Solutions to the Tutorial Problems in
the book "Magnetohydrodynamics of the Sun"
by ER Priest (2014)
CHAPTER 12

PROBLEM 12.1. Acceleration of an Isolated Horizontal Flux Rope.

Consider the equation of motion of an isolated line-tied flux rope of radius a with a purely poloidal field B_p at its surface. Show that the flux rope is accelerated either indefinitely or to a constant speed, depending on whether the current, radius or twist is held constant.

SOLUTION.
Model a flux rope at height h above the photosphere $y = 0$ as a line current of strength $2\pi I/\mu$, for which $B_y + iB_x = I/(Z - ih)$, where $Z = x + iy$ is the complex variable (Fig.12.5a). Photospheric line tying is modelled by adding an image flux rope with current $-2\pi I/\mu$ at a distance h below the photosphere to give a net resulting field

$$B_y + iB_x = \frac{I}{Z - ih} - \frac{I}{Z + ih} = \frac{2ih}{Z^2 + h^2}.$$

We assume this holds to within a distance a of the singularities at $z = \pm h$, namely the surface of the flux rope and its image. Inside the flux rope assume a purely azimuthal field

$$B_i = B_p \frac{r}{a},$$

where $B_p = I/a$ is the field at the surface of the flux rope.

The vertical equation of motion is then

$$M\frac{d^2h}{dt^2} = \frac{\pi I^2}{\mu h}.$$

If the flux rope starts from rest at $h = h_0$, say, this may be integrated to give

$$\tfrac{1}{2}Mv^2 = \int_{h_0}^{h} \frac{\pi I^2}{\mu h} dh, \qquad (1)$$

which determines $h(t)$ as a function of $I(t)$ once an extra assumption about the behaviour of I has been made.

On possibility is to assume $I =$ constant, for which Eq.(**??**) gives

$$v^2 = \frac{2\pi I^2}{\mu M} \log \frac{h}{h_0},$$

so that v increases indefinitely with h. This is at first a little surprising but arises because the magnetic energy of the system is increasing in time as the sources at infinity do work in an unrealistic way.

Another possibility is to assume that the flux tube radius (a) remains constant. The magnetic flux (ψ_0) crossing the y-axis below the flux rope is

$$\psi_0 = \int_0^{h-a} B_x(0, y)dy = I \log \left(\frac{2h}{a} - 1 \right),$$

and so, if this and a are held fixed during the eruption, it determines $I(h)$ as

$$I = \frac{\psi_0}{\log(2h/a - 1)} = I_0 \frac{\log(2h_0/a - 1)}{\log(2h/a - 1)}.$$

For $a \ll h_0$ the rise speed therefore increases with height like

$$v^2 = \frac{\pi \psi_0^2}{\mu M} \left(\frac{1}{\log(2h_0/a - 1)} - \frac{1}{\log(2h/a - 1)} \right),$$

In particular, at large heights it approaches a constant value of

$$v_\infty^2 = \frac{\pi \psi_0^2}{\mu M} \left(\frac{1}{\log(2h_0/a - 1)} \right) = \frac{\pi I_0^2}{\mu M} \log \left(\frac{2h_0}{a} - 1 \right).$$

In a similar way, it can be shown that, if the twist in the flux rope is held constant or if the prominence is modelled as a vertical current sheet rather than a line current, then the velocity at large heights is also constant (Priest and Forbes,1990).

PROBLEM 12.2. Instability of Horizontal Flux Rope.
Consider equilibria of a line-tied flux rope in a dipole background field. Prove that solutions on the lower branch of equilibria in Fig.12.5b are stable while those on the upper branch are unstable.

SOLUTION.
The equation of vertical motion for a line-tied horizontal flux rope treated as

a line current I of mass M at height h in the background field of a dipole of moment m at a depth h_b below the photosphere is given by Eq.12.1, namely,

$$M\frac{d^2h}{dt^2} = \frac{2\pi I}{\mu}\left(\frac{I}{2h} - \frac{m}{(h+h_b)^2}\right),$$

which may be rewritten

$$M\frac{d^2h}{dt^2} = \frac{\pi I^2}{\mu h_b}\left(\frac{h_b}{h} - \frac{2m/(Ih_b)}{(1+h/h_b)^2}\right).$$

This may be nondimensionalised by writing h in terms of h_b and t in terms of $[(\pi I^2)/(\mu h_b)]^{1/2}$ to give

$$\frac{d^2h}{dt^2} = \left(\frac{1}{h} - \frac{4c}{(1+h)^2}\right), \tag{2}$$

where $c = \frac{1}{2}m/(Ih_b)$.

The equilibria $h = h_0$, say, are given by setting the right-hand side equal to zero so that

$$(1+h_0)^2 - 4ch_0 = 0, \tag{3}$$

with solutions

$$h_0 = 2c - 1 \pm \sqrt{(2c-1)^2 - 1}.$$

Thus, we see that $h_0 = 1$ when $c = 1$ and there are two solutions when $c > 1$, one of them larger than 1 and the other smaller. When $c < 1$ there are no real solutions, in agreement with Fig.12.5b.

Now, in order to determine the stability of these two solutions, consider perturbations to the equilibria by writing

$$h = h_0(1+h_1),$$

where $h_1 \ll 1$. Then Eq.(**??**) becomes

$$h_0\frac{d^2h_1}{dt^2} = \frac{1}{h_0(1+h_1)} - \frac{4c}{(1+h_0+h_0h_1)^2},$$

or, after using Taylor's theorem to linearise the right-hand side,

$$h_0\frac{d^2h_1}{dt^2} = \frac{1}{h_0}(1-h_1) - \frac{4c}{(1+h_0)^2}\left(1 - \frac{2h_0h_1}{1+h_0}\right).$$

After using Eq.(**??**) to substitute for $4c$, this becomes

$$\frac{d^2h_1}{dt^2} = \frac{h_0 - 1}{{h_0}^2(1 + h_0)},$$

which has sinusoidal (i.e., stable) solutions when $h_0 < 1$ and exponentially growing (i.e., unstable) solutions when $h_0 > 1$, as required.

PROBLEM 12.3. Emergence of Magnetic Flux.
Consider a flux rope modelled as a line current (I) originally at location $(h, 0)$ in the magnetic field due to a line dipole at $(-d, 0)$ below the photosphere. Suppose new flux emerges in the form of a line dipole at $(-x_d, y_d)$. Solve Poisson's equation to find the flux function for the resulting equilibrium, following Lin, Forbes and Isenberg (2001) JGR 106, 25053.

SOLUTION.
Suppose the magnetic field is

$$(B_x, B_y) = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}\right)$$

and that the magnetic field possesses current sources $j_z(x, y)$ such that

$$\nabla^2 A = -\mu j_z(x, y). \tag{4}$$

This is to be solved subject to the boundary condition

$$A(x, 0) = \frac{md}{x^2 + d^2} + \frac{sy_d}{(x - x_d)^2 + y_d^2},$$

produced by one line dipole of strength m at (0,-d) below the photosphere and another of strength s at $(x_d, -y_d)$.

The line current at (x_h, y_h), say, corresponds to a current density

$$j_z(x, y) = I(h)\delta(x - x_h)\delta(y - y_h).$$

Now normalise distances with respect to d and replace the parameters m, s and I by normalised parameters $M = mc/(4I_0d)$, $S = sc/(4I_0d)$ and $J = I/I_0$, say. Then the expressions for $A(x, 0)$ and $j_z(x, y)$ become

$$A(x, 0) = \frac{4I_0}{c}\left[\frac{M}{x^2 + 1} + \frac{Sy_d}{(x - x_d)^2 + y_d^2}\right],$$

$$j_z(x,y) = \frac{JI_0}{d^2}\delta(x-x_h)\delta(y-y_h).$$

The general solution for the Dirichlet problem given by Eq.(**??**) together with the above two equations is

$$A(x,y) = \frac{1}{c}\int_{-\infty}^{+\infty}\int_0^{+\infty} G(x,y;u,v)j_z(u,v)dv\,du + \frac{1}{4\pi}\int_{-\infty}^{+\infty} A(u,0)\left[\frac{\partial G}{\partial v}\right]_{v=0} du,$$

where $G(x,y;u,v)$ is the 2D Green's function, namely,

$$G(x,y;u,v) = \log_e\left[\frac{(x-u)^2+(y+v)^2}{(x-u)^2+(y-v)^2}\right],$$

which satisfies $G(x,0;u,v) = G(x,y;u,0) = 0$.

After substituting for $A(x,0)$, $j_z(x,y)$ and $G(x,y;u,v)$, we find

$$A(x,y) = \frac{2I_0}{c}\mathcal{R}e\left[J\log_e\frac{z-z_h^*}{z-z_h} + \frac{2iM}{z+i} + \frac{2iS}{z-x_d+iy_d}\right],$$

where $\mathcal{R}e$ is the real part, $Z = x+iy$ and $Z_h = x_h + iy_h$, as required. Lin et al (2001) proceed to calculate the equilibrium locations of the current and the evolution of the system to a nonequilibrium point.

PROBLEM 12.4. Current Sheet below an Erupting Flux Rope.

(i) Find the magnetic field due to a flux rope modelled as a line current (I) at height h sitting in the corona in the magnetic field of a line dipole at depth d below the photosphere.

(ii) Suppose the flux rope erupts without reconnection and produces a current sheet stretching up from the photosphere to height q. Find the resulting magnetic field.

SOLUTION.
(i) In a similar way to PROBLEM 12.3, we follow Forbes and Isenberg (1991) Ap. J. 373, 294, and suppose the magnetic field is

$$(B_x, B_y) = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}\right)$$

with current sources $j_z(x,y)$ such that

$$\nabla^2 A = -\mu j_z(x,y). \tag{5}$$

This is to be solved subject to the boundary condition

$$A(x,0) = \frac{md}{x^2 + d^2},$$

produced by a line dipole of strength m at (0,-d) below the photosphere. The line current at $(0, h)$, say, corresponds to a current density

$$j_z(x,y) = I\delta(x)\delta(y-h).$$

The method of images gives the following simple solution to (**??**):

$$A(x,y) = \frac{\mu I}{2\pi}\mathcal{R}e\left[J\log_e\left(\frac{Z+ih}{Z-ih}\right) + \frac{2i}{Z+i}\right],$$

where $Z = x + iy$, $I_0 = m/(2d)$, $J = I/I_0$ and distances are normalised with respect to the length-scale d. The field is produced by a line current at $Z = ih$, an image current at $Z = -ih$ and a line dipole at $Z = -i$.

From this we find

$$B_x(0,y) = \frac{\mu I_0}{2\pi d}\left[\frac{J}{y+h} - \frac{J}{y-h} - \frac{2}{(y+1)^2}\right],$$

so that an X-line first appears at the origin when $h = 1,\ J = 1$.

(ii) Subsequently, suppose a current sheet stretches from the origin up to $(0, q)$. First of all, we use a conformal transformation

$$w = \sqrt{Z^2 + q^2},$$

from the Z-plane to the w-plane, where $w = u + iv$ and the current sheet is transformed to a line segment stretching from $(-q, 0)$ to $(q, 0)$ on the u-axis.

In the uv-plane, therefore, we have to solve

$$\nabla^2 A = -\mu j(u,v). \tag{6}$$

with

$$j(u,v) = (I_0/d^2)\delta(u)\delta(v - \sqrt{h^2 - q^2}),$$

subject to the boundary condition

$$A(u,0) = \frac{\mu I_0}{\pi}\begin{cases}1, & u^2 \le q^2,\\ (u^2 - q^2 + 1)^{-1}, & u^2 > q^2,\end{cases}$$

The general solution to (**??**) can be found by the method of Green's functions in the same way to PROBLEM 12.3 as

$$A(u,v) = \frac{1}{c}\int_{-\infty}^{+\infty}\int_{0}^{+\infty} G(u,v;u',v')j(u',v')du'\,dv' + \frac{1}{4\pi}\int_{-\infty}^{+\infty} A(u',0)\left[\frac{\partial G}{\partial v'}\right]_{v'=0} du',$$

where $G(u,v;u',v')$ is the 2D Green's function, namely,

$$G(u,v,u',v') = \log_e\left[\frac{(u-u')^2+(v+v')^2}{(u-u')^2+(v-v')^2}\right].$$

Substitution into this expression gives, after evaluating the integrals,

$$A(x,y) = \frac{\mu I_0}{2\pi}\mathcal{R}e\left[J\log_e\left(\frac{\sqrt{Z^2+q^2}+i\sqrt{h^2-q^2}}{\sqrt{Z^2+q^2}-i\sqrt{h^2-q^2}}\right)\right] +$$
$$+\left[\frac{2iZ^2}{\pi(Z^2+1)}\log_e\left(\frac{\sqrt{Z^2+q^2}+q}{\sqrt{Z^2+q^2}-q}\right)+\frac{2}{Z^2+1}\right]+$$
$$+\left[\frac{2i}{\pi\sqrt{q^2-1}}\frac{\sqrt{Z^2+q^2}}{Z^2+1}\log_e\left(\frac{q+\sqrt{q^2-1}}{q-\sqrt{q^2-1}}\right)\right].$$

Forbes and Isenberg (1991) proceed to determine q and J as functions of the reconnected flux and then determine the equilibrium locations of the flux rope, as well as the energetics and stability.

PROBLEM 12.5. The Hoop Force of a Toroidal Flux Rope.
Fill in the details of the proof for calculating the hoop force of a toroidal flux rope by:

(a) calculating A_ϕ for a toroidal current [Eq.(**??**) in the solution],

(b) approximating this close to the flux rope [Eq.(**??**)],

(c) showing that the poloidal flux function $\tilde{A} = \tilde{A}_0(r) + \tilde{A}_1(r,\theta)$ with $\tilde{A}_1(r,\theta) = -\Delta(r)B_{\theta 0}(r)\cos\theta \ll \tilde{A}_0$ represents a set of circular flux surfaces that are displaced by Δ,

(d) finding the field on the inner surface of the flux rope [Eq.(**??**)],

(e) finding the flux function outside the flux rope [Eq.(**??**)] and

(f) determining the free constants [Eq.(**??**)] by matching the field at the surface of the rope.

SOLUTION.

The discussion in Solar MHD is repeated here, with the proofs for parts (a)–(f) inserted and indicated. We consider the magnetic field of a simple isolated toroidal flux rope of major radius R_0, minor radius a and net toroidal current I. It is not in equilibrium but experiences a radially outwards *hoop force* of magnitude

$$F_{hoop} = \frac{\mu I^2}{4\pi R_0}\left(\log_e \frac{8R_0}{a} - \frac{3}{2} + \beta_p + \frac{l_i}{2}\right) \tag{7}$$

when $a \ll R_0$ (Shafranov, 1966). Here the constants β_p and l_i depend on the internal structure of the flux tube. They are the mean p and B_θ^2 over the volume of the flux rope and are given by

$$\beta_p = \frac{4\mu}{a^2 B_0^2}\int_0^a p\; r dr, \qquad l_i = \frac{\langle B_\theta \rangle^2}{B_0^2} = \frac{1}{\pi a^2 B_0^2}\int B_\theta^2\; dV = \frac{2}{a^2 B_0^2}\int_0^a B_\theta^2\; r dr, \tag{8}$$

where $B_0 = B_\theta(a)$ is the (zeroth-order) field at the flux rope surface ($r = a$) and local polar coordinates (r, θ) have been taken with respect to a point T on the major axis of the flux rope.

In order to prove Eq.(**??**), we consider two different coordinate systems, with the location of any point in a vertical plane given in terms of either (R, z) or (r, θ), where $R = R_0 + r\cos\theta,\ z = r\sin\theta$. Suppose the magnetic field is an axisymmetric magnetostatic equilibrium (independent of ϕ) with components that are written in terms of a poloidal flux function ($\tilde{A} = RA_\phi$) as

$$(B_R, B_\phi, B_z) = \frac{1}{R}\left(-\frac{\partial \tilde{A}}{\partial z}, b_\phi(\tilde{A}), \frac{\partial \tilde{A}}{\partial R}\right),$$

which satisfies $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ automatically, and for which $\tilde{A}$ is determined by the Grad-Shafranov equation (see Eqs. (3.61) and (3.62) in the book), namely,

$$\frac{\partial^2 \tilde{A}}{\partial R^2} - \frac{1}{R}\frac{\partial \tilde{A}}{\partial R} + \frac{\partial^2 \tilde{A}}{\partial z^2} = -\mu R^2 \frac{dp}{d\tilde{A}} - \frac{d}{d\tilde{A}}(\tfrac{1}{2}b_\phi^2).$$

In terms of coordinates (r, θ) these become

$$(B_r, B_\phi, B_\theta) = \frac{1}{R_0 + r\cos\theta}\left(-\frac{\partial \tilde{A}}{r\partial \theta}, b_\phi(\tilde{A}), \frac{\partial \tilde{A}}{\partial r}\right),$$

and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \tilde{A}}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \tilde{A}}{\partial \theta^2}\right)-\frac{1}{R_0+r\cos\theta}\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\tilde{A}$$
$$=-\mu(R_0+r\cos\theta)^2\frac{dp}{d\tilde{A}}-\frac{d}{d\tilde{A}}\left(\frac{b_\phi^2}{2}\right). \quad (9)$$

The proof of Eq.(**??**) involves two parts, as follows. Just as for a 2D line current, we evaluate the force on the toroidal flux rope as the product of its current (I) and the external vertical field (B_v) that exists at the location of the flux rope. The first part is to find the flux function [Eq.(**??**) below] of the field due to a toroidal current alone at small distance r from it. The second part is to find B_v by calculating the flux function [Eq.(**??**)] of the field just outside a toroidal flux rope in equilibrium and decomposing it into the fields of the toroidal current itself and of the vertical field needed to balance the hoop force.

Part 1 of Solution: FIELD of TOROIDAL CURRENT

If the torus is treated as a ring of current of radius R_0, then the vector potential $[A_\phi(R,\phi)\hat{\boldsymbol{\phi}}]$ at a point (R,ϕ) in polar coordinates in the plane of the ring relative to the ring's centre may be calculated as follows. In general, the vector potential ($\mathbf{A}$) is such that $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ and satisfies Poisson's equation $\nabla^2\mathbf{A}=-\mu\mathbf{j}$, which has solution $\mathbf{A}=[\mu/(4\pi)]\int\mathbf{j}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|dV'$. For our ring $\mathbf{j}\,dV'=IR_0\,d\phi'$, and at any point in its plane the only magnetic component is $B_z(R)$. Thus, the only component of $\mathbf{A}$ is $A_\phi(R)$ and it becomes

$$A_\phi=\frac{\mu I}{4\pi}\int\frac{ds'}{S}=\frac{\mu I R_0}{2\pi}\int_0^\pi\frac{\cos\phi'\,d\phi'}{(R^2+R_0^2-2R_0R\cos\phi')^{1/2}}, \quad (10)$$

where $S=[(R-R_0\cos\phi')^2+R_0^2\sin^2\phi']^{1/2}$ is the distance between a points $(R,0)$ on the x-axis and $(R_0\cos\phi',R_0\sin\phi')$ on the ring.

A_ϕ may be rewritten in terms of complete elliptic integrals of the 1st and 2nd kind with $k=2(R/R_0)^{1/2}/(1+R/R_0)$ as

$$A_\phi=\frac{\mu I}{4\pi}\frac{R_0+R}{R}[(2-k^2)K(k)-2E(k)], \quad (11)$$

where $K(k)=\int_0^{\pi/2}[1-k^2\sin^2x]^{-1/2}dx$ and $E(k)=\int_0^{\pi/2}[1-k^2\sin^2x]^{1/2}dx$.

PROOF (a) that Eqs.(??) and (??) are equivalent:
Let us work backwards. With the above definition of k, we have

$$2-k^2=\frac{2(R_0^2+R^2)}{R_0^2+R^2+2R_0R\sin\theta}$$

and

$$1-k^2\sin^2 x=\frac{R_0^2+R^2+2R_0R\sin\theta(1-2k^2\sin^2 x)}{R_0^2+R^2+2R_0R\sin\theta}$$

Then, with the above definitions of $K(k)$ and $E(k)$, we find

$$\begin{aligned}\frac{(2-k^2)K(k)-2E(k)}{k^2}&=\frac{1}{k^2}\int_0^{\pi/2}\frac{2-k^2}{(1-k^2\sin^2 x)^{1/2}}-2(1-k^2\sin^2 x)^{1/2}\ dx\\&=\int_0^{\pi/2}\frac{2\sin^2 x-1}{(1-k^2\sin^2 x)^{1/2}}\ dx\end{aligned}$$

Thus, Eq.(**??**) becomes

$$A_\phi=\frac{\mu I}{4\pi}\frac{R_0+R}{R}[(2-k^2)K(k)-2E(k)]=\frac{\mu I}{4\pi}\frac{R_0+R}{R}k^2\int_0^{\pi/2}\frac{2\sin^2 x-1}{(1-k^2\sin^2 x)^{1/2}}\ dx$$

Now, change the variable of integration from x to $\alpha=\pi/2-x$ to give

$$A_\phi=\frac{\mu I}{4\pi}\frac{R_0+R}{R}k^2\int_0^{\pi/2}\frac{1-2\sin^2\alpha}{[1-k^2(1-\sin^2\alpha)]^{1/2}}\ d\alpha$$

or, using the definition of $k^2=4R_0R/(R_0+R)^2$,

$$A_\phi=\frac{\mu IR_0}{\pi}\int_0^{\pi/2}\frac{1-2\sin^2\alpha}{[(R_0+R)^2-4R_0R(1-\sin^2\alpha)]^{1/2}}\ d\alpha$$

or

$$A_\phi=\frac{\mu IR_0}{4\pi}\int_0^{\pi}\frac{2(1-2\sin^2\alpha)}{[R_0^2+R^2-2R_0R+4R_0R\sin^2\alpha]^{1/2}}\ d\alpha.$$

Finally, change the variable from α to $\phi'=2\alpha$ such that $1-2\sin^2\alpha=\cos\phi'$ to give Eq.(**??**), namely,

$$A_\phi=\frac{\mu IR_0}{2\pi}\int_0^{\pi}\frac{\cos\phi'}{(R_0^2+R^2-2R_0R\cos\phi')^{1/2}}\ d\phi',$$

as required.

END OF PROOF (a)

By expanding in powers of r/R_0 it can be shown that near the inside of the flux rope this gives a flux function ($\tilde{A} = RA_\phi$) of approximately

$$\tilde{A} = \frac{\mu I}{4\pi}\left\{2R_0\left(\log_e \frac{8R_0}{r} - 2\right) - r\left(\log_e \frac{8R_0}{r} - 1\right)\right\}. \qquad (12)$$

PROOF (b) that Eq.(??) implies Eq.(??):
Using the relation $\tilde{A} = RA_\phi$, Eq.(**??**) may be written

$$\tilde{A} = \frac{\mu I}{4\pi} R_0 \left(1 + \frac{R}{R_0}\right)[(2-k^2)K(k) - 2E(k)],$$

Write $R = R_0 + r$, where $r \ll R_0$ and expand in powers of r/R_0 to give

$$k = \frac{2(1+r/R_0)^{1/2}}{2[1+r/(2R_0)]} = 1 - \frac{r^2}{8R_0^2} - \frac{r^3}{8R_0^3} + ...,$$

$$k^2 = 1 - \frac{r^2}{4R_0^2} - \frac{r^3}{4R_0^3} + ...,$$

$$1 - k^2 = \frac{r^2}{4R_0^2}\left(1 + \frac{r}{R_0} + ...\right),$$

$$(1-k^2)^{1/2} = \frac{r}{2R_0}\left(1 + \frac{r}{2R_0} + ...\right),$$

$$\frac{4}{(1-k^2)^{1/2}} = \frac{8R_0}{r}\left(1 - \frac{r}{2R_0} + ...\right),$$

and

$$\log_e \frac{4}{(1-k^2)^{1/2}} = \log_e \frac{8R_0}{r} - \frac{r}{2R_0} + ...$$

Now, according to the books on special functions by Spanier and Oldham (1987) page 612 or Gradshteyn and Ryzik (1980) page 905, for $r \ll R_0$ (i.e., when $k \approx 1$), the elliptic integrals behave like

$$K(k) \approx \log_e \frac{4}{(1-k^2)^{1/2}} + \left(\log_e \frac{4}{(1-k^2)^{1/2}} - 1\right)\left(\frac{1-k^2}{4}\right) + ...,$$

and

$$E(k) \approx 1 + \left(\log_e \frac{4}{(1-k^2)^{1/2}} - \frac{1}{2}\right)\left(\frac{1-k^2}{2}\right) + ...$$

and so, using the above expansions for $\log_e[4/(1-k^2)^{1/2}]$ and $1-k^2$, we find

$$K(k) \approx \log \frac{8R_0}{r} - \frac{r}{2R_0} + ...$$

and

$$E(k) \approx 1 + ...,$$

where the next terms are of order $r^2 \log_e r$.

But Eq.(**??**) is

$$A_\phi = \frac{\mu I}{4\pi}\frac{R_0+R}{R}[(2-k^2)K(k) - 2E(k)],$$

which with $\tilde{A} = RA_\phi$, $R = R_0 + r$ and the above expansions therefore approximates to

$$\tilde{A} = \frac{\mu I}{4\pi} 2R_0 \left(1 - \frac{r}{2R_0} + ...\right)\left[\left(1 + \frac{r^2}{4R_0^2} + ...\right)\left(\log_e \frac{8R_0}{r} - \frac{r}{2R_0} + ...\right) - 2 + ...\right],$$

or

$$\tilde{A} = \frac{\mu I}{2\pi} R_0 \left(\log_e \frac{8R_0}{r} - 2\right) + \frac{\mu I}{4\pi} r \left(-\log_e \frac{8R_0}{r} + 1\right) ...,$$

which is the same as Eq.(**??**), as required.

END OF PROOF (b)

Solution Part 2: FIELD OUTSIDE a TOROIDAL FLUX ROPE

The field both inside and outside the flux rope satisifies Eq.(**??**), which we expand in powers of the inverse aspect ratio ($\epsilon = a/R_0 \ll 1$), assuming $B_\phi = B_{\phi 0}R_0/R[1+O(\epsilon^2)]$ and $B_\theta \sim \epsilon B_{\phi 0}$, where $B_{\phi 0}$ is the potential toroidal field at major radius R_0. If we write $\tilde{A} = \tilde{A}_0(r) + \tilde{A}_1(r,\theta)$, the zeroth and first order contributions from Eq.(**??**) are

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\tilde{A}_0}{dr}\right) = -\mu R_0^2 \frac{dp}{d\tilde{A}_0} - b_\phi(\tilde{A}_0)\frac{db_\phi}{d\tilde{A}_0}, \qquad (13)$$

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \tilde{A}_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \tilde{A}_1}{\partial \theta^2}-\frac{\cos\theta}{R_0}\frac{d\tilde{A}_0}{dr} \\
&\qquad = -\frac{d}{dr}\left(\mu R_0^2\frac{dp}{d\tilde{A}_0}+b_\phi(\tilde{A}_0)\frac{db_\phi}{d\tilde{A}_0}\right)\frac{dr}{d\tilde{A}_0}\tilde{A}_1-2\mu R_0 r\cos\theta\frac{dp}{d\tilde{A}_0}. \quad (14)
\end{aligned}
$$

The next step is to seek a separable solution of Eq.(**??**) in the form

$$
\tilde{A}_1(r,\theta)=-\Delta(r)R_0B_{\theta 0}(r)\cos\theta, \qquad (15)
$$

where $B_{\theta 0}(r)=(1/R_0)d\tilde{A}_0/dr$. The resulting flux surfaces have cross-sections that are circles whose axes are displaced by a distance Δ (the *Shafranov shift*).

PROOF (c) that Flux Surfaces of Eq.(??) have Cross-Sections that are Displaced Circles:

A simple solution is to note that

$$
\tilde{A}=\tilde{A}_0(r)-\Delta(r)\frac{d\tilde{A}_0}{dr}\cos\theta
$$

may be rewritten

$$
\tilde{A}=\tilde{A}_0(r)-\Delta(r)\frac{\partial \tilde{A}_0}{\partial R}=\tilde{A}_0-\delta R\frac{\partial \tilde{A}_0}{\partial R}=\tilde{A}_0-\delta\tilde{A}_0.
$$

But $\tilde{A}_0 = \textit{constant}$ represents flux surfaces with circular cross-sections ($r =$ constant) and so $\tilde{A}_0-\delta A_0=\textit{constant}$ with $\delta\tilde{A}_0=\delta R\ \partial\tilde{A}_0/\partial R$ represents circular flux surfaces displaced by a distance δR.

A more detailed solution is to note that the equation of a circle whose axis is displaced a small distance Δ is

$$
(r\cos\theta-\Delta)^2+r^2\sin^2\theta=c^2
$$

or, if $\Delta \ll r$,

$$
r^2-2\Delta r\cos\theta=c^2.
$$

If we write $r=c+r_1$ where $r_1\ll c$, then after linearising this equation we find

$$
r_1=\Delta\cos\theta,
$$

so that $r=c+\Delta\cos\theta$ is the linearised equation for a circle with axis displaced by Δ.

Now what surfaces does

$$\tilde{A}_0(r) - \Delta(r)\frac{d\tilde{A}_0}{dr}\cos\theta = constant$$

represent? Linearise about $r = c$ and substitute

$$\tilde{A}_0(c + r_1) \approx \tilde{A}_{00}(c) + r_1 \left(\frac{d\tilde{A}_{00}}{dr}\right)_{r=c}$$

into the above equation to give

$$\tilde{A}_{00}(c) + r_1 \left(\frac{d\tilde{A}_{00}}{dr}\right)_{r=c} - \Delta\cos\theta \left(\frac{d\tilde{A}_{00}}{dr}\right)_{r=c} = \tilde{A}_{00}(c)$$

or

$$r_1 = \Delta\cos\theta,$$

as required.

END OF PROOF (c)

Inside the flux rope, Eq.(**??**) may be solved as follows to give the field at the inner surface of the flux rope to zeroth plus first order as

$$B_\theta(a) = B_0 \left[1 + \frac{a}{R_0}(\beta_p + \tfrac{1}{2}l_i - 1)\cos\theta\right], \tag{16}$$

where $B_0 = B_{\theta 0}(a)$ is the zeroth-order B_θ at $r = a$ and β_p and l_i are given by Eq.**??**.

PROOF (d) of Eq.(??):

For a solution of the form (**??**), namely, $\tilde{A}_1(r,\theta) = -\Delta(r)R_0 B_{\theta 0}(r)\cos\theta$, with $B_{\theta 0} = R_0^{-1}d\tilde{A}_0/dr$, Eq.(**??**) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}(\Delta B_{\theta 0})\right) + \frac{\Delta}{r^2}B_{\theta 0} - \frac{B_{\theta 0}}{R_0}$$
$$= \frac{d}{dr}\left(\mu R_0^2\frac{dp}{d\tilde{A}_0} + b_\phi(\tilde{A}_0)\frac{db_\phi}{d\tilde{A}_0}\right)\frac{\Delta}{R_0} + 2\mu r\cos\theta\frac{dp_0}{d\tilde{A}_0}.$$

Next, subsititute on the right-hand side for $\mu R_0^2(dp/d\tilde{A}_0) + b_\phi(\tilde{A}_0)(db_\phi/d\tilde{A}_0)$ from Eq.(**??**) and replace $dp/d\tilde{A}_0$ by $(dp/dr)(dr/d\tilde{A}_0) = (dp/dr)1/(R_0 B_{\theta 0})$ to give

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}(\Delta B_{\theta 0})\right) + \frac{\Delta}{r^2}B_{\theta 0} - \frac{B_{\theta 0}}{R_0}$$
$$= -\frac{d}{dr}\left(\frac{dB_{\theta 0}}{dr} + \frac{B_{\theta 0}}{r}\right)\Delta + \frac{2\mu r}{R_0}\frac{dp_0}{dr}\frac{1}{B_{\theta 0}}.$$

After multiplying through by $rB_{\theta 0}$ and rearranging terms, this becomes

$$\frac{d}{dr}\left(rB_{\theta 0}^2\frac{d\Delta}{dr}\right) = \frac{r}{R_0}\left(2\mu r\frac{dp_0}{dr} - B_{\theta 0}^2\right), \tag{17}$$

as the required differential equation for $\Delta(r)$ when $p_0(r)$ and $B_{\theta 0}(r)$ are known.

Integrating Eq.(**??**) gives

$$\frac{d\Delta}{dr} = \frac{2\mu}{rR_0B_{\theta 0}^2}\left[\int_0^r r^2\frac{dp_0}{dr} - \frac{rB_{\theta 0}^2}{2\mu}\right]dr$$

or, integrating the first term by parts,

$$\frac{d\Delta}{dr} = \frac{2\mu}{rR_0B_{\theta 0}^2}\left[r^2p_0 - \int_0^r\left(2p_0 + \frac{B_{\theta 0}^2}{2\mu}\right)rdr\right]$$

If we assume $p_0(a) = 0$, this may be evaluated at $r = a$ to give

$$\left(\frac{d\Delta}{dr}\right)_a = -\frac{a}{R_0}(\beta_p + \tfrac{1}{2}l_i), \tag{18}$$

where β_p and l_i are given by Eq.(**??**).

Now

$$B_\theta = \frac{1}{R}\frac{\partial\tilde{A}}{\partial r} = \frac{1}{R_0 + r\cos\theta}\frac{\partial\tilde{A}}{\partial r},$$

where, to zeroth plus first order,

$$\tilde{A}(r,\theta) = \tilde{A}_0 + \tilde{A}_1 = \tilde{A}_0 - \Delta(r)R_0B_{\theta 0}(r)\cos\theta,$$

so, if $\Delta(a) = 0$, the value of B_θ at $r = a$ to zeroth plus first order is

$$B_\theta(a) = B_{\theta 0}(a) \left[1 - \left(\frac{a}{R_0} + \left(\frac{d\Delta}{dr} \right)_a \right) \cos\theta \right].$$

Then, noting that $B_{\theta 0} = R_0^{-1} d\tilde{A}_0/dr$ and substituting $(d\Delta/dr)_a$ from Eq.(**??**) into this yields Eq.(**??**), as required.

END OF PROOF (d)

Outside the flux rope, the field is assumed potential and so the right-hand sides of Eqs.(**??**) and (**??**) vanish. The resulting flux function for $r \ll R_0$ for the field external to the flux rope is then to zeroth plus first order

$$\tilde{A} = \frac{\mu I}{4\pi} \left\{ \left[2R_0 \left(\log_e \frac{8R_0}{r} - 2 \right) + r \left(\log_e \frac{8R_0}{r} - 1 \right) \cos\theta \right] + r \left(\frac{c_1}{r^2} + c_2 \right) \cos\theta \right\}. \tag{19}$$

It may be proved as follows.

PROOF (e) of Eq.(??):

We now solve Eqs.(**??**) and (**??**) in the region outside the flux rope where the field is potential, so that those two equations for $\tilde{A}_0$ and $\tilde{A}_1$ reduce to

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{d\tilde{A}_0}{dr} \right) = 0, \tag{20}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \tilde{A}_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{A}_1}{\partial \theta^2} - \frac{\cos\theta}{R_0} \frac{d\tilde{A}_0}{dr} = 0. \tag{21}$$

The general solution of Eq.(**??**) is

$$\tilde{A}_0 = c \log_e r + k,$$

but this needs to be the same as the lowest order field of a toroidal current given by Eq.(**??**), namely,

$$\tilde{A}_0 = \frac{\mu I R_0}{2\pi} \left(\log_e \frac{8R_0}{r} - 2 \right). \tag{22}$$

In order to solve Eq.(**??**), we suppose $\tilde{A}_1$ has the form

$$\tilde{A}_1 = \frac{\mu I}{2\pi} F(r) \cos\theta, \tag{23}$$

so that Eq.(**??**) reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)-\frac{F}{r^2}=-\frac{1}{r},$$

This has general solution

$$F=\frac{c_1}{r}+\frac{C_2 r}{2}-\frac{1}{2}\log\frac{r}{R_0}.$$

in which we write $C_2 = c_2 - 1$, so that the solution (**??**) becomes

$$\tilde{A}_1=\frac{\mu I}{4\pi}\left[r\left(\log_e\frac{8R_0}{r}-1\right)+\frac{c_1}{r}+c_2 r\right]\cos\theta. \tag{24}$$

Writing the solution in this form has the advantage that the first part is simply the first-order part of the vacuum field outside a toroidal current in Eq.(**??**). Finally, summing Eqs.(**??**) and (**??**) gives us Eq.(**??**), as required, namely,

$$\tilde{A}=\frac{\mu I}{4\pi}\left\{\left[2R_0\left(\log_e\frac{8R_0}{r}-2\right)+r\left(\log_e\frac{8R_0}{r}-1\right)\cos\theta\right]+r\left(\frac{c_1}{r^2}+c_2\right)\cos\theta\right\}.$$

END OF PROOF (e)

Calculating B_r and B_θ from (**??**) and setting $B_r(a) = 0$ and $B_\theta(a)$ on the outer surface of the flux rope equal to the value (**??**) on the inner surface then determines the constants as

$$c_1 = a^2(l_i - 1)/2, \qquad c_2 = -\log_e(8R_0/a) + 3/2 - \beta_p - l_i/2, \tag{25}$$

as follows.

PROOF (f) of Eq.(??):

To zeroth plus first order, we have

$$B_\theta=\frac{1}{R_0+r\cos\theta}\left(\frac{d\tilde{A}_0}{dr}+\frac{\partial\tilde{A}_1}{\partial r}\right)\approx\frac{1}{R_0}\frac{d\tilde{A}_0}{dr}+\frac{1}{R_0}\frac{\partial\tilde{A}_1}{\partial r}-\frac{r}{R_0^2}\frac{d\tilde{A}_0}{dr}\cos\theta, \tag{26}$$

where at the edge of the flux rope we know that to zeroth plus first order

$$B_\theta(a)=B_0\left[1+\frac{a}{R_0}(\beta_p+\tfrac{1}{2}l_i-1)\cos\theta\right], \tag{27}$$

First of all, consider $B_r = -1/(Rr)\partial\tilde{A}/\partial\theta$, whose zeroth-order part vanishes and whose first-order part is

$$B_r = -\frac{1}{R_0 r}\frac{\partial \tilde{A}_1}{\partial\theta}.$$

Substituting for $\tilde{A}_1$ from Eq.(**??**) gives at first order

$$B_r = -\frac{\mu I}{4\pi R_0}\left\{\left[\left(\log_e\frac{8R_0}{r} - 1\right) + \left(\frac{c_1}{r^2} + c_2\right)\right]\sin\theta\right\}.$$

Adopting the boundary condition $B_r(a) = 0$ and so equating the above expression to zero at $r = a$ gives

$$\log_e\frac{8R_0}{a} - 1 + \frac{c_1}{a^2} + c_2 = 0 \tag{28}$$

as one relation between the unkown constants c_1 and c_2.

Next, consider $B_\theta(a)$. At zeroth order, substituting the expression for $\tilde{A}_0$ from Eq.(**??**) into Eq.(**??**) and equating it to the zeroth-order part of $B_\theta(a)$ from Eq.(**??**) gives

$$B_0 = -\frac{\mu I}{2\pi a}. \tag{29}$$

At first-order, substituting for $\tilde{A}_0$ and $\tilde{A}_1$ from Eq.(**??**) into Eq.(**??**) gives

$$\begin{aligned}
&\frac{1}{R_0}\frac{\partial\tilde{A}_1}{\partial r} - \frac{r}{R_0^2}\frac{d\tilde{A}_0}{dr}\cos\theta \\
&\qquad = \frac{\mu I}{4\pi R_0}\left[\left(\log_e\frac{8R_0}{r} - 1\right) - 1 - \frac{c_1}{r^2} + c_2\right]\cos\theta + \frac{r}{R_0^2}\frac{\mu I R_0}{2\pi r}\cos\theta \\
&\qquad\qquad = \frac{\mu I}{4\pi R_0}\cos\theta\left[\log_e\frac{8R_0}{r} - \frac{c_1}{r^2} + c_2\right].
\end{aligned}$$

Equating this at $r = a$ to the first-order part of Eq.(**??**) and using expression Eq.(**??**) for B_0 gives

$$-2(\beta_p + \tfrac{1}{2}l_i - 1) = \log_e\frac{8R_0}{a} - \frac{c_1}{a^2} + c_2 \tag{30}$$

as a second relation between c_1 and c_2.

Then the solution of Eq.(**??**) and Eq.(**??**) for c_1 and c_2 is

$$c_1 = a^2(l_i - 1)/2, \qquad c_2 = -\log_e(8R_0/a) + 3/2 - \beta_p - l_i/2,$$

namely, Eq.(**??**), as required.

END OF PROOF (f)

Finally, we note that the term in square brackets in Eq.(**??**) when evaluated at $\theta = \pi$ is identical with the approximation in Eq.(**??**) to the field of a toroidal current when $r \ll R_0$. At large r, by Eq.(**??**) the field of this current vanishes and so we are left from Eq.(**??**) with $\tilde{A} \approx (\mu I)/(4\pi)c_2 r\cos\theta$. This corresponds to a vertical field at R_0 of $B_v = (\mu I c_2)/(4\pi R_0)$, which therefore produces an outwards hoop force of $F_{hoop} = IB_v$, namely, Eq.(**??**), as required.

PROBLEM 12.6. Change of Current during Expansion of a Flux Rope.

Fill out the details of the proof of

$$I \sim \frac{1}{R_0\ \log_e(R_0/a)},$$

for the behaviour of the current (I) as a function of the major radius (R_0) of a flux ring.

SOLUTION.

Consider a simple isolated toroidal flux rope of major radius R_0, minor radius a and net toroidal current I. If the torus is treated as a ring of current of radius R_0, then the magnetic flux function $[A_\phi(R,\phi)]$ at a point (R,ϕ) in polar coordinates in the plane of the ring relative to the ring's centre may be calculated as follows.

In general, the vector potential $(\mathbf{A})$ is such that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and satisfies Poisson's equation

$$\nabla^2\mathbf{A} = -\mu\mathbf{j},$$

which has solution

$$\mathbf{A} = \frac{\mu}{4\pi}\int \mathbf{j}(\mathbf{r}')/|\mathbf{r} - \mathbf{r}'|dV'.$$

For our ring $\mathbf{j}\ dV' = IR_0\ d\phi'$, the magnetic field and vector potential are axisymmetric (independent of ϕ in cylindrical polar coordinates (R,ϕ,z) relative to the centre of the torus) and at any point in its plane the only magnetic component is $B_z(R)$. Thus, the only component of $\mathbf{A}$ there is $A_\phi(R)$.

At a more general point (R, ϕ, z) out of the plane of the ring, we have, according to Jackson's Electrodynamics book,

$$A_\phi = \frac{\mu I}{\pi} \frac{R_0}{\sqrt{R_0^2 + R^2 + z^2 + 2RR_0}} \left[\frac{(2-k^2)K(k) - 2E(k)}{k^2} \right],$$

in terms of complete elliptic integrals of the 1st and 2nd kind,

$$K(k) = \int_0^{\pi/2} [1 - k^2 \sin^2 \alpha]^{-1/2} d\alpha$$

and

$$E(k) = \int_0^{\pi/2} [1 - k^2 \sin^2 \alpha]^{1/2} d\alpha,$$

where

$$k^2 = \frac{4R_0 R}{R_0^2 + R^2 + z^2 + 2R_0 R}.$$

In particular, in the plane $(\theta = \pi/2)$ of the ring

$$A_\phi = \frac{\mu I}{4\pi} \frac{R_0 + R}{R} [(2-k^2)K(k) - 2E(k)],$$

where $k = 2(R/R_0)^{1/2}/(1 + R/R_0)$, in agreement with Eq.(12.7) in the book and Eq.(**??**) above.

Now, the total magnetic flux inside the ring is given by the value of RA_ϕ at the inner surface of the ring, but if we neglect the width of the flux rope and set $R = R_0$ then $k = 1$ and the elliptic integral $K(k)$ in A_ϕ becomes infinite. Instead, we take account of the flux rope width and evaluate RA_ϕ at the inner surface $R = R_0 - a$ of the ring, where R_0 is the major radius and $a \ll R_0$ is the minor radius.

From *An Atlas of Functions* by Spanier and Olham, 1987 edition, p612, we find that for k^2 near 1, $K(k) \approx \log_e(4/\sqrt{1-k^2})$ and $E(k) \approx 1$. Thus, when $R = R_0(1 - a/R_0)$ and $a \ll R_0$, we have

$$k = \frac{2\sqrt{1 - a/R_0}}{2 - a/R_0} \approx \left(1 - \frac{a}{2R_0} - \frac{a^2}{8R_0^2} + ..\right)\left(1 + \frac{a}{2R_0} + \frac{a^2}{4R_0^2} + ..\right) \approx 1 - \frac{a^2}{8R_0^2}.$$

Thus, $K(k) \approx \log_e(8R_0/a)$ and $E(k) \approx 1$, so that

$$A_\phi \approx \frac{\mu I}{2\pi}\left[\log_e \frac{R_0}{a} - 2\right]$$

at a point on the inside of the flux rope. (This agrees with Ex. 5.32 of Jackson (1998, Classical Electrodynamics, 3rd edition). Note that we have been working from first principles rather than invoking the expression $A \approx \bar{L}I$ in terms of self-inductance ($\bar{L}$), which is only approximate (Jackson, 1998, Sec 5.17A) and leads to a slightly different result.)

The flux through the ring is

$$\int B_z 2\pi R dR,$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and $\mathbf{A} = A_\phi \hat{\boldsymbol{\phi}}$ imply that

$$B_z = \frac{1}{R}\frac{\partial}{\partial R}(RA_\phi).$$

Thus, the flux through the ring becomes

$$\int B_z 2\pi R dR = 2\pi \int_0^{R_0-a} \frac{\partial}{\partial R}(RA_\phi) dR = 2\pi(R_0-a)A_\phi(R_0-a) \approx 2\pi R_0 A_\phi(R_0-a).$$

Therefore, the flux inside the ring is $\sim R_0 I \log_e(R_0/a)$. The condition that this be conserved (frozen flux) as the major radius changes during an eruption thus implies that the current behaves like

$$I \sim \frac{1}{R_0 \ \log_e(R_0/a)},$$

as required.

PROBLEM 12.7. Condition for Torus Instability.

Show that, if the total magnetic flux ($F = F_I + F_{ext}$) enclosed by a toroidal flux rope is held constant while both F_I and F_{ext} vary with R_0, then the condition for torus instability with an external field $B_{ext} = \hat{B}R_0^{-n}$ becomes $n > 3/2 - 1/(2c)$, where $c = \log_e(8R_0/a) - 1$.

SOLUTION.
The global equation of motion of the flux rope (of mass M, say) is

$$M\frac{d^2R_0}{dt^2} = \frac{\mu I^2 c}{4\pi R_0} - I\hat{B}R_0^{-n}, \tag{31}$$

where $c = \log_e(8R_0/a) - 1$ is regarded as constant during the eruption of the flux rope. The flux enclosed consists of two parts

$$F = F_I + F_{ext} = \mu R_0 I(c-1) - 2\pi\int_0^{R_0} B_{ext} R\, dR, \tag{32}$$

If F is held constant, then in terms of initial equilibrium current (I^*) and flux-rope radius (R_0^*), the current I is given as a function of the flux-rope radius R by

$$I = \frac{1}{R_0}\left\{I^* R_0^* + K[(R_0^*)^{2-n} - R_0^{2-n}]\right\},$$

With this expression for I, the equation of motion (**??**) becomes

$$\frac{d^2\bar{R}_0}{d\bar{t}^2} = \frac{1}{4(2-n)^2\bar{R}_0^3}\left[4-2n-(1+1/c)(1-\bar{R}_0^{2-n})\right](1-\bar{R}_0^{2-n})(3-2n-1/c),$$

in terms of dimensionless variables $\bar{R}_0 = R_0/R_0^*$ and $\bar{t}^2 = 4\pi t^2 R_0^* M/[\mu c(I^*)^2]$.

A linear perturbation from equilibrium ($\bar{R}_0 = 1$) in the form $\bar{R}_0 = 1 + r_0$, say, then implies

$$\frac{d^2 r_0}{d\bar{t}^2} = \left(2n - 3 + \frac{1}{c}\right)\frac{r}{2},$$

which gives torus instability if $n > 3/2 - 1/(2c)$, as required.

PROBLEM 12.8. Current (I) of a Toroidal Flux Rope.
Show that $Ia = constant$ for a linear force-free toroidal flux rope of current I and radius a, whose axial flux is constant and whose axial field vanishes on its surface.

SOLUTION.
Following Section 12.2.3, we assume the internal field of the flux rope is a linear force-free (i.e., Lundquist) field of the form

$$B_r = 0, \qquad B_\theta = \frac{B_0 J_1(\alpha r)}{J_1(\alpha a)}, \qquad B_\phi = \frac{B_0 J_0(\alpha r)}{J_1(\alpha a)},$$

where J_0 and J_1 are Bessel functions and the minor radius $r = a$ (where $B_\theta = B_0$) is located at the first zero of J_0, namely, $\alpha a \approx 2.405$, so that the axial field vanishes there (i.e., $B_\phi(\alpha a) = 0$).

Then the axial magnetic flux in the flux rope remains constant as its current and minor radius change, so that

$$\int_0^{2\pi}\int_0^a B_\phi r dr d\theta = constant.$$

But the force-free condition implies $\mu j_\phi = \alpha B_\phi$ and so

$$\frac{\mu}{\alpha}\int_0^{2\pi}\int_0^a j_\phi r dr d\theta = constant,$$

or, in other words,

$$\frac{\mu I}{\alpha} = constant,$$

where I is the total axial current.

However, $\alpha a = constant$ since this determines the edge of the flux rope, and so, eliminating a, the above equation implies

$$Ia = constant,$$

as required.

PROBLEM 12.9. Titov-Démoulin Model.
Show that, in the Titov-Démoulin model for the equilibrium of an active-region flux rope of current I, depth d, major radius R_0 and minor radius a, I reaches a maximum at about $R_0 \approx L/\sqrt{2}$ and torus instability sets in at about $R_0 \approx \sqrt{2}L$ when $d \ll R_0$ and $a \ll R_0$.

SOLUTION.
The proof follows the paper by Titov and Démoulin (1999). Equilibrium in their model is of the form

$$F_I + F_q = 0, \tag{33}$$

where

$$F_I = \frac{\mu I^2}{4\pi R_0}\log_e\left(\frac{8R_0}{a} - \frac{5}{4}\right)$$

is the hoop force and

$$F_q = -\frac{2qLI}{(R_0^2 + L^2)^{3/2}}$$

is the Lorentz force of the flux rope current I acting on the magnetic field at the flux rope created by the two magnetic charges $\pm q$.

This force balance determines the current as

$$I = \frac{8\pi qLR_0(R_0^2 + L^2)^{-3/2}}{\mu C}, \tag{34}$$

where

$$C = \log_e\left(\frac{8R_0}{a} - \frac{5}{4}\right).$$

Now, if we neglect the slowly varying logarithmic dependence in Eq.(**??**) and regard C as constant, independent of R_0, then the derivative with respect to R_0 of Eq.**??** is

$$\frac{dI}{dR_0} = \frac{8\pi qL}{\mu C}\frac{L^2 - 2R_0^2}{(R_0^2 + L^2)^{5/2}},$$

which implies that I reaches its maximum value at $L = \sqrt{2}R_0$.

In order to determine the stability, let us calculate the change in forces produced by an increase δR_0 in major radius, since if the net force is outward the equilibrium will be unstable. Thus, from the above definitions

$$\frac{\delta F_q}{F_I} + \frac{\delta F_I}{F_I} = \frac{\delta I}{I} + \frac{\delta R_0}{R_0}\left(\frac{2R_0^2 - L^2}{R_0^2 + L^2}\right) + \frac{1}{\log_e(8R_0/a - 5/4)}\left(\frac{\delta R_0}{R_0} - \frac{\delta a}{a}\right),$$

where the variations in minor radius (a) are given by conservation of toroidal flux [$\pi B_\phi a^2$ with $B_\phi = \mu I_0/(2\pi R_0)$] as

$$\frac{\delta a}{a} = \frac{\delta R_0}{2R_0},$$

and from the condition that the number of turns in the coronal part of the flux rope remain constant, namely, $N_{cor} = (N_t/\pi)\arccos(d/R_0) = constant$, where $N_t = IR_0^2/(I_0a^2)$ and toroidal flux conservation implies $a^2/R_0 = constant$. The condition $N_{cor} = constant$ may be written

$$\frac{\delta I}{I} = -\frac{\delta R_0}{R_0}\left(1 + \frac{d}{(R_0^2 - d^2)^{1/2}\arccos(d/R_0)}\right).$$

Combining these expressions gives

$$\frac{\delta F_q + \delta F_I}{F_I} = \frac{\delta R_0}{R_0}\left[\frac{R_0^2 - 2L^2}{R_0^2 + L^2} - \frac{d(R_0^2 - d^2)^{-1/2}}{\arccos(d/R_0)} + \frac{1/2}{\log_e(8R_0/a) - 5/4}\right].$$

If $d \ll R_0$ and $a \ll R_0$, the last two terms inside the square brackets are much smaller than unity and so the condition for instability, namely, that the force variation be positive, becomes $R_0 > \sqrt{2}L$, as required.