, 0}, {0, 0, 0, 1}}
```

$$\{\{\gamma\,\gamma1,\ -\beta\,\gamma\,\gamma1,\ -\beta1y\,\gamma1,\ 0\},\ \{-\beta\,\gamma,\ \gamma,\ 0,\ 0\},\ \{-\beta1y\,\gamma\,\gamma1,\ \beta\,\beta1y\,\gamma\,\gamma1,\ \gamma1,\ 0\},\ \{0,\ 0,\ 0,\ 1\}\}$$

```
lorentzP2R // MatrixForm
```

$$\begin{pmatrix} \gamma\,\gamma1 & -\beta\,\gamma\,\gamma1 & -\beta1y\,\gamma1 & 0 \\ -\beta\,\gamma & \gamma & 0 & 0 \\ -\beta1y\,\gamma\,\gamma1 & \beta\,\beta1y\,\gamma\,\gamma1 & \gamma1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It appears that we will need to make some radical assumptions to produce something that approximates an infinitesimal operation on a Lorentz matrix. But here is where things get really tricky. It is hard these days to find a simple derivation of this problem but Goldstein, Poole, and Safko make a try toward a solution. First we use the global reassignments that work. At least they work except for one term in the matrix that we have to mend by hand.

```
lorentzAP2 = Simplify[lorentzP2R /. {γ1 → 1, γ → γ2, β → β2x, β1y -> β2y}]
```

{{γ2, -β2x γ2, -β2y, 0}, {-β2x γ2, γ2, 0, 0}, {-β2y γ2, β2x β2y γ2, 1, 0}, {0, 0, 0, 1}}

```
lorentzAP2 // MatrixForm
```

$$\begin{pmatrix} \gamma 2 & -\beta 2x\,\gamma 2 & -\beta 2y & 0 \\ -\beta 2x\,\gamma 2 & \gamma 2 & 0 & 0 \\ -\beta 2y\,\gamma 2 & \beta 2x\,\beta 2y\,\gamma 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

There is only one term that does not agree with Goldstein et al. at this stage and that is (1, 3) where a factor of $\gamma 2$ is missing to make the matrix more symmetric with the (3, 1) term. This is a bit of a fudge but possible because of the "smallness" of β2y and β2x. This is what is meant by reverse engineering !

```
lorentzAP2[[1, 3]] = -β2y γ2;
```

```
lorentzAP2 // MatrixForm
```

$$\begin{pmatrix} \gamma 2 & -\beta 2x\,\gamma 2 & -\beta 2y\,\gamma 2 & 0 \\ -\beta 2x\,\gamma 2 & \gamma 2 & 0 & 0 \\ -\beta 2y\,\gamma 2 & \beta 2x\,\beta 2y\,\gamma 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We now have reach the next stage of Goldstein et al. But even this approximation for the L ''= L 'L matrix is still not a pure boost. If we start in S3 and work backwards with a boost in the x '' axis of -βx ''and a boost along the y '' axis of -βy" then the original XYZ matrix with no approximations becomes the inverse transformation to lorentz13. This is possible because the inverse of a lorentz matrix has an inverse given by negating the transpose. We use this to derive what the original matrix should be if not the L we supposed above.

Finding and Reducing the $L_{3\to 1}$ matrix to first order

```
lorentz31 =
 FullSimplify[lorentzXYZ /. {γ1 → γ2, β1z → 0, β1x → -β2x, β1y → -β2y, β1 -> β2}]
```

$$\left\{\{\gamma 2,\ \beta 2x\,\gamma 2,\ \beta 2y\,\gamma 2,\ 0\},\ \left\{\beta 2x\,\gamma 2,\ 1+\frac{\beta 2x^2\,(-1+\gamma 2)}{\beta 2^2},\ \frac{\beta 2x\,\beta 2y\,(-1+\gamma 2)}{\beta 2^2},\ 0\right\},\ \left\{\beta 2y\,\gamma 2,\ \frac{\beta 2x\,\beta 2y\,(-1+\gamma 2)}{\beta 2^2},\ 1+\frac{\beta 2y^2\,(-1+\gamma 2)}{\beta 2^2},\ 0\right\},\ \{0,0,0,1\}\right\}$$

```
lorentz31 // MatrixForm
```

$$\begin{pmatrix} \gamma 2 & \beta 2x\,\gamma 2 & \beta 2y\,\gamma 2 & 0 \\ \beta 2x\,\gamma 2 & 1+\frac{\beta 2x^2\,(-1+\gamma 2)}{\beta 2^2} & \frac{\beta 2x\,\beta 2y\,(-1+\gamma 2)}{\beta 2^2} & 0 \\ \beta 2y\,\gamma 2 & \frac{\beta 2x\,\beta 2y\,(-1+\gamma 2)}{\beta 2^2} & 1+\frac{\beta 2y^2\,(-1+\gamma 2)}{\beta 2^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But alas this is NOT what Goldstein et al. give for their approximation at this stage. So we have to do an additional manipulation. This time we alter the two diagonal terms that approximate one. Doing that leaves the two boost off diagonal term.

```
lorentz31[[3, 3]] = 1;
```

```
lorentz31[[2, 2]] = 1;
```

```
lorentz31A = FullSimplify[lorentz31 /. β2 → β2x]
```

$$\left\{\{\gamma 2,\ \beta 2x\,\gamma 2,\ \beta 2y\,\gamma 2,\ 0\},\ \left\{\beta 2x\,\gamma 2,\ 1,\ \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x},\ 0\right\},\ \left\{\beta 2y\,\gamma 2,\ \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x},\ 1,\ 0\right\},\ \{0,\ 0,\ 0,\ 1\}\right\}$$

This result is finally in the Goldstein et al. form at this stage

```
lorentz31A // MatrixForm
```

$$\begin{pmatrix} \gamma 2 & \beta 2x\,\gamma 2 & \beta 2y\,\gamma 2 & 0 \\ \beta 2x\,\gamma 2 & 1 & \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x} & 0 \\ \beta 2y\,\gamma 2 & \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Identifying the Source of the Asymmetry in Succesive Non-parallel Boosts

Although the matrices have been simplified somewhat the product of them is far from simple once again. But in anticipation of what is to come we set the variable name of this product to rotation.

```
rotation = FullSimplify[lorentzAP2.lorentz31A]
```

$$\left\{\left\{-\left(-1+\beta 2x^2+\beta 2y^2\right)\gamma 2^2,\ \frac{(\beta 2x-\beta 2y)\,(\beta 2x+\beta 2y)\,(-1+\gamma 2)\,\gamma 2}{\beta 2x},\ 0,\ 0\right\},\ \left\{0,\ \gamma 2-\beta 2x^2\,\gamma 2^2,\ \frac{\beta 2y\,\gamma 2\,\left(-1+\gamma 2-\beta 2x^2\,\gamma 2\right)}{\beta 2x},\ 0\right\},\ \left\{\beta 2y\,\gamma 2\,\left(1+\left(-1+\beta 2x^2\right)\gamma 2\right),\ -\frac{\beta 2y\,(-1+\gamma 2)\,\left(-1+\beta 2x^2\,\gamma 2\right)}{\beta 2x},\ 1-\beta 2y^2\,\gamma 2,\ 0\right\},\ \{0,\ 0,\ 0,\ 1\}\right\}$$

```
rotation // MatrixForm
```

$$\begin{pmatrix} -\left(-1+\beta 2x^2+\beta 2y^2\right)\gamma 2^2 & \frac{(\beta 2x-\beta 2y)\,(\beta 2x+\beta 2y)\,(-1+\gamma 2)\,\gamma 2}{\beta 2x} & 0 & 0 \\ 0 & \gamma 2-\beta 2x^2\,\gamma 2^2 & \frac{\beta 2y\,\gamma 2\,\left(-1+\gamma 2-\beta 2x^2\,\gamma 2\right)}{\beta 2x} & 0 \\ \beta 2y\,\gamma 2\,\left(1+\left(-1+\beta 2x^2\right)\gamma 2\right) & -\frac{\beta 2y\,(-1+\gamma 2)\,\left(-1+\beta 2x^2\,\gamma 2\right)}{\beta 2x} & 1-\beta 2y^2\,\gamma 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Some books describing the Lorentz matrices suggest that the 3 x 3 matrix representing the space coordiantes might take on the form of a rotation matrix and that may be the source of the asymmetry. Examining the above matrix we see that one off diagonal term (2, 3) is more complicated than the other (3, 2). So we expand the more complicated one.

$$\mathbf{Expand}\left[\frac{\beta 2y\,\gamma 2\,\left(-1+\gamma 2-\beta 2x^2\,\gamma 2\right)}{\beta 2x}\right]$$

$$-\frac{\beta 2y\,\gamma 2}{\beta 2x}+\frac{\beta 2y\,\gamma 2^2}{\beta 2x}-\beta 2x\,\beta 2y\,\gamma 2^2$$

Using this expression edited to first order we hand edit this matrix to produce the final "rotation" matrix

```
rotationA =
```

$$\left\{\{1, 0, 0, 0\}, \left\{0, 1, -\frac{\beta 2y\,\gamma 2}{\beta 2x} + \frac{\beta 2y\,\gamma 2^2}{\beta 2x}, 0\right\}, \left\{0, \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x}, 1, 0\right\}, \{0, 0, 0, 1\}\right\}$$

$$\left\{\{1, 0, 0, 0\}, \left\{0, 1, -\frac{\beta 2y\,\gamma 2}{\beta 2x} + \frac{\beta 2y\,\gamma 2^2}{\beta 2x}, 0\right\}, \left\{0, \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x}, 1, 0\right\}, \{0, 0, 0, 1\}\right\}$$

```
rotationA // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\beta 2y\,\gamma 2}{\beta 2x} + \frac{\beta 2y\,\gamma 2^2}{\beta 2x} & 0 \\ 0 & \frac{\beta 2y\,(-1+\gamma 2)}{\beta 2x} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The final approximation is to make $\gamma 2^2$=1 and since β = - β2x we finally arrive at the Goldstein et al. result

```
rotationB = FullSimplify[rotationA /. {γ2^2 → 1, β2x → -β, γ2 → γ}];
```

```
rotationB // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{\beta 2y\,(-1+\gamma)}{\beta} & 0 \\ 0 & \frac{\beta 2y - \beta 2y\,\gamma}{\beta} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix indeed now looks like a rotation matrix around the z-axis where the cosines are 1 and the sines are the off diagonal terms.

Summary

This effect is called Thomas precession and arises when combining Lorentz transformations that are not collinear. Many modern relativity books do not even have this in their indices. In its simplest form, it is often defined (Goldstein et al. for example who make the β substitution their final matrix to have the signs of a right hand system) as

$$\Delta\Omega = (\gamma 2 - 1)\,\frac{\beta 2y}{\beta}$$

$$\frac{\beta 2y\,(-1+\gamma 2)}{\beta}$$

How does this arise ? We suppose that S1 is a laboratory system at rest while S2 and S3 are two instantaneous rest systems a time Δt apart in a particle's motion. The velocity of the particle defines the x-axis v=β2x c and the Δv has only the component Δv=β2y c. Thus in velocity terms

$$\Delta\Omega 1 = (1 - \gamma)\ \frac{\mathbf{v} \times \Delta\mathbf{v}}{v^2}$$

$$\frac{(1 - \gamma)\ \mathbf{v} \times \Delta\mathbf{v}}{v^2}$$

This shows that if v and Δv are aligned there is no rotation. If we have a particle in motion on a close path (i.e circular for example) then the angular motion caused by dΩ/dt is ω. Using the series expansion of $\gamma 2 \sim 1 + \beta 2^2 / 2$ we find that in vector notation that if a is the acceleration in S1, v the velocity in S1, then some vector propery of the particle such as spin will appear to precess with frequency ω.

$$\omega = \frac{1}{2\, c^2}\ (\mathbf{a} \times \mathbf{v})$$

$$\frac{\mathbf{a} \times \mathbf{v}}{2\, c^2}$$

Our suspicion that Thomas precession is counterintutive is shown by the fact that classical mechanics would have ω defined as (v x a) so that the right hand rule is obeyed. It appears everytime there is an acceleration of a frame that is not collinear with the velocity.