hints and answers to selected exercises[1]

chapters 1-9

Medio and Lines

NONLINEAR DYNAMICS:

A PRIMER

[1]Revised 1 October 2001. M. Lines and A. Medio take all responsibility for the hints and answers. Please address comments, criticisms, corrections and suggestions to lines@dss.uniud.it

chapter 1

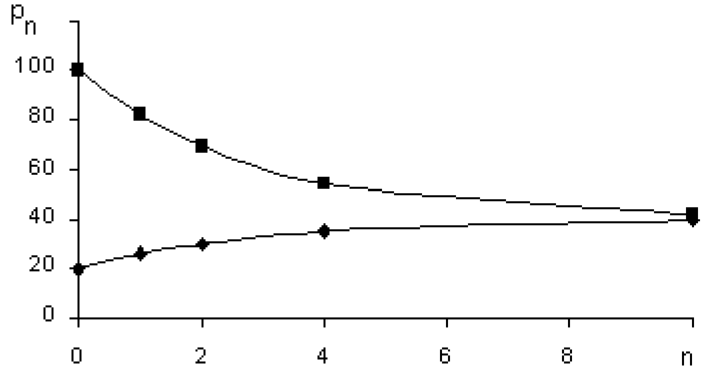
1.1 Given the parameter values $a = 10$, $b = 0.2$, $m = 2$, $s = 0.1$

(a) the general solutions: $p_n = 40 - 20(0.7)^n$ for $p_0 = 20$; $p_n = 40 + 60(0.7)^n$ for $p_0 = 100$.

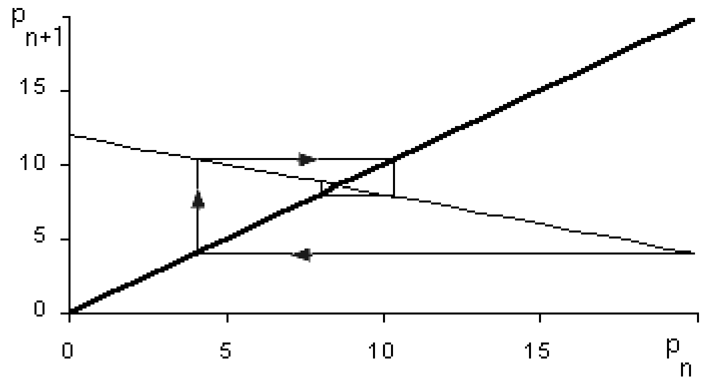
(b) the orbits are

n	p_n	p_n
0	20	100
1	26	82
2	30.2	69.4
4	≈ 35.2	≈ 54.5
10	≈ 39.4	≈ 41.7
100	≈ 40.0	≈ 40.0

(c)



1.2 Let $b = s = 0.7$ so that $\beta = -0.4$ and the dynamics are convergence with improper oscillations towards the equilibrium at 12/1.4.



1.5

(b) Let $x = z_1$, then

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = z_3$$
$$\dot{z}_3 = 2x - 0.4\ddot{x} = 2z_1 - 0.4z_3$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & -0.4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}.$$

1.6 Let $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, $z_4 = \dot{y}$. Then

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \ddot{x} = 1 - z_1$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = \dot{y} + y - 1 = z_4 + z_3 - 1$$

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

1.7

(a)

$$z_{n+1}^{(1)} = x_{n+1} = z_n^{(2)}$$
$$z_{n+1}^{(2)} = x_{n+2} = az_n^{(2)} - bz_n^{(1)} + 1$$

$$\begin{pmatrix} z_{n+1}^{(1)} \\ z_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix} \begin{pmatrix} z_n^{(1)} \\ z_n^{(2)} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b)

$$z_{n+1}^{(1)} = z_n^{(2)}$$
$$z_{n+1}^{(2)} = z_n^{(3)}$$
$$z_{n+1}^{(3)} = 0.2z_n^{(1)} - 4z_n^{(2)} + 4$$

$$\begin{pmatrix} z_{n+1}^{(1)} \\ z_{n+1}^{(2)} \\ z_{n+1}^{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.2 & -4 & 0 \end{pmatrix} \begin{pmatrix} z_n^{(1)} \\ z_n^{(2)} \\ z_n^{(3)} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}.$$

1.8

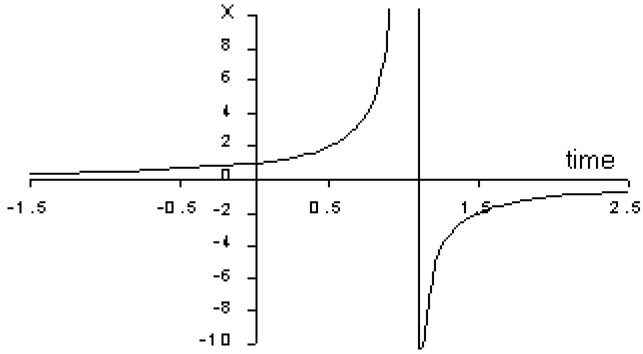
(c)

$$\frac{dx}{dt} = x^2$$
$$\int \frac{dx}{x^2} = \int dt$$
$$\frac{-1}{x} = t + c$$

from which we derive $x(t) = -1/(t+c)$ and $x_0 \equiv x(0) = -1/c$, or $c = -1/x_0$, giving the exact solution

$$x(t) = \frac{-1}{t - \frac{1}{x_0}} = \frac{x_0}{1 - tx_0}.$$

At $t = 1/x_0$, $x(t)$ is undefined.



1.10

$$\dot{x} = \mu x(1-x) \tag{1}$$
$$\frac{dx}{dt} = \mu x(1-x) \tag{2}$$
$$\int \frac{dx}{\mu x(1-x)} = \int dt \tag{3}$$
$$F(x) = \frac{1}{\mu} \ln\left(\frac{x}{\mu(1-x)}\right) = t + c \tag{4}$$

To check the correctness, perform

$$F'(x) = \frac{1}{\mu} \frac{\frac{\mu(1-x)+\mu x}{\mu^2(1-x)^2}}{\frac{x}{\mu(1-x)}} = \frac{1}{\mu x(1-x)}$$

To write the solution we first exponentiate both sides of (4)

$$\frac{x}{\mu(1-x)} = e^{\mu(t+c)} = e^{\mu c} e^{\mu t}.$$

The value of the integration constant can be fixed by using the initial condition at time $t = 0$. Letting $x(0) = x_0$ we have

$$\frac{x_0}{\mu(1-x_0)} = e^{\mu c}.$$

Then, for $t \neq 0$ we have

$$\frac{x}{\mu(1-x)} = \frac{x_0}{\mu(1-x_0)} e^{\mu t} \quad \text{and} \quad x = \frac{(1-x)x_0 e^{\mu t}}{1-x_0}.$$

Then

$$x\left[1 + \frac{x_0 e^{\mu t}}{1-x_0}\right] = \frac{x_0 e^{\mu t}}{1-x_0}.$$

and the exact solution is

$$x(t) = \frac{x_0 e^{\mu t}}{1 - x_0 + x_0 e^{\mu t}}.$$

For $x_0 = 0$, $x(t) = 0\ \forall t$, for $x_0 = 1$, $x(t) = 1\ \forall t$, and for $x_0 \in (0,1)$, $x(t) \in (0,1)$ for $-\infty < t < \infty$.

$$\lim_{t\to+\infty} x(t) = 1 \qquad \lim_{t\to-\infty} x(t) = 0$$

which implies that there are two equilibria 0 and 1, 0 is unstable, 1 is stable.

chapter 2

2.1 Let $F_1(n,x)$ and $F_2(n,x)$ be two functions such that $G[F_1(n,x)] = F_1(n+1,x)$, $G[F_2(n,x)] = F_2(n+1,x)$. Then, for $S = [\alpha F_1(n,x) + \beta F_2(n,x)]$,

$$\begin{aligned} G[S(n,x)] &= S(n+1,x) \quad \text{iff} \\ G[\alpha F_1(n,x) + \beta F_2(n,x)] &= \alpha F_1(n+1,x) + \beta F_2(n+1,x) \quad \text{or,} \\ G[\alpha F_1(n,x) + \beta F_2(n,x)] &= \alpha G[F_1(n,x)] + \beta G[F_2(n,x)] \end{aligned}$$

and the last equation is true for any $\alpha, \beta \in \mathbb{R}$ and for any $x \in \mathbb{R}^m$ if, and only if, G is linear.

2.3 This is the case of a complex conjugate pair of eigenvalues $(\alpha_j + i\beta_j, \alpha_j - i\beta_j)$ with corresponding pair of eigenvectors $(a_j + ib_j, a_j - ib_j)$. Consider the real solution $x_j(t)$ which is half of the sum of the conjugate solutions

$$x_j(t) = e^{\alpha_j t}[a_j \cos(\beta_j t) - b_j \sin(\beta_j t)].$$

If $x_j(t)$ is a solution then it must be that

$$\begin{aligned} \dot{x}_j(t) = e^{\alpha_j t}[-a_j\beta_j \sin(\beta_j t) - b_j\beta_j \cos(\beta_j t)] + \\ + e^{\alpha_j t}[\alpha_j a_j \cos(\beta_j t) - \alpha_j b_j \sin(\beta_j t)] \\ = A e^{\alpha_j t}[a_j \cos(\beta_j t) - b_j \sin(\beta_j t)]. \end{aligned} \tag{1}$$

If $a_j + ib_j$ is an eigenvector

$$A(a_j + ib_j) = (\alpha_j + i\beta)(a_j + ib_j) = (\alpha_j a_j - \beta_j b_j) + i(\beta_j a_j + \alpha_j b_j)$$

and

$$Aa_j = \alpha_j a_j - \beta_j b_j \qquad Ab_j = \beta_j a_j + \alpha_j b_j. \tag{2}$$

Substituting (2) into (1) we verify the result. A similar verification can be made for $x_{j+1}(t)$.

2.4 For real eigenvalues and eigenvectors, given the definitions of e^{tA}, $X(t)$, $X(0)$, and given that $Au_i = \lambda_i u_i$ and

$$e^{\lambda_i t} = \left(I + \lambda_i t + \frac{\lambda_i^2 t^2}{2!} + \cdots + \frac{\lambda_i^k t^k}{k!} + \cdots\right),$$

we have that

$$\begin{aligned} e^{tA}X(0) &= (u_1, \dots, u_n) + tA(u_1, \dots, u_n) + \frac{t^2}{2}A^2(u_1, \dots, u_n) + \\ &\quad + \cdots + \frac{t^k}{k!}A^k(u_1, \dots, u_n) + \cdots = \\ &= (u_1, \dots, u_n) + t(\lambda_1 u_1, \dots, \lambda_n u_n) + \frac{t^2}{2}(\lambda_1^2 u_1, \dots, \lambda_n^2 u_n) + \\ &\quad + \cdots + \frac{t^k}{k!}(\lambda_1^k u_1, \dots, \lambda_n^k u_n) + \cdots = \\ &= \left(e^{\lambda_1 t}u_1, \dots, e^{\lambda_n t}u_n\right) = \\ &= X(t). \end{aligned}$$

2.6 Because κ_i is a real, distinct eigenvalue of B and v_i is the corresponding real eigenvector, we have $Bv_i = \kappa_i v_i$. It is easy to verify that $x_i(n) = \kappa_i^n v_i$ is a solution to $x_{n+1} = Bx_n$ as follows

$$x_i(n+1) = \kappa_i^{n+1} v_i = \kappa_i^n \kappa_i v_i = \kappa_i^n B v_i = Bx_i(n).$$

2.7 The system is homogeneous with fixed point at $(\bar{x}, \bar{y}) = (0, 0)$.

(a) From the characteristic equation

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 3 & 0-\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

we have $\lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$, the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2$ and the fixed point $(0, 0)$ is a saddle-point. To find the associated eigenvectors, we substitute into $(A - \lambda I)u = 0$

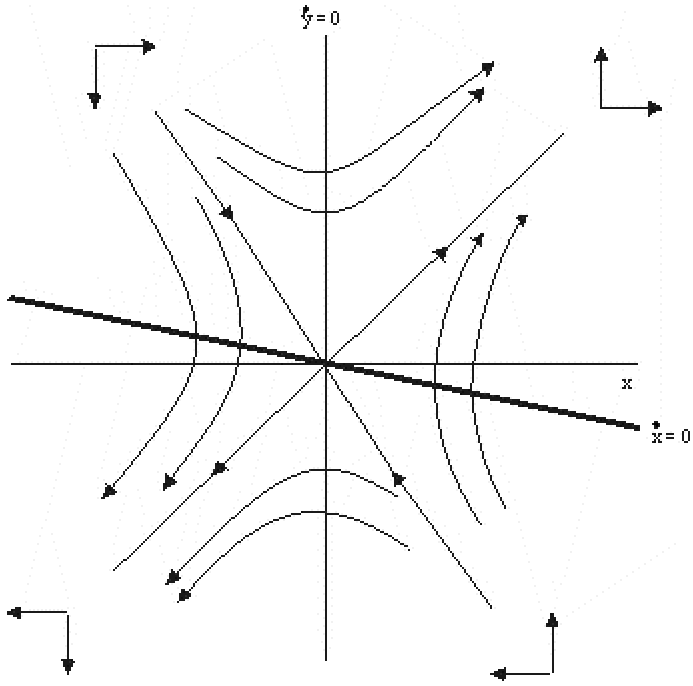
$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} u_2^{(1)} \\ u_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and, choosing $u_1^{(1)} = u_2^{(1)} = 1$, the eigenvectors are

$$\begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} u_2^{(1)} \\ u_2^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -3/2 \end{pmatrix}$$

(b) The stationaries are defined by $\dot{x} = 0$, that is, $y = -x/2$ and $\dot{y} = 0$, that is, $x = 0$.

(c)



2.8 For the homogeneous system we have

$$A = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} \qquad \lambda_1 = \lambda_2 = \lambda^* = -2$$

and the origin is a stable node. Substituting the single eigenvalue we have

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} u_1^{(1)} \\ u_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Using the procedure of Section 2.2 to get a second vector v such that $(A - \lambda I)v = u$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} v_1^{(1)} \\ v_1^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then writing the solution and using the given initial conditions we have

$$x(t) = (c_1 + c_2)e^{-2t} + tc_2e^{-2t}$$
$$y(t) = -c_1e^{-2t} - tc_2e^{-2t}$$

and $x(0) = c_1 + c_2$, $y(0) = -c_1 = 2$ so that $c_2 = 3$. The solution is then

$$x(t) = e^{-2t} + 3te^{-2t}$$
$$y(t) = 2e^{-2t} - 3te^{-2t}.$$

2.9

(b) We have the fixed point at $\dot{x} = \dot{y} = 0$ giving $(\bar{x}, \bar{y}) = (-4, 4)$. The eigenvalues $\lambda_1, \lambda_2 = -1/2, 1/2$ are one greater, one smaller than 0, so that $(-4, 4)$ is a saddle point.

2.10

(b) In equilibrium

$$\bar{x} = \frac{-1}{2}\bar{x} - 2 \qquad \bar{y} = \frac{1}{2}\bar{y} - 2$$

giving the fixed point at $(\bar{x}, \bar{y}) = (-4/3, -4)$. The eigenvalues λ_1, λ_2 are again $-1/2, 1/2$ and both smaller than one in absolute value, so that $(-4/3, 4)$ is a stable node. The fixed point differs from the continuous-time version in being stable (as opposed to the unstable saddle point) and being in a slightly different position in the plane. Also, due to the negative eigenvalues, there are improper oscillations.

2.11 Because $(\kappa, \bar{\kappa}) = \sigma \pm i\theta$ is a complex eigenvalue pair with a corresponding pair of eigenvectors $(v, \bar{v}) = p \pm iq$ of the matrix B, we can write

$$B(p + iq) = (\sigma + i\theta)(p + iq). \tag{1}$$

For $\kappa^n v = (\sigma + i\theta)^n(p + iq)$ to be a solution of (2.5) it must be

$$(\sigma + i\theta)^{n+1}(p + iq) = B(\sigma + i\theta)^n(p + iq) = (\sigma + i\theta)^n B(p + iq). \tag{2}$$

We write

$$\frac{(\sigma + i\theta)^{n+1}}{(\sigma + i\theta)^n} = \frac{(\sigma + i\theta)^n(\sigma + i\theta)}{(\sigma + i\theta)^n}$$

and let $(\sigma + i\theta)^n = \tilde{\sigma} + i\tilde{\theta}$. In the next step we use the definition of the **quotient** of two complex numbers z_1, z_2 as the complex number $(z_1\bar{z}_2)/(z_2\bar{z}_2)$ ($\bar{z}_i$ the complex conjugate of z_i). Let

$$(\sigma + i\theta)^n(\sigma + i\theta) = (\tilde{\sigma}\sigma - \theta\tilde{\theta}) + i(\tilde{\theta}\sigma + \theta\tilde{\sigma}) = z_1$$

and $(\sigma + i\theta)^n = \tilde{\sigma} + i\tilde{\theta} = z_2$. Then

$$\begin{aligned}\frac{(\sigma + i\theta)^{n+1}}{(\sigma + i\theta)^n} = \frac{z_1}{z_2} &= \frac{\left[(\tilde{\sigma}\sigma - \theta\tilde{\theta}) + i(\tilde{\theta}\sigma + \theta\tilde{\sigma})\right](\tilde{\sigma} - i\tilde{\theta})}{\tilde{\sigma}^2 + \tilde{\theta}^2} = \\ &= \frac{\sigma(\tilde{\sigma}^2 + \tilde{\theta}^2) + i\theta(\tilde{\sigma}^2 + \tilde{\theta}^2)}{\tilde{\sigma}^2 + \tilde{\theta}^2} = (\sigma + i\theta).\end{aligned} \tag{3}$$

Then, substituting (1) into (2) and using (3) we have the result.

2.12 (Third matrix only.)

$$\begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1.5 \end{pmatrix}$$

(a) The eigenvalues of B are 0.2, 0.5 and 1.5 and the dynamics are described by a saddle-node. Orbits initiating in the plane generated by the eigenvectors associated with the eigenvalues 0.2 and 0.5 converge towards the equilibrium at the origin. All other initial conditions lead to exponential expansion towards $\pm\infty$.

(b)

$$\mu_1 = 0.2 \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mu_2 = 0.5 \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mu_3 = 1.5 \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

2.13

(a) The procedure used in section 2.1 to transform an affine system into a linear system also transforms a nonhomogeneous into a homogeneous system. Let

$$\begin{pmatrix} w_n \\ z_n \end{pmatrix} = \begin{pmatrix} x_n + k_1 \\ y_n + k_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 4 \\ 1 & -3 \end{pmatrix}$$

so that

$$B\begin{pmatrix} w_n \\ z_n \end{pmatrix} = B\begin{pmatrix} x_n + k_1 \\ y_n + k_2 \end{pmatrix} = B\begin{pmatrix} x_n \\ y_n \end{pmatrix} + B\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \begin{pmatrix} w_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} x_{n+1} + k_1 \\ y_{n+1} + k_2 \end{pmatrix} &= B\begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \\ &= B\begin{pmatrix} w_n \\ z_n \end{pmatrix} - B\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} -8 \\ 4 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \end{aligned}$$

and the system in the new variables (w_n, z_n) is homogeneous if

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = (I - B)^{-1}\begin{pmatrix} 8 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ \frac{-5}{4} \end{pmatrix}.$$

The transformed system has the unique equilibrium at the origin, whereas the original system has the equilibrium at $(\bar{x}, \bar{y}) = (1, 5/4)$.

(b) $\mu_{1,2} \approx 4.53, -3.53$ The equilibrium is an unstable node as both eigenvalues are greater than one in absolute value.

(c)

$$v_1 \approx \begin{pmatrix} 1 \\ 0.13 \end{pmatrix} \quad v_2 \approx \begin{pmatrix} 1 \\ -1.88 \end{pmatrix}$$

(e) We have (approximately)

$$w_n = c_1(4.53)^n(1) + c_2(-3.53)^n(1)$$
$$z_n = c_1(4.53)^n(0.13) + c_2(-3.53)^n(-1.88)$$

and using the initial conditions

$$w_0 = c_1 + c_2 = -1$$
$$z_0 = 0.13c_1 - 1.88c_2 = 1$$

to calculate the arbitrary constants, we can write the solution as

$$w_n = -0.44(4.53)^n - 0.56(-3.53)^n$$
$$z_n = -0.06(4.53)^n + 1.05(-3.53)^n.$$

(f)

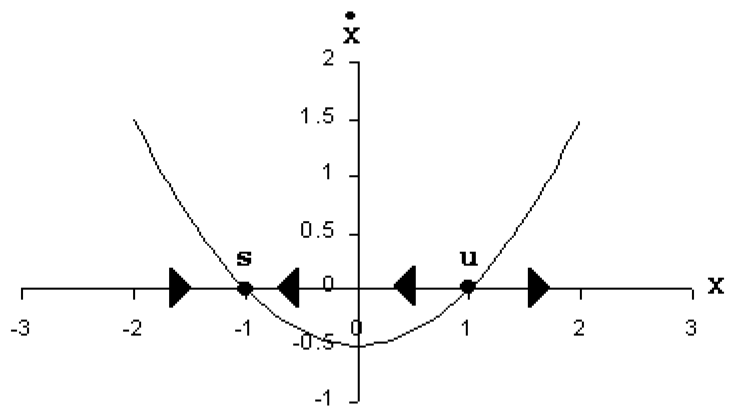
	w_n	z_n
$n = 0$	-1	1
$n = 1$	-0.02	-3.98
$n = 2$	-16.01	11.85
$n = 3$	-16.27	-51.76
$n = 10$	-1.77×10^6	9.71×10^4
$n = 100$	-1.8×10^{65}	2.4×10^{64}

The orbit approaches asymptotically to the eigenvector $v_1 = (1, 0.13)'$ associated with the largest eigenvalue in absolute value $\mu_1 \approx 4.53$. (The reader can verify that, after 100 iterations, we have $z_{100}/w_{100} \approx 0.13$).

chapter 3

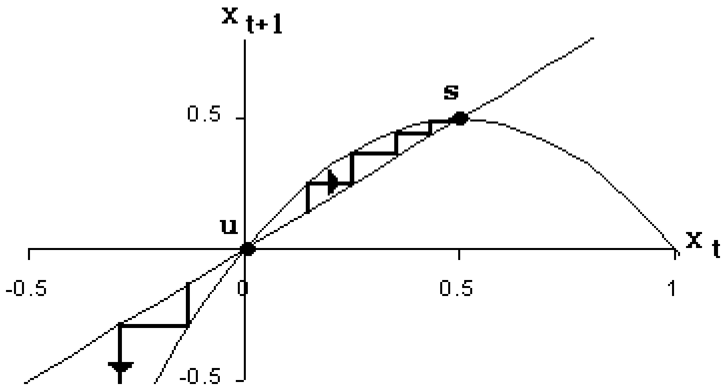
3.2

(b)



3.3

(c)



3.4

(a) The fixed points are found by setting $\dot{x} = \dot{y} = 0$ giving for the second equation $x^2 = y^2$. After substituting this relation into the first equation we have the x coordinate of the fixed point as the solution to the quadratic equation $x^2 - 3x + 2 = 0$. The fixed points are $(2,2); (2,-2); (1,1); (1,-1)$. The Jacobian matrix is

$$\begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}$$

giving the following respective eigenvalues (correct to two decimal places): 0.53, -7.53; $0.5 \pm i3.87$; $-0.44, -4.56$;$-2, 1$. The local stability properties of the fixed points are, respectively: unstable (the linearised system is a saddle point); unstable (the linearised system is an unstable focus); stable (the linearised system is a stable node); unstable (the linearised system is a saddle point).

(g) There is a fixed point at $(0,0)$, with eigenvalues $\alpha, -\gamma$ which is unstable (the linearised system is a saddle point). There is another fixed point at $(\gamma/\delta, \alpha/\beta)$. The Jacobian calculated at the equilibrium gives the imaginary eigenvalue pair $\pm i\sqrt{\gamma\alpha}$ which means that it is not hyperbolic and the Hartman-Grobman theorem does not apply. (The fixed point could behave locally like a stable or unstable focus or even a centre, see exercise 3.7(a) below.)

3.5

(a) For the transformed system

$$\dot{x} = y$$
$$\dot{y} = x^3 - x$$

there are three fixed points $(0,0),(1,0),(-1,0)$. The Jacobian matrix at the origin has eigenvalues $\pm i$ with dim $W^c = 2$ (the Hartman-Grobman Theorem is not applicable). The Jacobian matrix at fixed points $(1,0)$, $(-1,0)$ has eigenvalues $\pm\sqrt{2}$, giving dim $W^s =$ dim $W^u = 1$ (W^s, W^u and W^c denoting, respectively, stable, unstable and centre manifolds).

3.6

(a) The fixed points are $(0,9,0)$ and $(0,9,1)$ (excluding complex values for variables). The Jacobian matrix of the first is triagular and the eigenvalues are $1/6, 0, -9$ with dim $W^s = 2$, dim $W^u = 1$ (the linearised system is a saddle). The Jacobian matrix of the second is also triangular with eigenvalues $2, 1/6, -9$ with dim $W^s = 1$, dim $W^u = 2$ (the linearised system is a saddle). Both equilibria are therefore unstable.

3.7

(a)

$$\frac{\dot{y}}{\dot{x}} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{y(-\gamma + \delta x)}{x(\alpha - \beta y)}$$

and, after cross-multiplying, we can integrate both sides of the equation to obtain

$$\alpha \ln y - \beta y = -\gamma \ln x + \delta x + c$$

where c is a constant of integration. This equation defines curves in the state space (x, y) which are solutions of system (a).

(c) Again, variables can be separated giving

$$\int \left(1 + \frac{1}{y-1}\right) dy = 4 \int \frac{dx}{x}$$

whence

$$4 \ln x = y + \ln(y-1) + c$$

where c is a constant of integration. This equation defines curves in the state space (x, y) which are solutions of system (c).

3.10

(c) We have $\dot{V}(x,y) = x(-y - x^3) + y(x - y^3) = -x^4 - y^4$ which is negative for all real nonzero x and y, while $V(0,0) = 0$ and $V(x,y) > 0\ \forall x, y \neq 0$, and the equilibrium is asymptotically stable in $\mathbb{R}^2$.

(e) The transformed system is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + \frac{y^3}{4} - x. \end{aligned}$$

$\dot{V}(x,y) = y^2(y^2/4 - 1) < 0$ for $|y| < 2$, and the equilibrium is asymptotically stable for initial conditions such that $0 < V(x,y) < 2$ (that is, inside a circle around the origin with radius less than 2).

3.11

(a) Our apologies, the first equation is incorrectly given. The correct system is

$$\dot{x} = -x^3 - y$$
$$\dot{y} = 3x^3$$

Try the function

$$V(x,y) = \frac{1}{2}\dot{x}^2 + \frac{3}{4}x^4$$

and follow the answer to exercise 3.12.

(b) $V(x,y) = \frac{k}{2}x^2 + \frac{1}{2}y^2$. The origin is globally asymptotically stable for $k > 0$.

(c) $V(x,y) = \frac{1}{4}x^4 + \frac{1}{2}y^2$.

(d) $V(x,y) = \frac{1}{2}(x^2 + y^2)$. The origin is globally asymptotically stable for $k < 0$.

3.12 The unique fixed point for the system is at the origin. To prove its stability we apply the rule of thumb described in Section 3.3. Differentiating the first equation with respect to time and using the second we have

$$\ddot{x} + 3x^2\dot{x} + x^5 = 0.$$

Multiplying by $\dot{x}$ gives

$$\dot{x}\ddot{x} + 3x^2\dot{x}^2 + \dot{x}x^5 = 0$$

or

$$\frac{d}{dt}\Big(\frac{1}{2}\dot{x}^2 + \int_0^x s^5 ds\Big) = -3x^2\dot{x}^2.$$

Considering that $\int_0^x s^5 ds = (1/6)x^6$, set

$$V(x,y) = \Big(\frac{1}{2}\dot{x}^2 + \frac{1}{6}x^6\Big)$$

and

$$\dot{V}(x,y) = -3x^2\dot{x}^2 \leq 0.$$

$V(x,y)$ is positive definite for $\mathbb{R}^2 \setminus \{0\}$; $V(0,0) = 0$; $\dot{V}(x,y) \leq 0$ for $(x,y) \in \mathbb{R}^2$. From the fact that $\dot{V}(x,y) \leq 0$ it follows that, for any neighbourhood N of $(0,0)$, the subsets of the state space (x,y)

$$V_k(x,y) = \{(x,y) \mid V(x,y) \leq k\}$$

are invariant. On the other hand, the set

$$E = \{(x, y) \mid \dot{V}(x, y) = 0\}$$

includes the two curves $(x = 0)$ and $(y = -x^3)$, and the only invariant subset of E is the fixed point $(x = y = 0)$. Therefore, the La Salle Invariance Principle can be applied and the fixed point is not only stable but asymptotically so.

3.13 For equation (3.18)

$$[Df(z)]x = \frac{\partial}{\partial s} f(z) = \frac{\partial}{\partial s} f(sx).$$

Moreover

$$[f(z)]_0^1 = f(x) - f(0) = f(x).$$

For equation (3.19) we have

$$x^T [Df(z)]^T (z) Bx = \frac{\partial}{\partial s}\left(f(z)^T Bx\right) = \frac{\partial}{\partial s}\left(f(sx)^T Bx\right).$$

Also

$$[f(sx)^T Bx]_0^1 = f(x)^T Bx - f(0)^T Bx = f(x)^T Bx.$$

chapter 4

4.1 The ω-limit set of the points C, D, E are, respectively, the point A (a stable focus), the origin (a saddle), the point B (another stable focus).

4.2

(a) $\omega(x_0 > 0) = +\infty$, $\omega(x_0 < 0) = -\infty$, $\omega(x_0 = 0) = 0$; $\alpha(x) = 0$.

(c) $\omega(x_0 > 3) = +\infty$, $\omega(x_0 < 3) = 1$, $\omega(x_0 = 3) = 3$; $\alpha(x_0 > 1) = 3$, $\alpha(x_0 = 1) = 1$, $\alpha(x_0 < 1) = -\infty$.

4.3 Figure 4.3 representing the phase portrait of system (4.1)-(4.2) is incorrect. Please check the errata.

4.4 Denoting by ϕ the flow generated by each of the vector fields, we have:

(a) $\omega(0, 0) = (0, 0)$; $\omega(r, \theta)$ is the unit circle for $(r, \theta) \neq (0, 0)$. The positive limit set of the flow, $L^+(\phi)$, (here coinciding with the nonwandering set $\Omega(\phi)$) is the union of the origin and the unit circle.

4.5

(a) $\omega(0,0) = (0,0), \omega(1/2,0) = (0,0), \omega(1,0) =$ unit circle, $\omega(3/2,0) = \omega(2,0) = \omega(3,0)$ circle with radius 2.

4.6

(a) From the second equation we have

$$y(t) = -\frac{1}{t+c}$$

with $c < 0$ if $y(0) > 0$. Hence $y(t) \in (0,\infty)$ for $t \in (-\infty, -c)$. Notice that the solution is not defined for all $t \in \mathbb{R}$, cf. exercise 1.8(c).

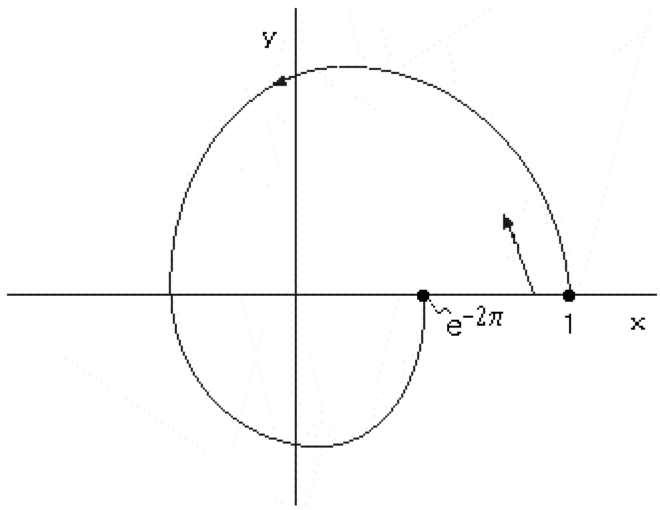
(b) The eigenvalues of the Jacobian matrix for the given system are $1 \pm i$. Then recalling equation 2.16, the solution starting at point $(x, y)= (1, 0)$ is

$$x(t) = e^{-t}\cos t$$
$$y(t) = e^{-t}\sin t$$

or in polar coordinates, putting $\theta = t$

$$\dot{r} = r$$
$$\dot{\theta} = 1.$$

Then the curve $f(\theta) = e^{-\theta}$ represents a solution from $(x, y)= (1, 0)$ to $(x, y)= (e^{-2\pi}, 0)$. Considering that along an orbit $dr/d\theta = -r$, all orbits starting from points on the segment $(x \in [e^{-2\pi}, 1); y = 0)$ point inside the closed set. Finally, no orbit can leave the closed set because orbits cannot intersect in the state space.



4.7

(a) The invariant set is the annulus $C\{(x,y) : \frac{1}{2} \leq r^2 \leq 1\}$ where r is the polar coordinate $r^2 = x^2 + y^2$. We have

$$\frac{d}{dt}\frac{r^2}{2} = r\dot{r} = \frac{d}{dt}\frac{x^2+y^2}{2} = x\dot{x} + y\dot{y}$$
$$= xy + y(-x) + y^2(1 - x^2 - 2y^2).$$

Then

$$x = \frac{y^2(1-x^2-2y^2)}{\sqrt{x^2+y^2}} > 0 \text{ if } 1 - x^2 - 2y^2 > 0.$$

At $r^2 = \frac{1}{2}$ we have $y^2 = \frac{1}{2} - x^2$ and $1 - x^2 - 2y^2 = x^2 > 0$. That is, from a circle with radius $r = \frac{1}{\sqrt{2}}$ the radius increases. On the other hand we have for $r^2 = 1$, $y^2 = 1 - x^2$ and $1 - x^2 - 2y^2 = x^2 - 1 \leq 0$ because $x^2 \leq 1$. That is, from the unit circle, the radius decreases or remains the same. Therefore, the annulus is invariant. Moreover there is only one fixed point, at the origin, and none in C. Then the conditions of theorem 4.2 are satisfied and there is at least one periodic solution in C.

(c) The exercise as it stands is not correct. The origin is an unstable fixed point and therefore we can define a circle C_1 around the origin, and sufficiently near it, such that all orbits starting from points on C_1 move outwards. Also, choosing the Lyapunov function $V(z) = \frac{1}{2}z^T z$ we can show that $\dot{V}(z) < 0$ for sufficiently large values of V (that is, for points whose Euclidean difference from the origin is sufficiently large). This measns that it is possible to define a second circle around the origin, C_2, such that all orbits starting on C_2 move inwards. However, without some additional restrictions on the matrix A, we cannot exclude the existence of fixed points inside the annulus between C_1 and C_2. (The reader can verify this statement using the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$). Thus, the conditions of the Poincaré-Bendixson theorem are not necessarily satisfied. What we need to require is that the two eigenvalues of A are complex conjugate with positive real part. In this case, the only fixed point is the origin. To see this, consider that, putting $\dot{z} = 0$, we have

$$\left(A - r^2 I\right) z = 0$$

and this equation has a nonzero solution for z if, and only if, r^2 is an eigenvalue of A, which cannot be true under our assumptions.

4.8 Take any two periodic points x_i and x_j, $0 \leq i < j < k$. We want to show that the matrices $DG^k(x_i)$ and $DG^k(x_j)$ have the same eigenvalues. Consider first that, for any k-periodic point x, it must be $G^k(G^l(x)) = G^l(G^k(x))$, that is, the two composite maps $G^k \circ G^l$ and $G^l \circ G^k$ are the same for any integer l. Then $G^k(G^{j-i}(x_i)) = G^{j-i}(G^k(x_i))$. Taking derivatives with respect to x and considering that $G^{j-i}(x_i) = x_j$ and $G^k(x_i) = x_i$, we have

$$DG^k(x_j)DG^{j-i}(x_i) = DG^{j-i}(x_i)DG^k(x_i).$$

Thus the matrices $DG^k(x_j)$ and $DG^k(x_i)$ are similar and have the same eigenvalues.

4.9

(a) The fixed point of the map $G(x_n) = 1 - x_n^2$ are solutions to

$$x^2 + x - 1 = 0 \qquad (i)$$

that is, $\bar{x}_1 = \frac{-1+\sqrt{5}}{2}$, $\bar{x}_2 = \frac{-1-\sqrt{5}}{2}$. The eigenvalue is simply $dG/dx_n = -2x_n$ giving $\kappa = 1 \mp \sqrt{5}$ at $\bar{x}_1$, $\bar{x}_2$, respectively, and both equilibria are unstable. The values x_0^*, x_1^* of the period-2 cycle are found by determining the fixed points of $G^2(x_n) = 2x_n^2 - x_n^4$ as solutions to

$$x^4 - 2x^2 + x = 0 \qquad (ii)$$

Since $\bar{x}_1$ and $\bar{x}_2$ are also fixed points of G^2 we can determine the remaining 2 fixed points by dividing (ii) by (i)

$$\frac{x^4 - 2x^2 + x}{x^2 + x - 1} = x^2 - x$$

giving fixed points of G^2 at 0 and 1 and a (local) period-2 cycle for G with $x_0^* = 0$, $x_1^* = 1$. The stability of the periodic points of G is the same as that of the fixed points of G^2, for which we have $dG^2/dx_n = 4x_n - 4x_n^3 = 0$ at either fixed point and the periodic cycle is (locally) stable.

4.10 Take the first oscillator, in x_1, x_2, with the variable angular rotation $\theta_1(t)$ and the variable radius $r_1(t)$. Let:

$$x_1 = r_1 \cos\theta_1$$
$$x_2 = r_1 \sin\theta_1$$

and, taking time derivatives, we have:

$$\dot{x}_1 = -r_1 \sin\theta_1 \dot{\theta}_1 + \cos\theta_1 \dot{r}_1 = -\omega_1 r_1 \sin\theta_1$$
$$\dot{x}_2 = r_1 \cos\theta_1 \dot{\theta}_1 + \sin\theta_1 \dot{r}_1 = \omega_1 r_1 \cos\theta_1$$

To solve for $\dot{\theta}_1$ and $\dot{r}_1$, apply Cramer's method to the above. Form the determinant of the coefficient matrix in which the first column is replaced by the RHS and divide that by the determinant of the coefficient matrix.

$$\dot{\theta}_1 = \frac{\begin{vmatrix} -\omega_1 r_1 \sin\theta_1 & \cos\theta_1 \\ \omega_1 r_1 \cos\theta_1 & \sin\theta_1 \end{vmatrix}}{\begin{vmatrix} -r_1 \sin\theta_1 & \cos\theta_1 \\ r_1 \cos\theta_1 & \sin\theta_1 \end{vmatrix}} =$$

$$= \frac{-\omega_1 r_1 \sin^2\theta_1 - \omega_1 r_1 \cos^2\theta_1}{-r_1 \sin^2\theta_1 - r_1 \cos^2\theta_1} = \frac{-\omega_1 r_1}{-r_1} = \omega_1$$

Repeating Cramer's method to solve for $\dot{r}_1$:

$$\dot{r}_1 = \frac{\begin{vmatrix} -r_1 \sin\theta_1 & -\omega_1 r_1 \sin\theta_1 \\ r_1 \cos\theta_1 & \omega_1 r_1 \cos\theta_1 \end{vmatrix}}{-r_1} =$$

$$= \frac{-\omega_1 r_1^2 \cos\theta_1 \sin\theta_1 + \omega_1 r_1^2 \cos\theta_1 \sin\theta_1}{-r_1} = 0$$

we find that the radius is constant, as was to be expected since the damping coefficient is zero.

4.12

i. The trace of the Jacobian matrix

$$\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z}$$

is equal to $(-\sigma - 1 - b) < 0$ and constant (it is not dependent on the state of the system). Therefore, for any three-dimensional volume V in $\mathbb{R}^3$, $dV(t)/dt < 0$. (Cf. appendix to chapter 4, pp. 128-129, especially equation (4.17).)

ii. See chapter 6, p. 186.

chapter 5

5.1 Hint: the result depends on the continuity of the functions f and g and on the fact that the eigenvalues of a matrix are continuous functions of its elements.

5.2 If $f(x;\mu) = xF(x;\mu)$ then

$$\frac{\partial^2 f(\bar{x};\mu_c)}{\partial x \partial \mu} = \frac{\partial F(\bar{x};\mu_c)}{\partial \mu} \neq 0.$$

Given the last relation, the implicit function theorem guarantees the existence of a function $\mu(x)$ defined sufficiently near $\bar{x}$, such that $F\big(x;\mu(x)\big)=0$ implies $\dot{x}=0$. In order to ensure that the curve of fixed points $\mu(x)$ does not coincide with $x=0$, we need to prove that

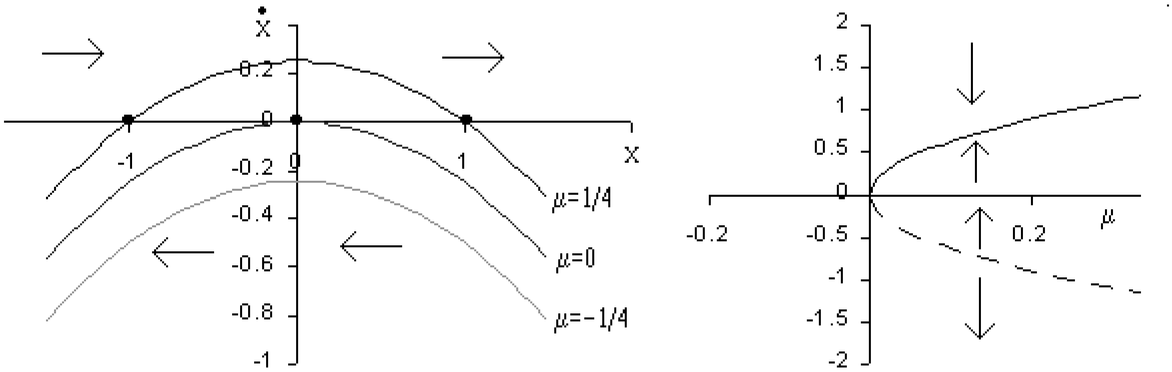
$$0<\left|\frac{d\mu(\bar{x})}{dx}\right|<\infty.$$

This is guaranteed if conditions (ii) and (iii') hold, since

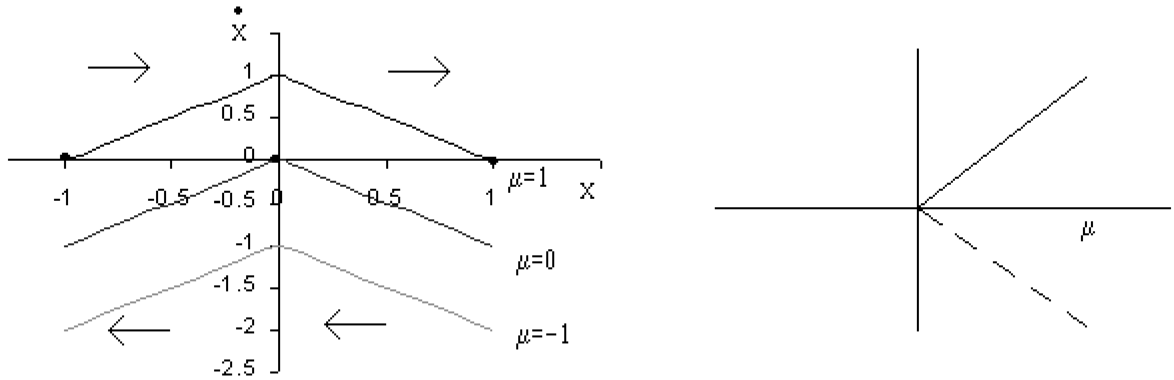
$$\frac{d\mu(\bar{x})}{dx}=-\frac{\partial F(\bar{x};\mu_c)/\partial x}{\partial F(\bar{x};\mu_c)/\partial \mu}=-\frac{\partial^2 f(\bar{x};\mu_c)/\partial x^2}{\partial^2 f(\bar{x};\mu_c)/\partial x\partial\mu}.$$

5.3

(a)



(c)



5.4

(a) The fixed points are $\bar{x}_{1,2} = \pm 2\sqrt{\mu}$ which are real for $\mu \geq 0$, $\bar{x}_1$ stable, $\bar{x}_2$ unstable. The eigenvalue becomes 0 at $x = 0$, that is for $\mu = 0$, the fixed points coincide and are nonhyperbolic. For this equation we have

$$(i) \qquad \frac{\partial f(0;0)}{\partial x} = 0$$

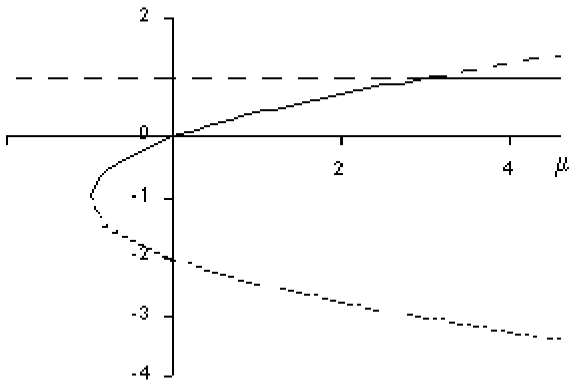
$$(ii) \qquad \frac{\partial^2 f(0;0)}{\partial x^2} = -\frac{1}{2} \neq 0$$

$$(iii) \qquad \frac{\partial f(0;0)}{\partial \mu} = 1 \neq 0.$$

and the conditions for a discontinuous fold bifurcation at $(\bar{x};\mu_c) = (0;0)$ are satisfied.

(c) The real fixed points are $\bar{x}_{1,2} = \pm\mu$, for $\mu > 0$, with $\bar{x}_1$ stable, $\bar{x}_2$ unstable. For $\mu = 0$, the only fixed point is $\bar{x} = 0$ but $\frac{\partial f}{\partial x}(0;0)$ is not defined.

5.6



5.7 There is a unique equilibrium at the origin. The Jacobian matrix calculated at that fixed point has eigenvalues

$$\lambda_{1,2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

which are complex for $|\mu| < 2$. Moreover, at $\mu = 0$ the real part is null and the eigenvalues are imaginary, that is the fixed point becomes nonhyperbolic and for $\mu > \mu_c$ the real part of the eigenvalues is positive. There are, of course, no other eigenvalues. We have $d\mathrm{Re}(\lambda)/d\mu = 1/2 \neq 0$ and there exists a Hopf bifurcation at $(\bar{x}, \bar{y}; \mu_c) = (0, 0; 0)$. Near the bifurcation value of μ there is a family of periodic solutions. For this system we can easily find a Lyapunov function to demonstrate the stability of the emerging periodic solutions. Let

$$V(x,y) = \frac{1}{2}(x^2 + y^2)$$

thus satisfying the first two conditions of such a function. There remains to study the time derivative of the Lyapunov function,

$$\dot{V}(x,y) = x\dot{x} + y\dot{y} = kx^2(x^2 + y^2) + \mu y^2.$$

At the bifurcation value $\mu = 0$ we have

$$\dot{V}(x,y) = kx^2(x^2 + y^2) < 0 \text{ if } k < 0.$$

Then for $k < 0$ and $\mu_c = 0$ the origin (a cycle of radius 0, as it were) is stable and so are the limit cycles that emerge after the bifurcation, and the Hopf bifurcation is supercritical. If $k > 0$ the bifurcation is subcritical, the limit cycles are unstable and are not observable, but serve as stability thresholds.

5.8

(c) The fixed points are $\bar{x}_{1,2} = \pm\sqrt{\mu}$. At $\mu = 0$ they coincide and are nonhyperbolic with an eigenvalue of 1. For $0 < \mu < 1$ the fixed points on the branch $\bar{x}_1$ are real and stable, for $\mu > 1$ they are unstable, at $\mu = 1$ the eigenvalue is -1 and $\bar{x}_1$ is nonhyperbolic. The fixed points on the branch $\bar{x}_2$ are real and unstable for $\mu > 0$. These considerations suggest a fold bifurcation at $(0;0)$ and a flip bifurcation at $(1;1)$. The conditions for a fold bifurcation at $(\bar{x};\mu_c) = (0;0)$

(*i*) $$\frac{\partial G(0;0)}{\partial x_n} = 1$$

(*ii*) $$\frac{\partial^2 G(0;0)}{\partial x_n^2} = -2 \neq 0$$

(*iii*) $$\frac{\partial G(0,;0)}{\partial \mu} = 1 \neq 0$$

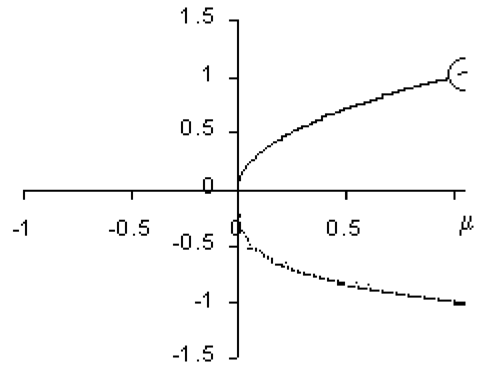
are satisfied. The conditions for a flip bifurcation at $(\bar{x};\mu_c) = (1;1)$ are satisfiec:

(*i′*) $$\frac{\partial G(1;1)}{\partial x_n} = -1$$

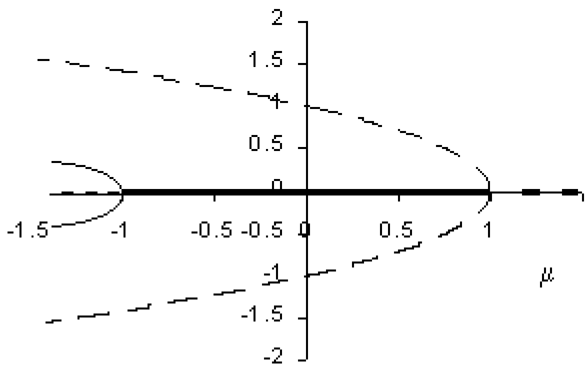
(*ii′*) $$\frac{\partial^2 G^2(1;1)}{\partial x_n^2} = 0 \quad \text{and} \quad \frac{\partial^3 G^2(1;1)}{\partial x_n^3} = -12 \neq 0$$

(*iii′*) $$\frac{\partial G^2(1;1)}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial^2 G^2(1;1)}{\partial \mu \partial x_n} = 2 \neq 0.$$

That is, the fixed point in nonhyperbolic with the eigenvalue of the linearised system at -1 and the conditions for a pitchfork bifurcation of G^2 are satisfied. For values of $\mu > 1$ the fixed points $\bar{x}_1$ are unstable, a period two cycle appears (and is stable) so that the pitchfork of G^2 is supercritical.



(d) The fixed points are $\bar{x}_1 = 0$, $\bar{x}_{2,3} = \pm\sqrt{1-\mu}$. The value of the first is independent of μ, is stable for $-1 < \mu < 1$, becomes nonhyperbolic at $\mu = 1$ and at $\mu = -1$, is unstable for $\mu > 1$ and $\mu < -1$. The other two equilibria are real and unstable for $\mu \leq 1$, nonhyperbolic for $\mu = 1$, complex for $\mu > 1$. These considerations suggest that a pitchfork bifurcation occurs at $(\bar{x}; \mu_c) = (0; 1)$ and a flip bifurcation occurs at $(\bar{x}; \mu_c) = (0; -1)$.



As regards the pitchfork bifurcation, the conditions

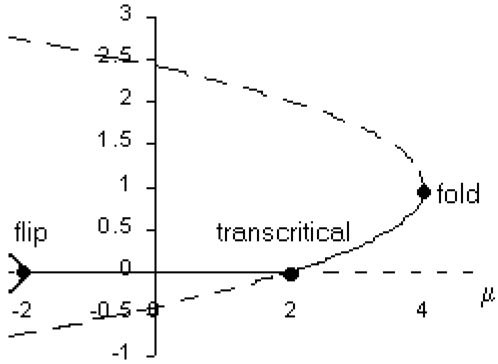
(i) $$\frac{\partial G(0;1)}{\partial x_n} = 1$$

(ii') $$\frac{\partial^2 G(0;1)}{\partial x_n^2} = 0 \quad \text{and} \quad \frac{\partial^3 G(0;1)}{\partial x_n^3} = 6 \neq 0$$

(iii') $$\frac{\partial G(0;1)}{\partial \mu} = 0 \quad \text{and} \quad \frac{\partial^2 G(0;1)}{\partial \mu \partial x_n} = 1 \neq 0.$$

are satisfied. The fixed points $\bar{x}_{2,3}$ exist for values of μ below the bifurcation value and the bifurcation is subcritical. G^2 is too long to calculate but numerical simulations confirm the flip bifurcation occurs at $(0;-1)$

5.9



5.11 Apart from the uninteresting trivial equilibrium $E_1 = (0,0)$, which is an unstable saddle point, there is a second equilibrium $(\bar{c}, \bar{l})$ at

$$E_2 = \left(\left(\frac{b-1}{b}\right)^{\frac{\mu}{\mu-1}}, \left(\frac{b-1}{b}\right)^{\frac{1}{\mu-1}} \right)$$

which is in the first quadrant under the assumptions. The Jacobian matrix calculated at the non–trivial fixed point is

$$DG\mid_{E_2} = \begin{pmatrix} 0 & \mu\frac{b-1}{b} \\ -b & b \end{pmatrix}$$

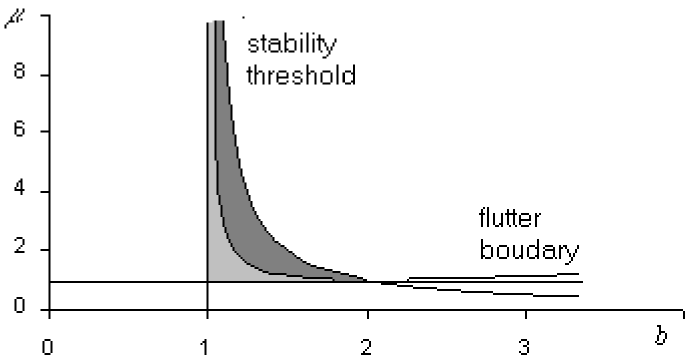
and the stability of this equilibrium is studied through the stability inequalities:

(i) $\quad 1 + b + (b-1)\mu > 0$

(ii) $\quad 1 - b + (b-1)\mu > 0$

(iii) $\quad 1 - (b-1)\mu > 0.$

Condition (i) is always satisfied eliminating the possibility of a flip bifurcation. Condition (ii) is also satisfied, under the hypotheses, and there will not be a fold bifurcation. Condition (iii) may or may not be satisfied, so that stability of fixed point E_2 depends on what specific values the parameters take on. The stability threshold, $1-(b-1)\mu = 0$, is drawn in the parameter space (b, μ) represented below. A Neimark bifurcation requires that the modulus of a pair of complex eigenvalues increases through 1. The **flutter condition** is the frontier between real and complex eigenvalues, which for E_2 is at $-4\mu + b^2/(b-1) = 0$, the second curve in the parameter space.



If b and μ are in the area bounded by $b = 1$, $\mu = 1$ and the flutter boundary, eigenvalues are real and less than 1 in absolute value (transient motion is monotonic towards the fixed point). In the parameter subspace between the flutter boundary and the stability threshold eigenvalues are complex with modulus less than 1 (transient motion is damped oscillations towards the fixed point). Nonhyperbolic fixed points are those on the curve representing the stability threshold. Then increasing either parameter, while holding the other constant, leads to first crossing the flutter condition and then the stability condition. There are no other eigenvalues and the first condition for a Neimark bifurcation is satisfied. Invariant circles, found for parameter values just to the right of the stability frontier, may be attracting or not, and the dynamics may be periodic or quasi–periodic. Numerical simulations suggest that, in this case the invariant circle is attracting and there are both types of cyclic behavior, depending on parameter values.

chapter 6

6.2

$$\sum_{i=-\infty}^{+\infty} \frac{|s_i - \bar{s}_i|}{2^{|i|}} \leq \sum_{i=-\infty}^{+\infty} \frac{1}{2^{|i|}} = 1 + 2R$$

$$R = \sum_{i=-\infty}^{1} \frac{1}{2^{|i|}} = \sum_{1}^{+\infty} \frac{1}{2^{|i|}}.$$

Writing R as a geometric series, we find that $R = 1$, whence the result.

6.4

(a) The distance between s and $\bar{s}$ will be

$$\bar{d}(s, \bar{s}) = \sum_{i=-\infty}^{\infty} = \frac{1}{2^0} + 2\sum_{i=1}^{\infty} 2^{-i}$$

where the infinite sum

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots=\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots=\frac{1}{1-\frac{1}{2}}-1=1.$$

Then this distance is the maximum distance between the two bi-infinte sequences and $\bar{d}=3$.

(c) The distances are $\bar{d}(\bar{s},\hat{s})=1/2$; $\bar{d}(\bar{s},\tilde{s})=3/2$; $\bar{d}(\hat{s},\tilde{s})=1$.

6.5

(a) The last number in the second part of the exercise should be $\frac{1}{128}$ so that the second question should reac:

$$d[T_+^n(s^*),s]\leq\frac{1}{128}.$$

The sequences agree in the first five places ($k=5$) after $n=10$ iterations, they agree in the first seven places ($k=7$) after $n=34$ iterations.

(b) The sequence $T_+^2(s^*)$

00 01 10 11 000 001 010 011 100 101 110 111 0000 0001

and the sequence s differ in the first element after the following iterations of the one-sided shift map $n=3$, 4, 6, 7, 12, 14, 17, 18, 19, 22.

6.7 and 6.8. To show conjugacy of the maps

$$\begin{array}{ll} F:[-1,1]\to[-1,1] & F(z)=2z^2-1 \\ G_4:[0,1]\to[0,1] & G(x)=4x(1-x) \\ G_\Lambda:[0,1]\to[0,1] & G_\Lambda(y)=\begin{cases} 2y, & \text{if } 0\leq y\leq\frac{1}{2}, \\ 2(1-y), & \text{if } \frac{1}{2}<y\leq 1. \end{cases} \end{array}$$

Consider the following diagram

$$\begin{array}{ccc} [0,\ 1] & \xrightarrow{G_\Lambda} & [0,\ 1] \\ {\scriptstyle x=h_1(y)}\downarrow & & \downarrow{\scriptstyle x=h_1(y)} \\ [0,\ 1] & \xrightarrow[G_4]{} & [0,\ 1] \\ {\scriptstyle z=h_2(x)}\downarrow & & \downarrow{\scriptstyle z=h_2(x)} \\ [-1,1] & \xrightarrow[F]{} & [-1,1] \end{array}$$

where

$$h_1(y) = \sin^2\left(\frac{\pi}{2}y\right) \qquad h_2(x) = 1 - 2x.$$

We know already that G_4 and G_Λ are topologically conjugate via the homeomorphism h_1 (see chapter 6, section 6.5, remark 6.7) and therefore we have

$$G_4 \circ h_1 = h_1 \circ G_\Lambda. \tag{1}$$

We can see that F and G_4 are topologically conjugate via the homeomorphism $h_2 : [0,1] \to [-1,1]$, and therefore

$$F \circ h_2 = h_2 \circ G_4. \tag{2}$$

The reader can verify that, for any $x \in [0,1]$

$$F[h_2(x)] = h_2[G_4(x)] \quad \text{or} \quad 2(1-2x)^2 - 1 = 1 - 2[4x(1-x)].$$

Finally, consider the homeomorphism $h \equiv h_2 \circ h_1 : [0,1] \to [-1,1]$. From (1) and (2) we have

$$F \circ h_2 \circ h_1 = h_2 \circ G_4 \circ h_1 = h_2 \circ h_1 \circ G_\Lambda$$

or $G \circ h = h \circ G_\Lambda$, which establishes the conjugacy between F and G_Λ.

6.9 (a) We need to demonstrate that $h \circ G_\Lambda = G_4 \circ h$, using the homeomorphism $h(x) = \sin^2\left(\frac{\pi x}{2}\right)$ (for $h[0,1]$ the inverse in continuous). For the LHS we have

$$\begin{aligned} 0 \le x \le \frac{1}{2} \quad & \sin^2\left(\frac{\pi}{2}2x\right) = \sin^2(\pi x) \\ \frac{1}{2} < x \le 1 \quad & \sin^2\left(\frac{\pi}{2}(2-2x)\right) = \sin^2(\pi - \pi x) = \sin^2(\pi x). \end{aligned}$$

For the RHS we have

$$4\sin^2\frac{\pi x}{2}\left(1 - \sin^2\frac{\pi x}{2}\right) = 4\sin^2\frac{\pi x}{2}\cos^2\frac{\pi x}{2} = \sin^2(\pi x).$$

(c) The map $h(x) = \frac{-\mu x}{a} + \frac{\mu - b}{2a}$, $a \neq 0$ is continuous and has a continous inverse over $[0,1]$. We need to demonstrate that $h \circ G_\mu = F \circ h$, that is,

$$\frac{-\mu\big(\mu[x(1-x)]\big)}{a} + \frac{\mu - b}{2a} = a\left(\frac{-\mu x}{a} + \frac{\mu - b}{2a}\right)^2 + b\left(\frac{-\mu x}{a} + \frac{\mu - b}{2a}\right) + c.$$

After a few manipulations we have that the logistic map is h-conjugate to F if the constant

$$c = \frac{b^2 + 2(\mu - b) - \mu^2}{4a}.$$

6.10

1. Exact solution of (i) as a function of t.

Write (i) as

$$\frac{dx}{dt} = 4x(1-x) \quad \text{whence} \quad \int \frac{dx}{4x-4x^2} = \int dt$$

and

$$\frac{1}{4} \ln \frac{x}{4x-4} = t + c$$

which is well-defined for $x \in (0,1)$ and where c is a constant of integration. Solving for c in terms of the initial condition $x_0 = x(0)$ we have

$$c = \frac{1}{4} \ln \frac{x_0}{4x_0 - 4}$$

and, after some elementary manipulations, we obtain x as a function of t and x as

$$x(t) = \frac{e^{4t}x_0}{e^{4t}x_0 - x_0 + 1}. \tag{1}$$

Differentiating (1) with respect to t, the reader can verify that $x(t)$ actually solves (i) for all $x \in (0,1)$. The solutions for $x_0 = 0$ and $x_0 = 1$ are trivial, namely $x(t;x_0) = x(t;0) = 0 \ \forall t$ and $x(t;x_0) = x(t;1) = 1 \ \forall t$. Notice that, as $t \to +\infty$, $x(t) \to 1$, the only stable fixed point.

2. From (ii), with a slight change in notation, we have

$$\frac{x(t+T) - x(t)}{T} = f\big(x(t)\big).$$

Taking the limit for $T \to 0$ and recalling the definition of derivative, we have

$$\lim_{T \to 0} \frac{x(t+T) - x(t)}{T} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} = \frac{dx(t)}{dt} = \dot{x} = f\big(x(t)\big)$$

where $T = \delta t$.

3. Hint: consider the map

$$G(x) = x + T4x(1-x)$$

with fixed points $\bar{x}_1 = 0$; $\bar{x}_2 = 1$. For any $T > 0$ $\bar{x}_1$ is unstable, while $\bar{x}_2$ is stable for values of $T < 1/2$. At $T = 1/2$, $G'(1) = -1$ and a flip bifurcation occurs. Find the two fixed points of the second iterate of G, G^2, different from 0 and 1, and calculate the value of T for which $(G^2)'(x)$, evaluated at either of the two fixed points, is equal to -1.

6.11

(a) $x_{n+1} = \sin^2(2^{n+1}\pi\theta) = 4\sin^2(2^n\pi\theta)\cos^2(2^n\pi\theta) = 4x_n(1-x_n)$

(b) If m is the Lebesgue measure $m\{\theta \in [0,1] | \theta \text{ is irrational}\} = 1$.

(c) The set $R = \{\theta \in [0,1] | \theta \text{ is rational }\}$ is dense in $[0,1]$.

chapter 7

7.1 By definition

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \ln|G'(x_j)| \quad G(x) = 2x \quad G_D(x) = 2x \bmod 1$$

and, since $G' = 2$ and $G'_D = 2$ both maps have an LCE of $\lambda(x_0) = \ln 2$. The difference is that the orbits of G_D are bounded and so, with positive LCE, chaotic, while the orbits of G are simply unstable and diverge to $\pm\infty$.

7.2 Using equation 5.28 we get the two periodic points of G $\{x_0^*, x_1^*\}$, at $\mu = 3.2$, as $\{3/4, 9/16\}$. Then using the results following 5.28 we have

$$\kappa(\bar{x}_3) = \kappa(\bar{x}_4) = \frac{\partial G^2(\bar{x}_3;\mu)}{\partial x_n} = -\mu^2 + 2\mu + 4 = 0.16.$$

The periodic orbit is therefore locally attracting. We use the derivative of G with respect to x, $3.2 - 6.4x$, to calculate the LCE for the orbit as

$$\begin{aligned}\lambda(x_0) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1} \ln|1.6| + \ln|-0.4| + \ln|1.6| + \cdots \\ &= \frac{1}{n}\left(\frac{n}{2}\ln 1.6 + \frac{n}{2}\ln 0.4\right) = \frac{1}{2}\ln 0.64 \approx -0.223.\end{aligned}$$

7.4 Notice that the first element in the vector on the RHS should be $\sqrt{3}+x$, rather than $1+x$, and the map should reac:

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{3}+x & \bmod 1 \\ \sqrt{2}+y & \bmod 1 \end{pmatrix}.$$

(a) The LCE of a typical orbit is 0. Hint: use the fact that the matrix

$$DG = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is independent of the state of the system.

(b) Topological transitivity follows from the fact that $\sqrt{2}$ and $\sqrt{3}$ are rationally independent. If, in the first of two equations of the map G we had left 1 instead of $\sqrt{3}$, the T^2 torus would be split into invariant circles, each corresponding to a constant value of x and orbits of G would be dense on each circle (cf. Katok and Hasselblatt (1995), Section 1.4, pp. 28-31).

7.5 Consider that

$$G'(x) = \begin{cases} \frac{2-a}{(1-ax)^2} & \text{if } 0 \le x \le \frac{1}{2}, \\ \frac{2-a}{[1-a(1-x)]^2}, & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

and, for the chosen value of a, $|G'(x)| > 1$, wherever defined.

7.6 The capacity dimension for the Kock snowflake is

$$D = \lim_{\epsilon \to 0} \frac{\log[N(\epsilon)]}{\log(1/\epsilon)} = \frac{\ln 4}{\ln 3} \approx 1.262.$$

7.7 The perimeter after i iterations is $P_i = N_i \cdot L_i$. We have

$$\begin{array}{cc} N_0 = 3 & L_0 = 1 \\ N_1 = 4 \cdot 3 & L_1 = 3^{-1} \\ N_2 = 4^2 \cdot 3 & L_2 = 3^{-2} \\ \cdots & \cdots \\ N_i = 4^i \cdot 3 & L_i = 3^{-i} \end{array}$$

so that

$$\lim_{i \to \infty} P_i = \lim_{i \to \infty} \frac{4^i \cdot 3}{3^i} = \lim_{i \to \infty} \left(\frac{4}{3}\right)^i \cdot 3 = \infty.$$

7.8 The capacity dimension for the Sierpinski triangle is

$$D = \lim_{\epsilon \to 0} \frac{\log[N(\epsilon)]}{\log(1/\epsilon)} = \frac{\ln 3}{\ln 2} \approx 1.585.$$

7.9 Because the number of pairs is equal to $N(N-1) \to N^2$ as $N \to \infty$.

chapter 8

8.1 The Schwarzian derivative

$$S(G) = \frac{G'''(x)}{G'(x)} - \frac{3}{2}\left(\frac{G''(x)}{G'(x)}\right)^2 < 0 \quad \text{where} \quad G'''(x) = \frac{d^3G}{dx^3}$$

for $G(x_n) = \mu x_n(1 - x_n)$ is

$$S = -\frac{3}{2}\left(\frac{-2\mu}{\mu - 2\mu x}\right)^2 < 0$$

and for $G(x_n) = \mu - x_n^2$

$$S = -\frac{3}{2}\left(\frac{-2}{-2x}\right)^2 < 0.$$

8.2 For the logistic map with $\mu = 0.8$ all initial values in [0,1] are attracted to the fixed point at 0. For $\mu = 2$ we have most of the interval attracted to the fixed point at $(\mu - 1)/\mu$, that is, $\omega\{(0,1)\} = 0.5$, but $\omega(0) = \omega(1) = 0$. For $\mu = 3.2$ most of the interval is attracted to the period-2 cycle $\omega\{(0,1)\} = \{0.513, 0.799\}$ but we have $\omega(0) = \omega(1) = 0$. For $\mu = 4$ most of the interval is attracted to the chaotic attractor which spans the interval, but $\omega(0) = \omega(1) = \omega(1/2) = 0$.

8.3

(a) A period-32 cycle implies that there are also cycles of period 16, 8, 4 and 2.
(b) No, 2^{10} is listed after $2^3 \cdot 3$ and existence of the period-24 cycle is not implied by existence of a cycle of 2^{10} periods.
(c) Existence of a period-$2^2 \cdot 11$ cycle implies existence of the period-$2^3 \cdot 5$ cycle and the period-2^5 cycle, but not the $2^2 \cdot 3$ cycle.

chapter 9

9.1 Because of symmetry we have

$$\lambda = \int_{[0,1]} \frac{\ln|G_4'(x)|}{\pi[(x(1-x)]^{1/2}} dx = 2\int_{[0,1/2]} \frac{\ln|4 - 8x|}{\pi[x(1-x)]^{1/2}} dx.$$

Employing the variable change $x = \sin^2\left(\frac{\pi}{2}y\right)$ we have

$$dx = \left[\pi \sin\left(\frac{\pi}{2}y\right) \cos\left(\frac{\pi}{2}y\right)\right] dy$$

and

$$\lambda = 2\int_{[0,1/2]} \ln\left(4\left[1-2\sin^2\left(\frac{\pi}{2}y\right)\right]\right)dy = 2\int_{[0,1/2]} \ln[4\cos(\pi y)]dy.$$

Then

$$\begin{aligned}\lambda &= 2\left(\int_{[0,1/2]} \ln 4dy + \int_{[0,1/2]} \ln[\cos(\pi y)]dy\right)\\ &= \ln 4 + 2\int_{[0,1/2]} \ln[\cos(\pi y)]dy.\end{aligned}$$

We now want to prove that

$$I = \int_{[0,1/2]} \ln[\cos(\pi y)]dy = -\frac{1}{2}\ln 2.$$

Put $\pi y = \pi(\frac{1}{2} - \alpha)$, $\alpha \in [0, 1/2]$ where $dy = -d\alpha$. Substituting, we have

$$I = \int_{[0,1/2]} \ln\left[\cos\left(\frac{\pi}{2} - \pi\alpha\right)\right](-d\alpha) = \int_{[0,1/2]} \ln[\sin(\pi\alpha)]d\alpha$$

therefore

$$\begin{aligned}2I &= \int_{[0,1/2]} \ln[\cos(\pi y)]dy + \int_{[0,1/2]} \ln[\sin(\pi y)]dy\\ &= \int_{[0,1/2]} \ln[\cos(\pi y)\sin(\pi y)]dy\\ &= \int_{[0,1/2]} \ln\left[\sqrt{\cos^2(\pi y)\sin^2(\pi y)}\right]dy\\ &= \int_{[0,1/2]} \ln\left[\frac{\sin(2\pi y)}{2}\right]dy\\ &= \int_{[0,1/2]} \ln[\sin(2\pi y)]dy - \frac{1}{2}\ln 2\end{aligned}$$

and putting $\hat{y} = 2y$ $d\hat{y} = 2dy$

$$\begin{aligned}&= \int_{[0,1]} \ln[\sin(\pi\hat{y})]\frac{1}{2}d\hat{y} - \frac{1}{2}\ln 2\\ &= \frac{1}{2}\int_{[0,1]} \ln[\sin(\pi\hat{y})]d\hat{y} - \frac{1}{2}\ln 2\\ &= I - \frac{1}{2}\ln 2\end{aligned}$$

and we have

$$I = \frac{1}{2}\ln 2.$$

Finally,

$$\lambda = \ln 4 - 2\left(\frac{1}{2}\ln 2\right) = 2\ln 2 - \ln 2 = \ln 2.$$

9.3 Hint: show that the measure μ can be written as

$$\alpha\mu_1 + (1-\alpha)\mu_2$$

where $\alpha \in (0,1)$ and μ_1 and μ_2 are invariant measures.

9.4 Please notice the following correction: in the definition of the map T the T on the RHS must be replaced with G_Λ so that the definition should reac: Let T be the map

$$T : M \to M$$
$$T\big(\{x^i\}\big) = \{G_\Lambda(x^i)\} \quad i \geq 0$$

(a) First show that T is invertible. Suppose T is not invertible, that is, there exist sequences $\{y^i\}$, $\{z^i\}$ and $\{x^i\}$, $i \geq 0$ such that

$$T(\{y^i\}) = T(\{z^i\}) = \{x^i\}.$$

Then, from the definition of T, we must have

$$\begin{aligned}
G_\Lambda(y^0) &= G_\Lambda(z^0) \\
G_\Lambda(y^1) &= y^0 = G_\Lambda(z^1) = z^0 \\
G_\Lambda(y^2) &= y^1 = G_\Lambda(z^2) = z^1 \\
&\vdots \\
G_\Lambda(y^N) &= y^{N-1} = G_\Lambda(z^N) = z^{N-1} \\
&\vdots
\end{aligned}$$

whence

$$\{y^i\} = \{z^i\}.$$

To prove that μ is T-invariant, we must show that for any A, $\mu\big(T^{-1}(A)\big) = \mu(A)$.

Consider that for a sequence $\{x^i\} = (x^0, x^1, x^2, \dots)$

$$T^{-1}(\{x^i\}) = (x^1, x^2, x^3, \dots).$$

Hence

$$T^{-1}(A) = \{\{x^i\}_1^\infty \in M | x^{k+1} \in G_\Lambda^{-1}(I) \subset [0,1]\}.$$

But G_Λ preserves the Lebesgue measure m and therefore for any I

$$\mu\big(T^{-1}(A)\big) = m\big(G_\Lambda^{-1}(I)\big) = m(I) = \mu(A).$$

(b) Notice that the condition $x^0 \in I_0, \dots, x^r \in I_r$ is the same as

$$x^r \in \bar{I} = \big(I_r \cap G_\Lambda^{-1}(I_{r-1}) \cap \ldots \cap G_\Lambda^{-r+1}(I_1) \cap G_\Lambda^{-r} I_0\big)$$

or

$$G_\Lambda(x^r) \in I_{r-1}, G_\Lambda^2(x^r) \in I_{r-2}, \dots, G_\Lambda^{r-1}(x^r) \in I_1, G_\Lambda^r(x^r) \in I_0.$$

Hence, applying the definition of μ we get the result.

9.7 Hint.

First of all, the reader can easily verify that G preserves the Lebesgue measure m. Next, partition the state space into two disjoint subspaces $\{0 \leq x < 1/2\}$ and $\{1/2 \leq x < 1\}$ and call them respectively, H(ead) and T(ail).

Suppose now that a sequence of values of x generated by successive applications of the map G are observed, and that at each step we record the state of the system in one or the other of the two subspaces so that either $x \in H$ or $x \in T$. If we choose the initial value at random, the probability of H, given by

$$m\left(\{x \in [0,1) | \{0 \leq x < 1/2\}\right)$$

and the probability of T

$$m\left(\{x \in [0,1) | \{1/2 \leq x < 1\}\right)$$

are both equal to 0.5.

On the other hand, $G(x) \in H$ or $G(x) \in T$, according to whether $x \in G^{-1}(H)$ or $x \in G^{-1}(T)$, where

$$G^{-1}(H) = \{0 \leq x < 1/4\} \cup \{1/2 \leq x < 3/4\}$$
$$G^{-1}(T) = \{1/4 \leq x < 1/2\} \cup \{3/4 \leq x < 1\}$$

Thus the probability of a given sequence of two elements, say $\{T, H\}$ is

$$m\left\{x \in [0,1) | x_{i_0} \in T, x_{i_1} = G(x_{i_0}) \in H\right\}$$

or

$$m\left\{x \in [0,1) | x_{i_0} \in T \cap G^{-1}(H)\right\}$$

which the reader can verify is equal to 0.25. (Analogously, the probability is the same for any other sequence of 2 elements $\{H,H\}, \{H,T\}, \{T,T\}$.)

Generalising, if we consider one-sided infinite sequences of H and T, the probability that the k elements of a finite given subsequence have *given* values (H or T) is equal to 2^{-k}. But this is the same probability we have for repeated (independent) tosses of a fair coin with equal 0.5 probability of Head or Tail at each toss.

9.9 $\bar{x} = 1/2$; $x_0 = 0$; $\mu = \delta_0$.

9.10 Properties (1) and (2) of remark 1.2 clearly hold.
Property (3) (the triangle inequality) is

$$|z_1 - z_2| \leq |z_1 - z_3| + |z_3 - z_2|$$

for any three points z_i on S^1, as defined in remark 4.4 Recalling that

$$z_i = e^{i2\pi\theta_i} = \cos(2\pi\theta_i) + i\sin(2\pi\theta_i)$$
$$|\alpha + i\beta| = \sqrt{\alpha^2 + \beta^2}, \quad \cos^2\theta + \sin^2\theta = 1 \quad \forall\theta$$

we have

$$|z_i - z_j| = \sqrt{2\big(1 - [\cos(2\pi\theta_i)\cos(2\pi\theta_j) + \sin(2\pi\theta_i)\sin(2\pi\theta_j)]\big)}$$

for any pair i, j, $i \neq j$. On the other hand, for any point z_i on S^1 (the radius is one)

$$y_i = \sin(2\pi\theta_i) \qquad x_i = \cos(2\pi\theta_i)$$

where x_i and y_i are the Cartesian coordinates. Using the Euclidean distance, simple calculations show that the triangle inequality in S^1 coincides with the triangle inequality in $\mathbb{R}^2$, namely

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}.$$

As a matter of fact, the Euclidean metric on $\mathbb{R}^2$ and the metric $d(z_1, z_2) = |z_1 - z_2|$ on S^1 are (Lipschitz) equivalent, cf. Sutherland (1999), pp. 38-42.

9.11 In exercises 6.7-6.8 it was shown that the map $F(x) = 2x^2 - 1$ is topologically conjugate to the tent map G_Λ and therefore $F \circ h = h \circ G_\Lambda$, where $h : [0,1] \to [-1,1]$, $x = h(y) = 1 - 2\sin^2[\pi y/2]$. In view of this, for any $I \subset [-1,1]$, the sets

$$\{y \in [0,1] | F[h(y)] \in I\}$$
$$\{y \in [0,1] | h[G_\Lambda(y)] \in I\}$$

are the same and therefore $h^{-1}[F^{-1}(I)] = G_\Lambda^{-1}[h^{-1}(I)]$. We can now show that the probability measure $\mu = mh^{-1}$ (where m is the Lebesgue measure) is preserved by F. This requires that for any measurable subset $I \subset [-1, 1]$

$$\mu(I) = m[h^{-1}(I)] = m\{h^{-1}[F^{-1}(I)]\} = m\{G_\Lambda^{-1}[h^{-1}(I)]\}. \tag{1}$$

But this is true for any measurable subset $h^{-1}(I) \subset [0, 1]$ because G_Λ preserves the Lebesgue measure. The result in (1) also shows that F and G_Λ are isomorphic.

The still unspecified probability measure μ can be represented as

$$\mu(I) = \int_{h^{-1}(I)} dy = \int_I |\left(h^{-1}\right)'(x)| dx.$$

Considering that

$$\left(h^{-1}\right)'(x) = \frac{1}{h'[y(x)]}$$

and using the definition of $h(y)$, we have

$$\mu(I) = \int_I \frac{dx}{\pi[x(1-x)]^{1/2}}$$

that is, the same invariant measure as for the logistic map G_4.