Problems in

# QUANTUM FIELD THEORY

## Links with BOOK I

Although it is meant to be self-contained and can be used on its own, this collection of problems in quantum field theory has an obvious lineage with the textbook "*Quantum Field Theory, From Basics to Modern Topics (CUP, 2019, ISBN 9781108480901)*", thereafter referred to as "**Book I**". The purpose of this document is to highlight the connections between Book I and the present book, in particular by indicating for each problem the chapters of Book I for which it provides a natural extension. Chapter numbers such as I-1, I-2, ... refer to chapters in Book I.

First, let us indicate which problems of the present book may be interesting as an extension of the topics treated in the chapters of Book I (some problems may be relevant for more than one chapter of Book I). Note that, in some cases, a given problem is relevant for the physics topic of that chapter, but requires techniques developed in a later chapter to be solved (in that case, we also list the problem under that chapter).

**Chapter I-1:** 1,2,3,4,24

**Chapter I-2:** 4,5,6,7,51,64,77

**Chapter I-3:** 2,5,8,9,10,11,12,13,14,26,40,43,45

**Chapter I-4:** 15,22,26

**Chapter I-5:** 15,18,19,20,21,22,23,25,28,29,48,49,50

**Chapter I-6:** 27,38

**Chapter I-7:** 2,9,10,26,27,30,31,32,33,34,35,36,37,38,39,46,48,49,50,69,72

**Chapter I-8:** 5,11,12,37,38,40,41,42,43,44,45,48,49,50,51,52,61,62,63,64,67,68

**Chapter I-9:** 17,46,47

**Chapter I-10:** 16,17,47,83

**Chapter I-13:** 22,48,49,50

**Chapter I-14:** 51,52,53,55,56,57,58,59,60,61,62,63

**Chapter I-15:** 27,38

**Chapter I-16:** 22,25,28,29,69,70

**Chapter I-17:** 18,37,38,71,72,73,74,75,82,83

**Chapter I-18:** 12,76,77,78,79,80,81,82,83

**Chapter I-19:** 53,54,57,64,65,66,67,68

We now discuss each problem to place it in the context of Book I:

- **Problem 1** establishes a crucial relationship between the field operators $\phi$ (in the Heisenberg representation) and $\phi_{\rm in}$ (in the interaction representation), namely that the former obeys the interacting equation of motion if the latter obeys the free Klein–Gordon equation. This result, stated in Chapter I-1, is essential for setting up perturbation theory in QFT, since it allows to collect all the dependence on the interactions into evolution operators whose power expansion is straightforward.

- In **Problem 2**, we derive an explicit form of the elements of the *little group* for massless particles. This problem, closely related to the discussion of the little group in Chapter I-1, is best solved by using the exact Baker-Campbell-Hausdorff formula derived in Chapter I-7 (and thus it may also be viewed as a concrete application of the BCH formula in a context not related to Lie algebras). More importantly, the result is then used in **Problem 9** in order to show that, in a theory with massless spin-1 bosons, the Lorentz invariance of scattering amplitudes implies a property that may be viewed as a weak form of the Ward–Takahashi identity. This observation, due to Weinberg, is extended to gravity in **Problem 10**. These results complement in an interesting way the discussion of Abelian gauge theories in Chapter I-3, since they show that the structure of quantum field theories with massless bosons is strongly constrained by Lorentz invariance, with little leeway to construct anything but a gauge theory.

- **Problem 3** establishes some formal relationships between various equivalent expressions for the time evolution operator and the S-matrix, either in terms of the field $\phi_{\rm in}$ or in terms of the field $\phi_{\rm out}$. These results are then used in **Problem 4**, in which it is shown that the expression for the S-matrix as the time-ordered exponential of a local interaction term is to a large extent a direct consequence of causality. Both problems complement the foundational aspects of quantum field theory discussed in Chapter I-1.

- In **Problem 5**, we derive a set of conditions, known as the *Landau equations*, for a given loop integral to have infrared or collinear singularities. An explicit multi-loop integration is studied in **Problem 6**, which provides another point of view on these conditions. Both problems are related to Chapter I-2, that deals with perturbation theory. In addition, **Problem 5** is relevant in the discussion of the soft and collinear divergences that occur in the perturbative expansion of QED (Chapter I-3) and QCD (Chapter I-8).

- **Problem 7** establishes *Weinberg's convergence theorem* in the simple case of scalar field theory. This is a crucial result, stated without proof in Chapter I-2, for the discussion of renormalization since it clarifies the role of the superficial degree of divergence as a tool for assessing whether a particular diagram is ultraviolet divergent.

- The electron anomalous magnetic moment is calculated at one loop in **Problem 8**. This classic QED calculation of great historical importance, which is a straightforward application of the results of Chapter I-3, has now been pushed to five loops and provides one of the most precise agreements between theory and experiment in all of physics.

- **Problem 11** derives the *Lee–Nauenberg theorem*, an important result about soft and collinear singularities which states that such divergences are removed by summing transition probabilities over degenerate initial and final states, thereby providing a link between the finiteness of an observable and its practical measurability. This classic result is relevant both in QED (Chapter I-3) and QCD (Chapter I-8).

- In **Problem 12**, we discuss the external classical field approximation, thanks to which a heavy charged object may be replaced by its classical Coulomb field. As a consequence, the multiple interactions of a probe with this object may be handled by solving a classical equation of motion. This approximation has many applications in QED and QCD (Chapters I-3 and I-8), and it also plays an important role in the discussion of strong fields (Chapter I-18).

- **Problems 13** and **14** are devoted to a derivation of the *Low–Burnett–Kroll theorem*, a result that states that the emission probability of a soft photon is proportional to the probability of the underlying hard process, at the first two orders in the energy of the emitted photon. This result (and its extension to arbitrarily many emitted photons – see Chapter I-3) is very well known for the leading term, but the fact that this is also true for the next order is not. This lesser known extension is nevertheless essential, since it ensures that the non-factorizable terms are finite in the soft photon limit.

- *Coherent states* are introduced in **Problem 15** and their main properties established. Coherent states defined with a homogeneous background field play a role in the discussion of spontaneous symmetry breaking, in Chapter I-4. They also appear in Chapter I-18, where coherent states with a strong background field are discussed.

- **Problems 16** and **17** study the running coupling in a scalar field theory with two fields, and in a QCD-like theory at two-loop order (this problem is thus also relevant for Chapters I-8 and I-9). These two problems are applications of Chapter I-10, on the renormalization group. **Problem 16**, although somewhat academic, illustrates how certain symmetries may be restored at large/short distance, a phenomenon that one may calculate via the RG flow.

- **Problem 18** addresses the quantization of a theory in which the classical Hamiltonian is not separable, i.e., is not the sum of a function of momenta and a function of the coordinates. *Weyl quantization* is introduced and its main properties are studied, as well as its consequence on the path integral. This problem generalizes the derivation of the path integral of Chapter I-5, where the Hamiltonian was assumed separable for simplicity. It also introduces the concept of *Wigner transform*, which is reused in Chapter I-17.

- The Legendre transform that relates the quantum effective action $\Gamma$ to the generating functional of connected correlation functions can be obtained from the requirement that $\Gamma$ shall reproduce the full theory solely from trees. In **Problem 19**, we provide a combinatorial interpretation of the Legendre transform which illustrates its action at the level of Feynman graphs. Although this is not a full alternative to the functional approach of Chapter I-5, it provides some additional insight about what is going on at the diagrammatic level.

- **Problem 20** introduces the *coherent state path integral*, by which one may resolve the identity in terms of coherent states instead of states of fixed coordinates or fixed momenta. This formalism is used in **Problem 21** in an alternative, more intuitive, derivation of the bosonic path integral of Chapter I-5.

- In **Problem 22**, the concept of coherent state is extended to a spin, where it also has the interpretation of a "nearly classical state" (i.e., loosely speaking, an arrow pointing in some direction of three-dimensional space). A path integral expression is derived for the transition amplitude between two such spin coherent states, that exhibits the so-called *Berry phase*, or geometrical phase. Spin coherent states are relevant in the context of magnetic phase transitions, discussed in Chapter I-4.

- As an illustration of *zeta function regularization*, **Problem 23** derives a regularized expression of the determinant of the Klein–Gordon operator. This problem provides an illustration of a technique only briefly introduced in Chapter I-5.

- In **Problem 24**, we derive the *Casimir energy* between two parallel walls in scalar field theory by the traditional approach, which consists in computing the dependence of the zero-point vacuum energy on the separation between the walls. This approach requires only the techniques and knowledge of Chapter I-1, together with a careful handling of the zero-point energy. In **Problem 25**, we re-derive the Casimir energy as the free energy resulting from an explicit modeling of the field–wall interaction. This also provides a concrete application of zeta function regularization, introduced in Chapter I-5.

- The *Gross–Neveu model*, a model of spontaneous chiral symmetry breaking, is introduced in **Problem 26**. In addition to its phenomenological interest (which could complement the discussion of chiral symmetry in Chapters I-4 and I-7), this illustrates a standard technique called *bosonization*, by which a local fermionic interaction may be removed via the introduction of an extra auxiliary bosonic field.

- **Problem 27** derives the *D'Hoker–Gagné formula*, which provides a fermionic path integral representation for the trace of a Wilson line in $\mathrm{SU}(N)$. This illustrates the fact that path integrals may be used for ordering any type of non-commuting objects. This problem fits naturally in the context of Chapters I-6 and I-7, and has also applications in the worldline formalism discussed in Chapter I-15.

- In **Problem 28**, we introduce the framework of *stochastic quantization*, in which Euclidean correlation functions may be obtained as noise averages of products of fields that obey a Langevin equation with Gaussian white noise. These ideas are explored further in **Problem 29**, in the case of a non-Euclidean toy model, in order to introduce the concept (and problems) of *complex Langevin equations*. These ideas complement naturally the discussion of functional quantization of Chapter I-5, and are also related to some aspects of lattice field theory (Chapter I-16).

- **Problem 30** derives some basic identities obeyed by the generators of the fundamental representation of $\mathfrak{su}(n)$, and by the structure constants $f^{abc}$ and $d^{abc}$. More elaborate identities are obtained in **Problems 31** and **32**. These problems are standard applications of the properties of Lie algebras (Chapter I-7), in the important case of $\mathfrak{su}(n)$ which is relevant for QCD (Chapter I-8).
- In **Problem 33**, which may be used as practice for Chapter I-7, we prove an identity obeyed by the generators of $\mathfrak{su}(2)$, which plays a role in calculations related to coherent spin states.
- **Problems 34** and **35** explore how Wilson loops and Wilson lines change when the contour on which they are defined is deformed in various ways. This wisdom is then applied in **Problem 36** in order to obtain the *Makeenko–Migdal loop equations*. These problems complement the introduction of Wilson lines and Wilson loops of Chapter I-7.
- In **Problem 37**, that may complement Chapters I-7 and I-8, we introduce the concept of a point-like classical particle carrying a non-Abelian charge, and derive the *Wong equations* that describe how such a charge evolves in an external gauge potential. These equations are obtained from a fermionic worldline integral (Chapters I-6 and I-15) in **Problem 38**. These ideas can be used to make the derivation of hard thermal loops (see Chapter I-17) more transparent and intuitive.
- The well-known Stokes's theorem relates the integral of a vector field along a closed path and the integral of its curl over a surface bounded by this path. The goal of **Problem 39** is to obtain a similar formula that relates a non-Abelian gauge potential and the corresponding field strength. This problem is closely related to the Wilson lines and Wilson loops introduced in Chapter I-7.
- In **Problem 40**, we consider for didactic purposes the quantization of scalar QED in a non-linear gauge, which leads to complications (non-linear gauge boson couplings, ghosts) quite similar to those encountered in the quantization of non-Abelian gauge theories (Chapter I-8), while keeping calculations simpler.
- How gauge invariant scattering amplitudes are obtained from the LSZ reduction formulas that contain gauge-dependent fields is discussed in **Problem 41**. This point is relevant for perturbation theory in non-Abelian gauge theories, discussed in Chapter I-8.
- **Problem 42** is devoted to studying the large-$N$ limit of $SU(N)$ Yang–Mills theory. It is shown that the leading contribution comes from the subset of planar graphs. This approximation, that naturally complements Chapter I-8, is often used as a toy model of Yang-Mills theory in which analytical calculations can be carried out much further than in the full theory.
- Soft gluon radiation from a quark–antiquark dipole is calculated in **Problem 43**, with emphasis on the antenna patterns of this emission. In **Problem 44**, we derive an auxiliary identity useful in this study. **Problem 45** is devoted to a derivation of the *Low–Burnett–Kroll theorem* in QCD in a simple case. These problems, that naturally belong to Chapter I-8, extend to the non-Abelian case the discussion of soft radiation of Chapter I-3.

- In **Problem 46**, we discuss quadratic and cubic operators that are invariant under global gauge transformations. This problem threads on some techniques developed in Chapter I-7, and its results play an important role in the discussion of renormalizability of Yang–Mills theory in Chapter I-9.

- **Problem 47**, relevant for both Chapters I-9 and I-10, studies some aspects of *Banks–Zaks fixed points*, which are perturbative infrared fixed points in asymptotically free theories.

- Several aspects of instantons are studied in **Problems 48**, **49** and **50**. They are first discussed in the simpler setting of ordinary quantum mechanics in **Problem 48**. Then, it is shown in **Problem 49** that in Yang–Mills theory instantons interpolate between pure gauges of different topological indices, and **Problem 50** is devoted to studying the ground states of Yang–Mills theories and their topological classification. These problems, which use functional methods (Chapter I-5), results about Lie groups and algebras (Chapter I-7) and some topological invariants discussed in Chapter I-13, complement Chapters I-7 and I-8 in a non-perturbative direction. In particular, they provide a physical interpretation for the term in $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$ that one may add to the Yang-Mills Lagrangian.

- **Problem 51** is devoted to counting the tree graphs that contribute to $n$-point amplitudes in various quantum field theories (scalar field theories, Yang–Mills theory). The result of this problem should be a good motivation for Chapter I-14, but it requires nothing more than the techniques exposed in Chapters I-2 and I-8 (and possibly I-5). In the Yang-Mills case, the method is extended to counting also color-ordered graphs. **Problem 64** generalizes this to one-loop graphs, and thus fits better in Chapter I-19.

- In **Problem 52**, we apply the BCFW recursion in order to obtain an expression with two terms only for the six-point $1^-2^-3^-4^+5^+6^+$ tree amplitude. The purpose of this problem is to illustrate the power of the techniques of Chapter I-14, but it could in principle (with considerably more effort) also be addressed with the standard perturbative techniques of Chapter I-8.

- Some frequently used identities for traces in the spinor-helicity formalism, relevant for Chapters I-14 and I-19, are derived in **Problem 53**.

- Various properties of polarization vectors in the spinor-helicity formalism are established in **Problems 54** and **55**, both useful in Chapters I-14 and I-19.

- **Problem 56** discusses the important *photon decoupling identity*, a relationship among color ordered amplitudes – introduced in Chapter I-14 – that reduces the number of independent ones.

- In **Problem 57**, we calculate some tree amplitudes with gluons and a pair of colored scalar and antiscalar particles. This problem uses the techniques of Chapter I-14 in order to depart from the purely gluonic case. These results will be used to derive some one-loop amplitudes by the generalized unitarity method in **Problems 67** and **68**.

- The color decomposition of tree amplitudes with gluons and a quark–antiquark pair is discussed in **Problem 58**. In **Problem 59**, we explain how to deal with massless quarks in the spinor-helicity formalism, and determine the expression for quark–antiquark–gluon amplitudes. Four- and five-point amplitudes containing a quark–antiquark pair are calculated in **Problem 60**. These problems show that the spinor-helicity framework introduced in Chapter I-14 is not limited to gluons, and they also give a glimpse of some simple supersymmetric relations that relate amplitudes that differ by a substitution $gg \to q\overline{q}$.

- In **Problem 61**, we apply the spinor-helicity formalism and the Parke–Taylor formula in order to derive the expression for the $gg \to gg$ tree-level scattering amplitude. This problem provides a very concrete application of the results of Chapter I-14. Note that this is a very classic QCD calculation that can also be done – with greater effort – using the standard perturbation theory of Chapter I-8. Along the way, we obtain the *Kleiss–Kuijf relation* for a very simple case.

- **Problem 62** studies the limit where a gluon becomes soft in an amplitude (this is the spinor-helicity analogue of the eikonal approximation), and in **Problem 63** the collinear limit is considered. The study of soft and collinear divergences in QCD is quite standard using the techniques of Chapter I-8, but the tools of Chapter I-14 used in this problem provide a convenient alternative to obtain the same results.

- One-loop amplitudes are complicated to calculate, even with the modern methods discussed in Chapter I-19 (even more so if one insists on analytical results). For this reason, we restrict ourselves to a few particularly simple yet non-trivial examples. In **Problems 65** and **66**, we first show that one-loop amplitudes with only gluons of positive helicity are ultraviolet finite and rational.

- In **Problem 67**, we use the generalized unitarity method to obtain a very simple expression for the $1^+2^+3^+4^+$ amplitude with a scalar loop. This is extended in **Problem 68** to the one-loop $1^+2^+3^+4^+5^+$ amplitude. These are among the simplest one-loop amplitudes one may calculate using the techniques of Chapter I-19.

- **Problem 69** derives the *Banks–Casher relation*, which relates the expectation value of the chiral condensate to the spectral density at null energy of the Dirac operator. This property is relevant in the discussion of chiral symmetry breaking in the QCD ground state (Chapter I-7), and also often appears in the context of lattice QCD (Chapter I-16).

- In **Problem 70**, that fits in Chapter I-16, we discuss the *Nielsen–Ninomiya theorem*, which states that massless fermions on a lattice must have equal numbers of right-handed and left-handed degrees of freedom. This obstruction prevents the formulation of lattice chiral gauge theories.

- In **Problem 71**, we determine the expression of the Wigner transform of the convolution product of two two-point functions. This result plays a role in the derivation of kinetic equations from the Kadanoff–Baym equations in Chapter I-17, and is also essential in the phase-space formulation of quantum mechanics.

- The interplay between *center symmetry* and the deconfinement transition in Yang–Mills theory is studied in **Problem 72**. Although the result of this problem is relevant for Chapter I-7, it probably fits better in Chapter I-17 because of the finite temperature aspects.

- The hard thermal loop contribution to the photon polarization tensor is derived in **Problem 73** using classical charged particles and a kinetic equation that describes the evolution of their distribution under the effect of an external electromagnetic field. This approach is extended in **Problem 74** to the gluon hard thermal loop. These problems provide an alternative to the standard perturbative calculation of the hard thermal loop 2-point function sketched in Chapter I-17.

- We study photon radiation by a hot plasma of charged particles in **Problem 75**. For soft photons, the spectrum is modified by the *Landau–Pomeranchuk–Migdal effect*. This calculation makes use of several concepts and tools introduced in Chapter I-17, namely hard thermal loops and kinetic equations.

- In **Problem 76**, that fits in Chapter I-18, we discuss how a microcanonical equilibrium distribution may emerge semi-classically from the non-linearity of the field equation of motion. The relationship between the equation of state and a microcanonical distribution is also studied.

- The combinatorial aspects of multi-particle production are studied in **Problem 77**. These are very generic properties, true no matter what the underlying quantum field theory is. We also derive a diagrammatic interpretation in terms of cutting rules for some of the quantities that appear in this discussion. This problem relates mostly to Chapter I-18 (although most of it may be solved without any result from that chapter), and marginally to Chapter I-2 in the final question that discusses cutting rules.

- In **Problem 78**, that threads on the techniques developed in Chapter I-18, these questions are made more quantitative, by studying a functional that generates the complete distribution of produced particles. From this, we determine the first moment of the distribution in terms of classical fields.

- The next two problems are devoted to the study of the *Schwinger mechanism* in scalar QED. In **Problem 79**, we follow the approach of **Problem 78** to calculate the functional that generates the full distribution of produced particles. These results are rederived in **Problem 80** using *Bogoliubov transformations*. These problems fit well with Chapter I-18 from a technical standpoint, but the result is interesting to anyone studying QED.

- In **Problem 81**, we use Bogoliubov transformations to relate the vacuum states of two observers whose relative motion is uniformly accelerated. This leads to the *Unruh effect*. By analogy, the same line of reasoning leads to *Hawking radiation*, which is a thermal spectrum seen by an observer at rest in the gravitational field of a black hole. This problem is loosely connected with Chapter I-18, but may be considered on its own after the concept of Bogoliubov transformation has been introduced (see, e.g., the preceding problem).

- **Problem 82** proposes a field theory approach to *Anderson localization*, namely the fact that electrons do not propagate in a disordered medium. In **Problem 83**, we present simple scaling arguments suggesting that strong localization always happens in large one- and two-dimensional systems, and depends on the disorder strength in three dimensions. Both problems are loosely connected with Chapters I-17 and I-18, and the latter is also an application of renormalization group techniques (Chapter I-10).