Reliability and Availability Engineering: Modeling, Analysis, Applications
Chapter 10 - Continuous Time Markov Chain: Reliability Models

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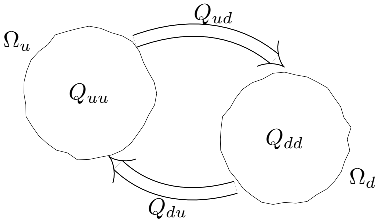
Dependability model

The main assumption in a dependability model is that the state space can be partitioned into a subset of up states and a subset of down states.

$$\Omega_u \quad \text{and} \quad \Omega_d \quad \text{such that} \quad \Omega_u \cup \Omega_d = \Omega$$

The states in Ω_u are the up states in which the structure function of the system is equal to 1, and the states in Ω_d are the down states in which the structure function of the system is equal to 0.

From the above, the infinitesimal generator matrix of the CTMC can be partitioned in the following way



$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_{uu} & \boldsymbol{Q}_{ud} \\ \boldsymbol{Q}_{du} & \boldsymbol{Q}_{dd} \end{bmatrix}$$

Reliability model

In an availability model both $\boldsymbol{Q}_{ud}$ and $\boldsymbol{Q}_{du}$ must have non-zero entries.

In a reliability model, the states in Ω_d are absorbing so that $\boldsymbol{Q}_{du}$ and $\boldsymbol{Q}_{dd}$ are zero matrices (matrices with all entries equal to 0).

The system reliability at time t is defined as the sum of the state probabilities at time t over the up states, and the unreliability at time t as the sum over the down state at time t.

$$R(t) = \sum_{i \in \Omega_u} \pi_i(t) \quad ; \quad F(t) = 1 - R(t) = \sum_{j \in \Omega_d} \pi_j(t)$$

For the reliability models only the transient solution is meaningful being trivial the steady state solution.

MRM for Reliability

To compute the reliability as reward measures, we need to define a reward structure for the model.

To do this, assign to each state $i \in \Omega$ a reward variable given by:

$$\begin{cases} r_i = 1 & \text{if} \quad i \in \Omega_u \\ r_i = 0 & \text{if} \quad i \in \Omega_d \end{cases}$$

In other words, r_i equals the value of the structure function of the system in state i.

Grouping the reward variables in vector $\boldsymbol{r}$, the system reliability can be rewritten in matrix form as the expected reward rate at time:

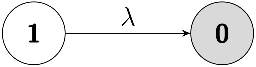
$$R(t) = \boldsymbol{\pi}(t)\, \boldsymbol{r}^T$$

This chapter thus concentrates on evaluating metrics for *non-irreducible* Markov chains.

A single component system (A two state CTMC)

The CTMC of the reliability model of this system has only two states.

In state 1 component is up, in state 0 is down. The transition rate from state 1 to state 0 is the failure rate λ.



From the problem specification, the transition rate matrix is:

$$Q_R = \begin{array}{c c} & \begin{array}{cc} 1 & 0 \end{array} \\ \begin{array}{c} 1 \\ 0 \end{array} & \left| \begin{array}{cc} 0 & \lambda \\ 0 & 0 \end{array} \right| \end{array}$$

Adjusting the diagonal entries, the infinitesimal generator is:

$$Q = \begin{array}{c c} & \begin{array}{cc} 1 & 0 \end{array} \\ \begin{array}{c} 1 \\ 0 \end{array} & \left| \begin{array}{cc} -\lambda & \lambda \\ 0 & 0 \end{array} \right| \end{array}$$

The Markov equation becomes:

$$\left[\frac{d\,\pi_1(t)}{d\,t}, \frac{d\,\pi_0(t)}{d\,t} \right] = (\pi_1(t), \pi_0(t)) \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$

A single component system (A two state CTMC)

Expanding the above Equation:

$$\begin{cases} \dfrac{d\,\pi_1(t)}{d\,t} & = & -\lambda\,\pi_1(t) \\ \dfrac{d\,\pi_0(t)}{d\,t} & = & \lambda\,\pi_1(t) \end{cases}$$

with initial probability $\boldsymbol{\pi}(0) = (1, 0)$

Solving, we obtain:

$$\begin{cases} \pi_1(t) & = & e^{-\lambda\,t} \\ \pi_0(t) & = & 1 - e^{-\lambda\,t} \end{cases}$$

The system reliability is: $R(t) = \pi_1(t) \quad F(t) = \pi_0(t) = 1 - R(t)$

The MTTF is: $\mathrm{MTTF} = \int_0^\infty R(t)dt = \int_0^\infty e^{-\lambda\,t}dt = 1/\lambda$

A single component system: Mass at origin

Suppose the system is initially in a failed state with some probability q.

We can then solve the above differential equations with the initial probability vector $\boldsymbol{\pi}(0) = (1 - q, q)$, to get:

$$\begin{cases} R(t) = \pi_1(t) = (1-q)e^{-\lambda t} \\ F(t) = \pi_0(t) = 1 - (1-q)e^{-\lambda t} \end{cases}$$

The reliability function starts at $1 - q$ at $t = 0$ and then decays to zero as t approaches infinity.

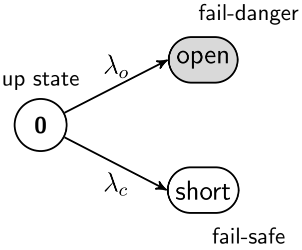
The unreliability (or the time to failure distribution) function has a mass at origin equal to q.

The MTTF in this case is:

$$\text{MTTF} = \int_0^\infty R(t)dt = \int_0^\infty (1-q)e^{-\lambda t}dt = (1-q)/\lambda$$

Defective distribution: Two failure modes

The Figure depicts a 3-state CTMC with 1 up state and two failure modes



$$\left[\frac{d\,\pi_u(t)}{d\,t}, \frac{d\,\pi_o(t)}{d\,t}, \frac{d\,\pi_c(t)}{d\,t}\right] = (\pi_u(t), \pi_o(t), \pi_c(t)) \begin{bmatrix} -(\lambda_o+\lambda_c) & \lambda_o & \lambda_c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

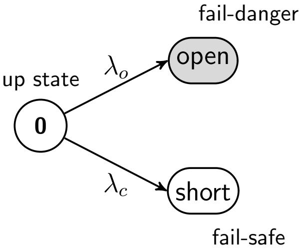
$$\pi_u(t) = e^{-(\lambda_o+\lambda_c)\,t}$$

$$\pi_o(t) = \frac{\lambda_o}{\lambda_o+\lambda_c} - \frac{\lambda_o}{\lambda_o+\lambda_c} e^{-(\lambda_o+\lambda_c)\,t}$$

$$\pi_c(t) = \frac{\lambda_c}{\lambda_o+\lambda_c} - \frac{\lambda_c}{\lambda_o+\lambda_c} e^{-(\lambda_o+\lambda_c)\,t}$$

Defective distribution: Two failure modes

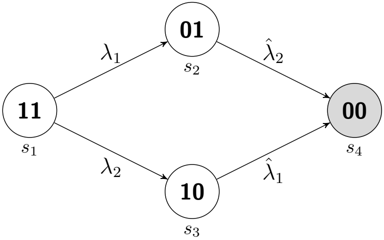
The Figure depicts a 3-state CTMC with 1 up state and two failure modes



The distribution of time to reach state o (starting with state u at time 0) is defective with a defect equal to $\frac{\lambda_c}{\lambda_o+\lambda_c}$.

Similarly, the time to reach state c (starting with state u at time 0) is defective with a defect equal to $\frac{\lambda_o}{\lambda_o+\lambda_c}$.

A two component system: dependent components



λ_1 is the failure rate of component 1 in state s_1 and $\hat{\lambda}_1$ ($\lambda_1 \neq \hat{\lambda}_1$) the failure rate of component 1 in state s_3.

λ_2 is the failure rate of component 2 in state s_1 and $\hat{\lambda}_2$ ($\lambda_2 \neq \hat{\lambda}_2$) the failure rate of component 2 in state s_2.

$$
\begin{array}{cc}
 & \begin{array}{cccc} (1,1) & (1,0) & (0,1) & (0,0) \end{array} \\
\boldsymbol{Q} = \begin{array}{c} (1,1) \\ (1,0) \\ (0,1) \\ (0,0) \end{array} & \left| \begin{array}{cccc}
-(\lambda_1 + \lambda_2) & \lambda_1 & \lambda_2 & 0 \\
0 & -\hat{\lambda}_2 & 0 & \hat{\lambda}_2 \\
0 & 0 & -\hat{\lambda}_1 & \hat{\lambda}_1 \\
0 & 0 & 0 & 0
\end{array} \right|
\end{array}
$$

A two component system: dependent components

$$\left[\frac{d\,\pi_1(t)}{d\,t}, \frac{d\,\pi_2(t)}{d\,t}, \frac{d\,\pi_3(t)}{d\,t}, \frac{d\,\pi_4(t)}{d\,t}\right] =$$

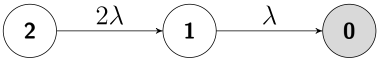
$$[\pi_1(t), \pi_2(t), \pi_3(t), \pi_4(t)] \begin{bmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & \lambda_2 & 0 \\ 0 & -\hat{\lambda}_2 & 0 & \hat{\lambda}_2 \\ 0 & 0 & -\hat{\lambda}_1 & \hat{\lambda}_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the following set of coupled differential equations is obtained:

$$\begin{cases} \dfrac{d\,\pi_1(t)}{d\,t} & = & -(\lambda_1 + \lambda_2)\,\pi_1(t) \\ \dfrac{d\,\pi_2(t)}{d\,t} & = & \lambda_1\,\pi_1(t) - \hat{\lambda}_2\,\pi_2(t) \\ \dfrac{d\,\pi_3(t)}{d\,t} & = & \lambda_2\,\pi_1(t) - \hat{\lambda}_1\,\pi_3(t) \\ \dfrac{d\,\pi_4(t)}{d\,t} & = & \hat{\lambda}_2\,\pi_2(t) + \hat{\lambda}_1\,\pi_3(t) \end{cases}$$

A two component system: Identical components

If the components are identical and independent, the CTMC state diagram can be simplified.



The label inside each state indicates the number of working components. Kolmogorov differential equations in this case become:

$$\left[\frac{d\,\pi_2(t)}{d\,t}, \frac{d\,\pi_1(t)}{d\,t}, \frac{d\,\pi_0(t)}{d\,t}\right] = [\pi_2(t), \pi_1(t), \pi_0(t)] \begin{bmatrix} -2\,\lambda & 2\,\lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

A two component system: Identical components

Assuming that State 2 is the initial state, we obtain, by direct integration:

$$\begin{cases} \pi_2(t) &= e^{-2\lambda t} \\ \pi_1(t) &= 2\,(e^{-\lambda t} - e^{-2\lambda t}) \\ \pi_0(t) &= 1 - \pi_1(t) - \pi_2(t) = 1 - 2e^{-\lambda t} + e^{-2\lambda t}\,. \end{cases} .$$

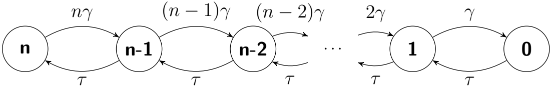
Check that the reliability is HYPO$(2\lambda, \lambda)$

$$R(t) = \pi_2(t) + \pi_1(t) = 2e^{-\lambda t} - e^{-2\lambda t}\,.$$

The same expression that has been obtained in the previous chapters since we have assumed independence across the two components.
The MTTF can be computed to yield:

$$\text{MTTF} = \int_0^\infty R(t)dt = \int_0^\infty [2e^{-\lambda t} - e^{-2\lambda t}]dt = \frac{1.5}{\lambda}\,.$$

Telecommunication Switching System Model



The transient probability of being in State i at time t, $\pi_i(t)$, can be computed by solving the following differential equations:

$$\begin{cases} \dfrac{d\pi_0(t)}{dt} = -\tau\pi_0(t) + \gamma\pi_1(t) \\ \dfrac{d\pi_k(t)}{dt} = -(\tau + k\gamma)\pi_k(t) + \tau\pi_{k-1}(t) + (k+1)\gamma\pi_{k+1}(t), \quad k = 1, 2, .., n-1 \\ \dfrac{d\pi_n(t)}{dt} = -n\gamma\pi_n(t) + \tau\pi_{n-1}(t)\,. \end{cases}$$

Assuming that the system is up as long as at least l trunks are functioning, the instantaneous availability $A(l,t)$ and the expected interval availability $A_I(l,t)$ are given as:

$$A(l,t) = \sum_{k=l}^{n} \pi_k(t) \qquad A_I(l,t) = \frac{\sum_{k=l}^{n} \int_0^t \pi_k(x)\,dx}{t}\,.$$

Convolution Integration Method

The Kolmogorov differential equations can be put in the form of coupled system of integral equations:

$$p_{ij}(t) = \delta_{ij} e^{q_{ii} t} + \int_0^t \sum_k p_{ik}(x) q_{kj} e^{q_{jj}(t-x)} dx$$

where $p_{ij}(t)$ are the entries of the transition probability matrix $\boldsymbol{P}(t)$, and δ_{ij} is the Kronecker delta function.

The above equation can be specialized to obtain the unconditional probabilities of states at time t:

$$\pi_j(t) = \pi_j(0) e^{q_{jj} t} + \int_0^t \sum_k \pi_k(x) q_{kj} e^{q_{jj}(t-x)} dx$$

These equations can be solved relatively easily for acyclic CTMCs and the method is known as the *convolution integration method*.

A two component system: Convolution integration

We apply the convolution integration method to the two component system model:

$$\pi_1(t) = \pi_1(0)e^{-(\lambda_1+\lambda_2)t} = e^{-(\lambda_1+\lambda_2)t}$$

$$\begin{aligned}
\pi_2(t) &= \pi_2(0)e^{-\hat{\lambda}_2\, t} + \int_0^t \pi_1(x)\lambda_1 e^{-\hat{\lambda}_2\,(t-x)}dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2 - \hat{\lambda}_2}\left(e^{-\,\hat{\lambda}_2\, t} - e^{-\,(\lambda_1+\lambda_2)\,t}\right)
\end{aligned}$$

$$\begin{aligned}
\pi_3(t) &= \frac{\lambda_2}{\lambda_1 + \lambda_2 - \hat{\lambda}_1}\left(e^{-\,\hat{\lambda}_1\, t} - e^{-\,(\lambda_1+\lambda_2)\,t}\right) \\
\pi_4(t) &= 1 - \pi_1(t) - \pi_2(t) - \pi_3(t).
\end{aligned}$$

Solution with Laplace Transforms

Indicating by $\mathcal{L}[f(t)] = f^*(s)$ the Laplace transform of a fuunction $f(t)$, the following relation holds:

$$\mathcal{L}\left[\frac{f(t)}{d\,t}\right] = f^*(s) - f(0)$$

By taking the Laplace transform of the Kolmogorov equation, we get:

$$\begin{cases} s\,\pi_1^*(s) - \pi_1(0) & = & \pi_1^*(s)\,q_{11} + \pi_2^*(s)\,q_{21} + \ldots + \pi_n^*(s)\,q_{n1} \\ s\,\pi_2^*(s) - \pi_2(0) & = & \pi_1^*(s)\,q_{12} + \pi_2^*(s)\,q_{22} + \ldots + \pi_n^*(s)\,q_{n2} \\ \ldots & & \ldots \end{cases}$$

After rearranging:

$$\begin{cases} \pi_1^*(s)\,(s - q_{11}) - \pi_2^*(s)\,q_{21} - \ldots - \pi_n^*(s)\,q_{n1} & = & \pi_1(0) \\ -\pi_1^*(s)\,q_{12} + \pi_2^*(s)\,(s - q_{22}) - \ldots - \pi_n^*(s)\,q_{n2} & = & \pi_2(0) \\ \ldots & & \ldots \end{cases}$$

in matrix form:

$$\boldsymbol{\pi}^*(s)\,(s\,\boldsymbol{I} - \boldsymbol{Q}) = \boldsymbol{\pi}(0) \qquad ; \qquad \boldsymbol{\pi}^*(s) = \boldsymbol{\pi}(0)\,(s\,\boldsymbol{I} - \boldsymbol{Q})^{-1}$$

A two component system: Laplace transform solution

$$
\begin{cases}
s\,\pi_1^*(s) - 1 & = & -(\lambda_1 + \lambda_2)\,\pi_1^*(s) \\
s\,\pi_2^*(s) & = & \lambda_1\,\pi_1^*(s) - \hat{\lambda}_2\,\pi_2^*(s) \\
s\,\pi_3^*(s) & = & \lambda_2\,\pi_1^*(s) - \hat{\lambda}_1\,\pi_3^*(s) \\
s\,\pi_4^*(s) & = & \hat{\lambda}_2\,\pi_2^*(s) + \hat{\lambda}_1\,\pi_3^*(s)
\end{cases}
$$

The symbolic solution in the transform domain is:

$$
\begin{cases}
\pi_1^*(s) & = & \dfrac{1}{s + \lambda_1 + \lambda_2} \\
\pi_2^*(s) & = & \dfrac{\lambda_1}{(s + \lambda_1 + \lambda_2)\,(s + \hat{\lambda}_2)} \\
\pi_3^*(s) & = & \dfrac{\lambda_2}{(s + \lambda_1 + \lambda_2)\,(s + \hat{\lambda}_1)} \\
\pi_4^*(s) & = & \dfrac{\lambda_1\hat{\lambda}_2(s + \hat{\lambda}_1) + \hat{\lambda}_1\lambda_2(s + \hat{\lambda}_2)}{s\,(s + \lambda_1 + \lambda_2)\,(s + \hat{\lambda}_1)(s + \hat{\lambda}_2)}
\end{cases}
$$

A two component system: time-domain solution

Using the partial fraction expansion:

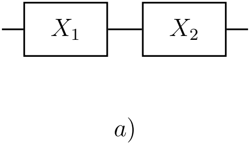
$$\begin{cases} \pi_1(t) & = & e^{-(\lambda_1 + \lambda_2)\, t} \\ \pi_2(t) & = & \dfrac{\lambda_1}{\lambda_1 + \lambda_2 - \hat{\lambda}_2} \left(e^{-\hat{\lambda}_2\, t} - e^{-(\lambda_1 + \lambda_2)\, t} \right) \\ \pi_3(t) & = & \dfrac{\lambda_2}{\lambda_1 + \lambda_2 - \hat{\lambda}_1} \left(e^{-\hat{\lambda}_1\, t} - e^{-(\lambda_1 + \lambda_2)\, t} \right) \\ \pi_4(t) & = & 1 - \pi_1(t) - \pi_2(t) - \pi_3(t) \end{cases}$$

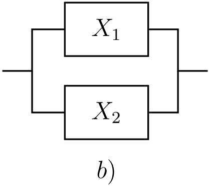
If we assume that the two components are statistically independent

$$\lambda_1 = \hat{\lambda}_1 \quad \text{and} \quad \lambda_2 = \hat{\lambda}_2$$

the above state probabilities can be expressed as the product of the individual probabilities of the single components.

Series/parallel system of two components





If the two components are connected in series, then:

$$R_{ser}(t) = \pi_1(t) = e^{-(\lambda_1 + \lambda_2)\, t}$$

If the components are connected in parallel, the only system down state is State 4, so that:

$$\begin{aligned} R_{par}(t) &= \pi_1(t) + \pi_2(t) + \pi_3(t) = \\ &= e^{-(\lambda_1+\lambda_2)\,t} + \frac{\lambda_1}{\lambda_1 + \lambda_2 - \hat{\lambda}_2}\left(e^{-\hat{\lambda}_2\, t} - e^{-(\lambda_1+\lambda_2)\,t}\right) \\ &+ \frac{\lambda_2}{\lambda_1 + \lambda_2 - \hat{\lambda}_1}\left(e^{-\hat{\lambda}_1\, t} - e^{-(\lambda_1+\lambda_2)\,t}\right). \end{aligned}$$

Series/parallel system of two components: MTTF

The above results can be obtained in terms of expected reward rate at time t from by assigning the following reward rate vectors:

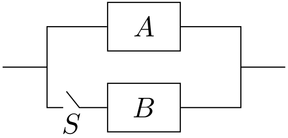
$$\boldsymbol{r}_{series} = [1,\, 0,\, 0,\, 0] \qquad \boldsymbol{r}_{par} = [1,\, 1,\, 1,\, 0]\,.$$

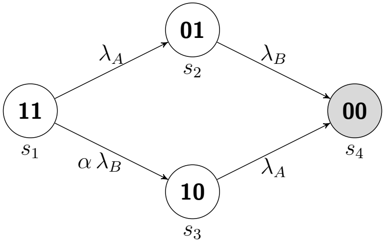
The MTTFs are:

$$\text{MTTF}_{ser} = \int_0^\infty R_{ser}(t)dt = \frac{1}{(\lambda_1 + \lambda_2)}\,,$$

$$\begin{aligned}\text{MTTF}_{par} &= \int_0^\infty R_{par}(t)dt \\ &= \frac{1}{(\lambda_1 + \lambda_2)} + \frac{\lambda_1}{\hat{\lambda}_2\,(\lambda_1 + \lambda_2)} + \frac{\lambda_2}{\hat{\lambda}_1\,(\lambda_1 + \lambda_2)}\,.\end{aligned}$$

Standby configuration





Based on the value of the dormancy factor α we distinguish three cases:

$\alpha = 0 \rightarrow$ *cold standby* : The failure rate of the standby component B is equal to 0 when dormant.

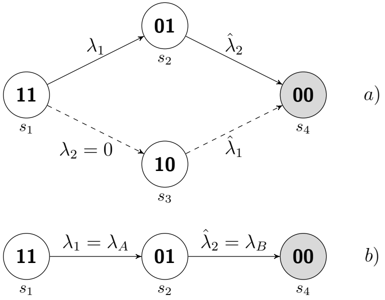
$(0 < \alpha < 1) \rightarrow$ *warm standby*

$\alpha = 1 \rightarrow$ *hot standby* : The failure rate of the standby component B is the same when dormant and when in operation. This case is the same as the parallel configuration.

Cold Standby: $\alpha = 0$

By substituting in Figure $\lambda_1 = \lambda_A$, $\lambda_2 = 0$ and $\hat{\lambda}_2 = \lambda_B$, the state probabilities become:

$$\begin{cases} \pi_1(t) & = & e^{-\lambda_A t} \\ \pi_2(t) & = & \dfrac{\lambda_A}{\lambda_A - \lambda_B} \left(e^{-\lambda_B t} - e^{-\lambda_A t} \right) \\ \pi_3(t) & = & 0 \qquad \text{non reachable} \\ \pi_4(t) & = & 1 - \pi_1(t) - \pi_2(t) - \pi_3(t) \end{cases}$$



Cold Standby - 2

Finally the cold standby system reliability is:

$$R_{CS}(t) = \pi_1(t) + \pi_2(t) = \frac{\lambda_A}{\lambda_A - \lambda_B} e^{-\lambda_B t} - \frac{\lambda_B}{\lambda_A - \lambda_B} e^{-\lambda_A t},$$

and the system MTTF by integration:

$$\begin{aligned} \mathrm{MTTF}_{CS} &= \int_0^\infty R_{CS}(t)dt = \frac{\lambda_A}{\lambda_A - \lambda_B}\frac{1}{\lambda_B} - \frac{\lambda_B}{\lambda_A - \lambda_B}\frac{1}{\lambda_A} \\ &= \frac{1}{\lambda_A} + \frac{1}{\lambda_B}. \end{aligned}$$

Cold Standby: Equal components

If the two components have equal failure rates ($\lambda_A = \lambda_B = \lambda$) the above Equations are undefined. One method of resolution is to use L'Hospital's rule or to resort to Laplace transforms.
The system reliability is (Erlang-2 distribution):

$$R_{CS-2}(t) = \pi_1(t) + \pi_2(t) = (1 + \lambda\, t)\, e^{-\lambda\, t}$$

and the system mean time to failure is:

$$\mathrm{MTTF}_{CS-2} = \int_0^\infty R_{CS-2}(t) dt = \frac{2}{\lambda}$$

Warm Standby

By substituting the proper values of the failure rates, we obtain:

$$
\begin{aligned}
R_{WS}(t) &= \pi_1(t) + \pi_2(t) + \pi_3(t) = \\
&= e^{-(\lambda_A + \alpha\lambda_B)t} + \frac{\lambda_A}{\lambda_A - (1-\alpha)\lambda_B}\left(e^{-\lambda_B t} - e^{-(\lambda_A + \alpha\lambda_B)t}\right) \\
&+ e^{-\lambda_A t} - e^{-(\lambda_A + \alpha\lambda_B)t} \\
&= e^{-\lambda_A t} + \frac{\lambda_A}{\lambda_A - (1-\alpha)\lambda_B}\left(e^{-\lambda_B t} - e^{-(\lambda_A + \alpha\lambda_B)t}\right)
\end{aligned}
$$

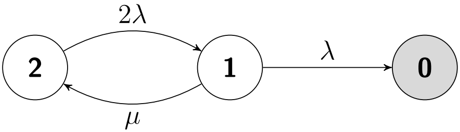
The above Equation reduces to the reliability of a parallel system of independent components when $\alpha = 1$.

Warm Standby - 2

System MTTF is obtained by integration of the reliability expression:

$$
\begin{aligned}
\mathrm{MTTF}_{WS} &= \int_0^{\infty} R_{WS}(t)dt \\
&= \frac{1}{(\lambda_A + \alpha\,\lambda_B)} + \frac{\lambda_A}{\lambda_A - (1-\alpha)\,\lambda_B}\left(\frac{1}{\lambda_B} - \frac{1}{(\lambda_A + \alpha\,\lambda_B)}\right) \\
&+ \frac{1}{\lambda_A} - \frac{1}{(\lambda_A + \alpha\,\lambda_B)} \\
&= \frac{\lambda_A\,\lambda_B + \lambda_A^2 + \alpha\,\lambda_B^2}{\lambda_A\,\lambda_B\,(\lambda_A + \alpha\,\lambda_B)}
\end{aligned}
$$

Two component reliability model with repair



Kolmogorov differential equations in matrix form for the example are:

$$\left[\frac{d\,\pi_2(t)}{d\,t}, \frac{d\,\pi_1(t)}{d\,t}, \frac{d\,\pi_0(t)}{d\,t}\right] = [\pi_2(t), \pi_1(t), \pi_0(t)] \begin{bmatrix} -2\,\lambda & 2\,\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & 0 \end{bmatrix}.$$

The scalar version of the system of equations are:

$$\begin{cases} \dfrac{d\,\pi_2(t)}{d\,t} & = & -\,2\,\lambda\,\pi_2(t) + \mu\,\pi_1(t) \\ \dfrac{d\,\pi_1(t)}{d\,t} & = & 2\,\lambda\,\pi_2(t) - (\,\lambda + \mu\,)\,\pi_1(t) \\ \dfrac{d\,\pi_0(t)}{d\,t} & = & \lambda\,\pi_1(t)\,. \end{cases}$$

Two component reliability model with repair

Taking Laplace transforms on both sides, we get:

$$\begin{cases} s\,\pi_2^*(s) - 1 & = & -2\,\lambda\,\pi_2^*(s) + \mu\,\pi_1^*(s) \\ s\,\pi_1^*(s) & = & 2\,\lambda\,\pi_2^*(s) - (\,\lambda + \mu\,)\,\pi_1^*(s) \\ s\,\pi_0^*(s) & = & \lambda\,\pi_1^*(s)\,. \end{cases}$$

Solving this system of algebraic equations is s-domain, we get:

$$\pi_0^*(s) = \frac{2\,\lambda^2}{s\,[\,s^2 + (3\,\lambda + \mu)\,s + 2\,\lambda^2\,]}\,,$$

and by an inversion via partial fraction expansion, we get an expression for system reliability:

$$R(t) = 1 - \pi_0(t) = \frac{\alpha_1}{\alpha_1 - \alpha_2}\,e^{-\,\alpha_2\,t} - \frac{\alpha_2}{\alpha_1 - \alpha_2}\,e^{-\,\alpha_1\,t}$$

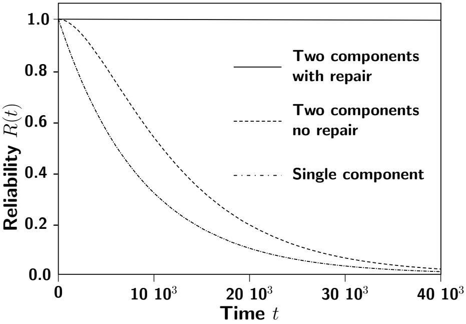
Two component reliability model with repair

α_1 and α_2 are the roots of the equation:

$$s^2 + (3\lambda + \mu)s + 2\lambda^2 = (s + \alpha_1)(s + \alpha_2).$$

$$\alpha_1, \alpha_2 = \frac{(3\lambda + \mu) \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}}{2}.$$

Compare the effect of redundancy combined with repair over redundancy without repair ($\lambda = 1/8760\ f/h$ and $\mu = 1/2\ r/h$).



CTMC with Absorbing States

CTMC with Absorbing States

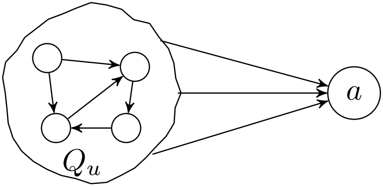
An absorbing state is a state with no outgoing arcs and the corresponding row of the infinitesimal generator matrix has only zero entries.

In the partitioned infinitesimal generator matrix $\boldsymbol{Q}$ only the matrices $\boldsymbol{Q}_{uu}$ and $\boldsymbol{Q}_{ud}$ have non-zero entries.

Absorbing states appear in many problems:

- The reliability of a system is the probability of continuous operation in an interval $(0, t]$. The failure states for the system in the corresponding CTMC model must be absorbing states.
- Absorbing states have an independent interest as a first passage time problems.

CTMC with a Single Absorbing State



$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ \boldsymbol{0} & 0 \end{bmatrix}$$

$$\boldsymbol{a}^T = -\boldsymbol{Q}_u \boldsymbol{e}^T$$

The Markov equation in partitioned form can be written as:

$$\left[\frac{d\,\boldsymbol{\pi}_u(t)}{d\,t} \quad \frac{d\,\pi_a(t)}{d\,t} \right] = [\boldsymbol{\pi}_u(t) \;\; \pi_a(t)] \left[\begin{array}{c|c} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ - & - \\ \boldsymbol{0} & 0 \end{array} \right],$$

where $\boldsymbol{\pi}_u(t)$ refers to the partition of the state probability vector over the state in Ω_u, and $\pi_a(t)$ is the probability of the absorbing State a.

CTMC with Absorbing States

Solving the above equation in partitioned form, we get:

$$\begin{cases} \dfrac{d\,\boldsymbol{\pi}_u(t)}{d\,t} & = & \boldsymbol{\pi}_u(t)\,\boldsymbol{Q}_u \\ \dfrac{d\,\pi_a(t)}{d\,t} & = & \boldsymbol{\pi}_u(t)\,\boldsymbol{a}^T \end{cases}$$

$$\begin{cases} \boldsymbol{\pi}_u(t) & = & \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t} \\ \dfrac{d\,\pi_a(t)}{d\,t} & = & \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t}\,\boldsymbol{a}^T\,, \end{cases}$$

where $\boldsymbol{\pi}_u(0)$ is the partition of the initial probability vector $\boldsymbol{\pi}(0)$ over the transient states and $\pi_a(0) = 0$.
Given that T_a is the time to absorption, we can compute its Cdf $F_a(t)$ as:

$$\begin{aligned} F_a(t) &= P\{T_a \le t\} = P\{Z(t) = a\} = \pi_a(t) \\ &= 1 - \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t}\,\boldsymbol{e}^T\,. \end{aligned}$$

T_a is called a *continuous Phase-Type* (*PH*) random variable and $F_a(t)$ is a *PH* distribution.

CTMC with Absorbing States

Taking Laplace transform:

$$[\, s\,\boldsymbol{\pi}_u^*(s) - \boldsymbol{\pi}_u(0) \quad s\,\pi_a^*(s)\,] = [\,\boldsymbol{\pi}_u^*(s) \quad \pi_a^*(s)\,] \left[\begin{array}{c|c} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ - & - \\ \boldsymbol{0} & 0 \end{array} \right]$$

$$\begin{cases} s\,\boldsymbol{\pi}_u^*(s) - \boldsymbol{\pi}_u(0) & = \quad \boldsymbol{\pi}_u^*(s)\,\boldsymbol{Q}_u \\ s\,\pi_a^*(s) & = \quad \boldsymbol{\pi}_u^*(s)\,\boldsymbol{a}^T \end{cases}$$

$$\begin{cases} \boldsymbol{\pi}_u^*(s) & = \quad \boldsymbol{\pi}_u(0)(s\,\boldsymbol{I} - \boldsymbol{Q}_u)^{-1} \\ \pi_a^*(s) & = \quad \dfrac{1}{s}\,\boldsymbol{\pi}_u^*(s)\,\boldsymbol{a}^T = \dfrac{1}{s}\,\boldsymbol{\pi}_u(0)(s\,\boldsymbol{I} - \boldsymbol{Q}_u)^{-1}\,\boldsymbol{a}^T . \end{cases}$$

Hence, the transform of the Cdf and the density of T_a are:

$$\begin{aligned} F_a^*(s) &= \frac{1}{s}\,\boldsymbol{\pi}_u(0)(s\,\boldsymbol{I} - \boldsymbol{Q}_u)^{-1}\,\boldsymbol{a}^T \\ f_a^*(s) = s\,F_a^*(s) &= \boldsymbol{\pi}_u(0)(s\,\boldsymbol{I} - \boldsymbol{Q}_u)^{-1}\,\boldsymbol{a}^T . \end{aligned}$$

Expected Time to Absorption

Since T_a is the r.v. representing the time to absorption, the expected time to absorption can be computed as:

$$\begin{aligned} E[T_a] &= \int_0^\infty (1 - F_a(t))\, d\, t \\ &= \int_0^\infty \boldsymbol{\pi}_u(0)\, e^{\boldsymbol{Q}_u\, t}\, \boldsymbol{e}^T\; d\, t \\ &= \boldsymbol{\pi}_u(0)\, \left[\boldsymbol{Q}_u^{-1}\, e^{\boldsymbol{Q}_u\, t}\, \boldsymbol{e}^T\right]_0^\infty \\ &= \boldsymbol{\pi}_u(0)\, (-\boldsymbol{Q}_u)^{-1}\, \boldsymbol{e}^T\, . \end{aligned}$$

Computing $E[T_a]$ from the above equation involves the inversion of matrix $\boldsymbol{Q}_u$.

Expected Time to Absorption

A simpler algorithm for the computation of $E[T_a]$ can be obtained considering the expected state occupancy in the transient states before absorption.

$$\left[\frac{d\,\boldsymbol{b}_u(t)}{d\,t} \;\; \frac{d\,b_a(t)}{d\,t} \right] = [\,\boldsymbol{b}_u(t)\;\; b_a(t)\,] \left[\begin{array}{c|c} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ - & - \\ \boldsymbol{0} & 0 \end{array} \right] + [\boldsymbol{\pi}_u(0)\,\pi_a(0)]\,.$$

By considering only the partition over the transient states,

$$\frac{d\,\boldsymbol{b}_u(t)}{d\,t} = \boldsymbol{b}_u(t)\,\boldsymbol{Q}_u + \boldsymbol{\pi}_u(0)\,,$$

and letting $(t \rightarrow \infty)$, since states in $\boldsymbol{Q}_u$ are transient, we get:

$$\lim_{t\rightarrow\infty} b_i(t) = \tau_i \quad \text{and} \quad \lim_{t\rightarrow\infty} \frac{d\,b_i(t)}{d\,t} = 0 \;, \qquad i \in \boldsymbol{Q}_u\,,$$

where $\tau_i = E[T_i]$ is the expected total (possibly over many sojourns) time spent in the transient State i before absorption.

Expected Time to Absorption

Grouping the expected times τ_i into a vector $\boldsymbol{\tau} = [\tau_i]$, we have:

$$\boldsymbol{\tau}\, \boldsymbol{Q}_u = -\boldsymbol{\pi}_u(0)\,.$$

The total expected time till absorption is the sum of the expected times in the transient states:

$$E[T_a] = \boldsymbol{\tau}\, \boldsymbol{e}^T\,.$$

$E[T_a]$ can be calculated directly starting from the CTMC specification, without computing and integrating the reliability $R(t)$.
Also no matrix inversion needs to be carried out as the linear system of Equations is easier to solve.

Expected Time to Absorption: Two Components

The MTTF of a series/parrallel system of two components was computed in Slide 22

The same results can be obtained by solving the linear Equation over the transient states to get:

$$
\begin{aligned}
-\tau_1(\lambda_1 + \lambda_2) &= -1 \qquad & \tau_1 &= \frac{1}{\lambda_1 + \lambda_2} \\
\lambda_1 \tau_1 - \hat{\lambda}_2 \tau_2 &= 0 \qquad & \tau_2 &= \frac{\lambda_1}{\hat{\lambda}_2 (\lambda_1 + \lambda_2)} \\
\lambda_2 \tau_1 - \hat{\lambda}_1 \tau_3 &= 0 \qquad & \tau_3 &= \frac{\lambda_2}{\hat{\lambda}_1 (\lambda_1 + \lambda_2)}
\end{aligned}
$$

$$
\text{MTTF} = \tau_1 + \tau_2 + \tau_3 = \frac{1}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\hat{\lambda}_2 (\lambda_1 + \lambda_2)} + \frac{\lambda_2}{\hat{\lambda}_1 (\lambda_1 + \lambda_2)}
$$

The 3 summands in the MTTF expression above are the expected times spent in States 1, 2 and 3, respectively, till absorption.

Expected Time to Absorption: Warm Standby

The linear equations over the transient states become:

$$[\tau_1\ \tau_2\ \tau_3]\begin{bmatrix} -(\lambda_A+\alpha\lambda_B) & \lambda_A & \alpha\lambda_B \\ 0 & -\lambda_B & 0 \\ 0 & 0 & -\lambda_A \end{bmatrix} = -[1\ 0\ 0]$$

whose solution yields:

$$\begin{aligned}
\tau_1 &= \frac{1}{\lambda_A+\alpha\lambda_B} \\
\tau_2 &= \frac{\lambda_A}{\lambda_B(\lambda_A+\alpha\lambda_B)} \\
\tau_3 &= \frac{\alpha\lambda_B}{\lambda_A(\lambda_A+\alpha\lambda_B)},
\end{aligned}$$

finally (compare this with Slide 28),

$$\text{MTTF} = \tau_1+\tau_2+\tau_3 = \frac{\lambda_A\lambda_B+\lambda_A^2+\alpha\lambda_B^2}{\lambda_A\lambda_B(\lambda_A+\alpha\lambda_B)}.$$

Expected Time to Absorption: Warm Standby

The above Equation provides known results in the limiting cases $\alpha = 1$ and $\alpha = 0$.

$\alpha = 1$: *hot standby* or independent parallel case:

$$\text{MTTF}_{(\alpha=1)} = \frac{1}{\lambda_A} + \frac{1}{\lambda_B} - \frac{1}{\lambda_A + \lambda_B}$$

$\alpha = 0$: *cold standby* or simply standby:

$$\text{MTTF}_{(\alpha=0)} = \frac{1}{\lambda_A} + \frac{1}{\lambda_B}$$

MTTF: Two components with repair

The infinitesimal generator can be partitioned as follows:

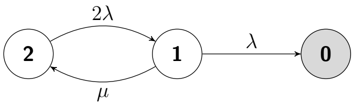
$$\mathbf{Q} = \left[\begin{array}{c|c} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ \hline \mathbf{0} & 0 \end{array} \right] = \left[\begin{array}{cc|c} -2\lambda & 2\lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ \hline 0 & 0 & 0 \end{array} \right]$$

Assuming as initial probability $\pi_2(0) = 1$, the linear system becomes:

$$[\tau_2 \; \tau_1] \left[\begin{array}{cc} -2\lambda & 2\lambda \\ \mu & -(\lambda+\mu) \end{array} \right] = -[1 \; 0]$$

whose solution is:

$$\tau_2 = \frac{\lambda+\mu}{2\lambda^2} \quad , \quad \tau_1 = \frac{1}{\lambda}$$



finally,

$$\text{MTTF} = \tau_2 + \tau_1 = = \frac{3\lambda+\mu}{2\lambda^2}$$

MTTF: Two components with repair

If $\mu = 0$ (2 parallel components without repair) we get:

$$E\,[\,T_a\,] \;=\; \text{MTTF} \;=\; \frac{3}{2\,\lambda}$$

Note that by using redundancy by itself, the MTTF increases by 50% from $1/\lambda$ to $1.5/\lambda$.

By adding repair on top of redundancy the increase in the MTTF is by several orders of magnitude: $1.5/\lambda + \mu/(2\lambda^2)$ since μ will generally be several orders of magnitude larger than λ in practice.

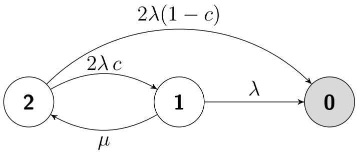
With the following data ($\lambda = 1/8760\ f/h$ and $\mu = 1/0.5\ r/h$), we get:

$$\begin{aligned} \text{MTTF}_{single} &= 8760\ h \\ \text{MTTF}_{par\text{-}no\text{-}rep} &= 13,140\ h \\ \text{MTTF}_{par\text{-}with\text{-}rep} &= 1.92 \times 10^{7}\ h\,. \end{aligned}$$

MTTF: Two components with repair and coverage

The detection and recovery process may complete successfully with a coverage probability c, and with probability $(1-c)$ the recovery process does not complete successfully, and the system incurs a complete failure and moves to the down State 0.

$$\boldsymbol{Q} = \left[\begin{array}{c|c} \boldsymbol{Q}_u & \boldsymbol{a}^T \\ -- & -- \\ \boldsymbol{0} & 0 \end{array}\right] = \left[\begin{array}{cc|c} -2\lambda & 2\lambda c & 2\lambda(1-c) \\ \mu & -(\lambda+\mu) & \lambda \\ \hline 0 & 0 & 0 \end{array}\right].$$



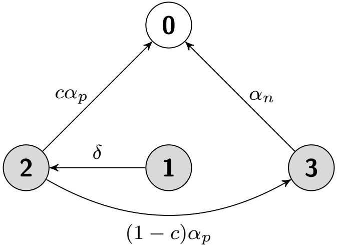
Initial probability $\pi_2(0) = 1$,

$$\tau_2 = \frac{\lambda+\mu}{2\lambda(\lambda+\mu(1-c))}$$

$$\tau_1 = \frac{2\lambda c}{2\lambda(\lambda+\mu(1-c))},$$

$$\text{MTTF} = E[T_a] = \tau_2 + \tau_1 = = \frac{\lambda+\mu+2\lambda c}{2\lambda(\lambda+\mu(1-c))}.$$

Defects Per Million for an application server - 1



The Figure shows the state transitions after a failure has occurred.

The cumulative distribution function for the time to absorption to State T_d 0 is HYPO(δ, α_p) with probability c, while with probability $1-c$, is HYPO$(\delta, \alpha_p, \alpha_n)$.

$$F_{T_d}(t) = \pi_0(t) = c\left(1 - \frac{\alpha_p}{\alpha_p - \delta} e^{-\delta t} + \frac{\delta}{\alpha_p - \delta} e^{-\alpha_p t}\right)$$

$$+ (1-c)\left(1 - \frac{\alpha_p}{\alpha_p - \delta}\frac{\alpha_n}{\alpha_n - \delta} e^{-\delta t} - \frac{\delta}{\delta - \alpha_p}\frac{\alpha_n}{\alpha_n - \alpha_p}\right.$$

$$\left. e^{-\alpha_p t} - \frac{\delta}{\delta - \alpha_n}\frac{\alpha_p}{\alpha_p - \alpha_n} e^{-\alpha_n t}\right), \qquad t \geq 0\,.$$

Defects Per Million for an application server - 2

The mean number of new calls dropped due to a server failure is:

$$
\begin{aligned}
n_a &= \int_{w_i}^{\infty} \lambda(t - w_i) d\pi_0(t) \\
&= \left[\text{MTTA} - w_i + \int_0^{w_i} \pi_0(t) dt \right] \lambda \,,
\end{aligned}
$$

where MTTA denotes the mean time to absorption to State 0, which is given by:

$$
\begin{aligned}
\text{MTTA} &= \int_0^{\infty} t \; d\pi_0(t) \\
&= c\left(\frac{1}{\delta} + \frac{1}{\alpha_p} \right) + (1 - c)\left(\frac{1}{\delta} + \frac{1}{\alpha_p} + \frac{1}{\alpha_n} \right) \\
&= \frac{1}{\delta} + \frac{1}{\alpha_p} + \frac{1 - c}{\alpha_n} \,.
\end{aligned}
$$

Defects Per Million for an application server - 3

The MTTA could also be obtained by solving the linear system:

$$[\tau_1 \ \tau_2 \ \tau_3] \begin{bmatrix} -\delta & \delta & 0 \\ 0 & -\alpha_p & (1-c)\alpha_p \\ 0 & 0 & -\alpha_n \end{bmatrix} = -[1 \ 0 \ 0]$$

whose solution is: $\tau_1 = \dfrac{1}{\delta} \qquad \tau_2 = \dfrac{1}{\alpha_p} \qquad \tau_3 = \dfrac{1-c}{\alpha_n}$.

The mean number of lost calls n_a is:

$$n_a = \frac{1}{\delta - \alpha_p}\left[\frac{\delta}{\alpha_p} e^{-\alpha_p w_i} - \frac{\alpha_p}{\delta} e^{-\delta w_i}\right]\lambda$$

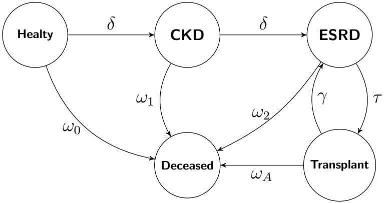
$$+ \frac{1-c}{\delta-\alpha_p}\left[\frac{\delta}{\alpha_n - \alpha_p} e^{-\alpha_p w_i} + \frac{\alpha_p}{\delta - \alpha_n} e^{-\delta w_i} - \frac{\delta - \alpha_p}{\alpha_n} \frac{\delta}{\delta - \alpha_n} \frac{\alpha_p}{\alpha_n - \alpha_p} e^{-\alpha_n w_i}\right]\lambda$$

The DPM caused by application server failure is:

$$DPM = \pi_0 \gamma n_a \frac{10^6}{\lambda}$$

Renal disease model - 1

Reliability modeling techniques provide new frontiers in predicting health care outcomes.



1. Healthy: Normal renal function,
2. CKD: Chronic Kidney Disease without renal failure,
3. ESRD: End-Stage Renal Disease, an administrative term for patients with renal failure,
4. Transplant: Renal failure patients who have successfully received a transplant,
5. Deceased: An absorbing state.

Renal disease model - 2

Assuming all patients begin healthy, the solution for healthy states is straightforward. In the Laplace transform domain, we get (where H indicates the healthy state and C the CKD state):

$$
\begin{aligned}
s\pi_H^*(s) - 1 &= -(\delta + \omega_0)\pi_H^*(s) = \frac{1}{s + \delta + \omega_0} \\
s\pi_C^*(s) &= -(\delta + \omega_1)\pi_C^*(s) + \delta\pi_H^*(s) \\
\pi_C^*(s) &= \frac{\delta}{s + \delta + \omega_1}\pi_H^*(s) = \frac{\delta}{(s + \delta + \omega_0)(s + \delta + \omega_1)} \\
&= \frac{\delta}{\omega_0 - \omega_1}\left[\frac{1}{s + \delta + \omega_1} - \frac{1}{s + \delta + \omega_0)}\right]
\end{aligned}
$$

Inverting to the time domain, we get:

$$
\begin{aligned}
\pi_H(t) &= e^{-(\delta + \omega_0)t} \\
\pi_C(t) &= \frac{\delta}{\omega_0 - \omega_1}\left[e^{-(\delta + \omega_1)t} - e^{-(\delta + \omega_0)t}\right]
\end{aligned}
$$

Renal disease model - 3

The parameter values reported in the Table are derived and are based on the latest available statistics from United States Renal Data System (USRDS) annual report [*].

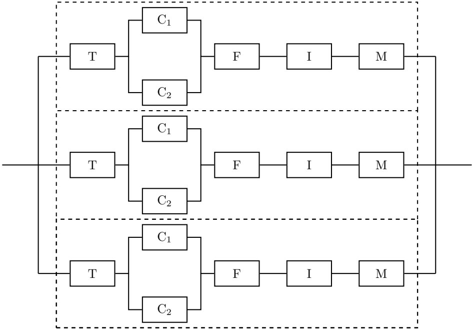
Description	Symbol	Value *event/year* (*event/year*)
Decline	δ	0.1887
Transplant	τ	0.1786
Graft Rejection	γ	0.0050
Prognosis-Healthy	ω_0	0.0645
Prognosis-CKD	ω_1	0.1013
Prognosis-ESRD	ω_2	0.2174
Prognosis-Transplant	ω_T	0.0775

[*] R. Fricks, A. Bobbio, and K. Trivedi, "Reliability models of chronic kidney disease," in *Proceedings IEEE Annual Reliability and Maintainability Symposium*, 2016.

Reliability of a multivoltage high speed train - 1

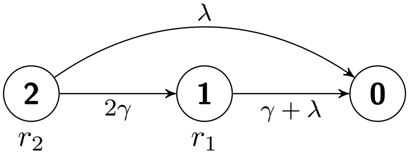
The RBD of a multivoltage propulsion system designed for the Italian High Speed Railway System is in the Figure.

The system consists of three equivalent modules in parallel redundant configuration. Each module is modeled as a series of 4 blocks (transformer T, filter F, inverter I and motor M) and two parallel converters (C_1 and C_2) that feed the induction motors.



Reliability of a multivoltage high speed train - 2

A single module can be represented by a 3-state MRM model, that accounts for the power level delivered in each configuration:



- a fully operational state delivering maximum power ($r_2 = 2200$ KW) when all components are working;
- a degraded state delivering half of the power ($r_1 = 1100$ KW) when all the series components and one converter are working;
- a failed state delivering no power.

$$\boldsymbol{Q} = \begin{bmatrix} -(2\gamma + \lambda) & 2\gamma & \lambda \\ 0 & -(\gamma + \lambda) & \gamma + \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Reliability of a multivoltage high speed train - 3

Solving the transient equations, we obtain:

$$\begin{cases} \pi_2(t) &= e^{-(2\gamma+\lambda)\,t} \\ \pi_1(t) &= 2\,e^{-(\gamma+\lambda)\,t} - 2\,e^{-(2\gamma+\lambda)\,t} \\ \pi_0(t) &= 1 - \pi_2(t) - \pi_1(t) = 1 + e^{-(2\gamma+\lambda)\,t} - 2\,e^{-(\gamma+\lambda)\,t}\,. \end{cases}$$

The result is the same as the one obtained from the RBD solution.

The MRM can combine the system reliability with its performance in terms of power delivered.

We can write down the expected power $E[X(t)]$ available at time t:

$$E[X(t)] = \sum_{i=0}^{2} r_i\,\pi_i(t) = r_2\,\pi_2(t) + r_1\,\pi_1(t)$$

the expected accumulated energy $E[Y(t)]$ delivered in the interval $(0, t]$:

$$E[Y(t)] = \sum_{i=0}^{2} \int_0^t r_i\,\pi_i(x)dx = r_2 \int_0^t \pi_2(x)dx + r_1 \int_0^t \pi_1(x)dx\,.$$

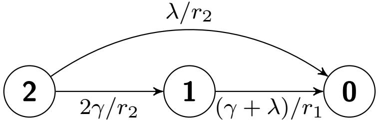
Reliability of a multivoltage high speed train - 4

The expected accumulated reward (energy delivered) until absorption (failure) is then easily computed as:

$$E[Y(\infty)] = r_2 \int_0^\infty \pi_2(x)dx + r_1 \int_0^\infty \pi_1(x)dx$$
$$= \frac{r_2}{2\gamma+\lambda} + \frac{2r_1}{\gamma+\lambda} - \frac{2r_1}{2\gamma+\lambda}\,.$$

We can also obtain the Cdf of, $Y(\infty)$, the reward accumulated till absorption abbreviated as here Y using a method proposed by Beaudry [*].

Beaudry's method consists in dividing the transition rates of the CTMC by the corresponding reward rates.



Reliability of a multivoltage high speed train - 5

The state probabilities $\pi_2^{(R)}(r)$ and $\pi_1^{(R)}(r)$ in the scaled CTMC are:

$$\begin{cases} \pi_2^{(R)}(r) = e^{-\frac{(2\gamma+\lambda)}{r_2}\, r} \\ \pi_1^{(R)}(r) = \dfrac{2r_1\gamma}{\gamma(2r_1 - r_2) + \lambda(r_1 - r_2)} \cdot \left(e^{-\frac{\gamma+\lambda}{r_1} r} - e^{-\frac{2\gamma+\lambda}{r_2} r}\right) \\ \pi_0^{(R)}(r) = 1 - \pi_2^{(R)}(r) - \pi_1^{(R)}(r) \end{cases}$$

and, finally:

$$P(Y(\infty) \geq a) = \pi_1^{(R)}(a) + \pi_2^{(R)}(a)$$

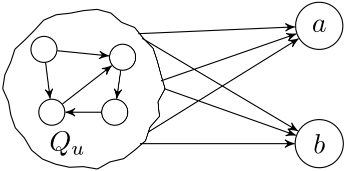
[*] M. Beaudry, "Performance-related reliability measures for computing systems," *IEEE Transactions on Computers*, vol. C-27, pp. 540–547, 1978.

CTMC with Multiple Absorbing States

Multiple absorbing states in dependability modeling arise from two principal reasons:

- the presence of multiple failure causes or failure modes,
- the different effects that some faults can have on the system.

To simplify the analysis, we assume, in the sequel, that the CTMC has two absorbing states, only, say a and b.



$$\left[\frac{d\,\boldsymbol{\pi}_u(t)}{d\,t}\;\; \frac{d\,\pi_a(t)}{d\,t}\;\; \frac{d\,\pi_b(t)}{d\,t}\right] = [\boldsymbol{\pi}_u(t)\;\; \pi_a(t)\;\; \pi_b(t)] \left[\begin{array}{c|cc} \boldsymbol{Q}_u & \boldsymbol{a}^T & \boldsymbol{b}^T \\ - & - & - \\ \boldsymbol{0} & 0 & 0 \\ \boldsymbol{0} & 0 & 0 \end{array}\right]$$

CTMC with Multiple Absorbing States

Solving the above equations in partitioned form, we get:

$$\begin{cases} \dfrac{d\,\boldsymbol{\pi}_u(t)}{d\,t} = \boldsymbol{\pi}_u(t)\,\boldsymbol{Q}_u \\ \dfrac{d\,\pi_a(t)}{d\,t} = \boldsymbol{\pi}_u(t)\,\boldsymbol{a}^T \\ \dfrac{d\,\pi_b(t)}{d\,t} = \boldsymbol{\pi}_u(t)\,\boldsymbol{b}^T \end{cases}$$

$$\begin{cases} \boldsymbol{\pi}_u(t) = \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t} \\ \dfrac{d\,\pi_a(t)}{d\,t} = \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t}\,\boldsymbol{a}^T \\ \dfrac{d\,\pi_b(t)}{d\,t} = \boldsymbol{\pi}_u(0)\,e^{\boldsymbol{Q}_u\,t}\,\boldsymbol{b}^T \end{cases}$$

Define T_a and T_b as the times to absorption to states a and b, respectively. Corresponding Cdfs $F_a(t)$ and $F_b(t)$ are now defective distributions, and the mean time to absorption to either State a or State b are undefined (infinity).

CTMC with Multiple Absorbing States

$$
\begin{aligned}
F_a(t) &= P\{\tau_a \leq t\} = P\{Z(t) = a\} = \pi_a(t) \\
F_b(t) &= P\{\tau_b \leq t\} = P\{Z(t) = b\} = \pi_b(t)\,.
\end{aligned}
$$

The Cdf $F_a(t)$ can be obtained by integrating the differential equation:

$$
\frac{d\,\pi_a(t)}{d\,t} = \boldsymbol{\pi}_u(0)\, e^{\boldsymbol{Q}_u\, t}\, \boldsymbol{a}^T\,,
$$

whose solution is:

$$
\begin{aligned}
\pi_a(t) &= \int_0^t \boldsymbol{\pi}_u(0)\, e^{\boldsymbol{Q}_u\, x}\, \boldsymbol{a}^T\, dx = \left[\boldsymbol{\pi}_u(0)\left(e^{\boldsymbol{Q}_u\, x}\, \boldsymbol{Q}_u^{-1}\right) \boldsymbol{a}^T\right]_0^t \\
&= \boldsymbol{\pi}_u(0)\,(-\boldsymbol{Q}_u^{-1})\,\boldsymbol{a}^T + \boldsymbol{\pi}_u(0)\, e^{\boldsymbol{Q}_u\, t}\, \boldsymbol{Q}_u^{-1}\, \boldsymbol{a}^T\,.
\end{aligned}
$$

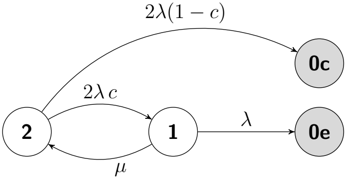
As $t \to \infty$, the probability of final absorption in State a becomes:

$$
\pi_a(\infty) = \lim_{t\to\infty} \pi_a(t) = \boldsymbol{\pi}_u(0)\,(-\boldsymbol{Q}_u^{-1})\,\boldsymbol{a}^T\,.
$$

Different failure causes

We consider a system whose failure was caused either by the exhaustion of the redundancy or due to imperfect coverage in detecting/recovering from a failure.

These two causes are separated out in the Figure where the State labeled ($0e$) indicates a failure caused by the exhaustion of the redundancy while State labeled ($0c$) indicates a failure caused by imperfect coverage.



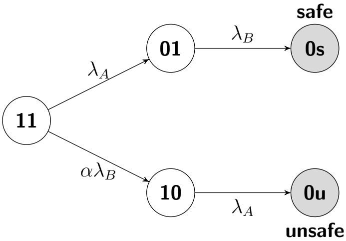
$$\pi_{0e} = \frac{\lambda c}{\lambda + (1-c)\mu}$$

$$\pi_{0c} = \frac{(1-c)(\lambda+\mu)}{\lambda + (1-c)\mu}$$

$$\pi_{0e} + \pi_{0c} = 1$$

Warm standby: Safety analysis - 1

In a safety analysis we wish to distinguish two cases: When the primary unit A fails whether or not the standby unit B is up.



Safe State $0s$ is reached when unit B can recover the failure of the primary unit A. Unsafe State $0u$ is reached if the standby unit had failed prior to the occurrence of primary unit failure.

We partition the generator matrix of this CTMC:

$$
\boldsymbol{Q} = \left[\begin{array}{c|cc} \boldsymbol{Q}_u & \boldsymbol{a}^T & \boldsymbol{b}^T \\ - & - & - \\ \boldsymbol{0} & 0 & 0 \\ \boldsymbol{0} & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|cc} -(\lambda_A + \alpha\lambda_B) & \lambda_A & \alpha\lambda_B & 0 & 0 \\ 0 & -\lambda_B & 0 & \lambda_B & 0 \\ 0 & 0 & -\lambda_A & 0 & \lambda_A \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

Warm standby: Safety analysis - 2

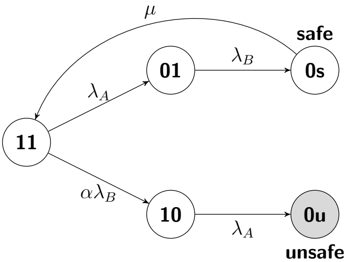
$$-\boldsymbol{Q}_u^{-1} = \frac{1}{\lambda_A\,\lambda_B(\lambda_A + \alpha\lambda_B)} \begin{bmatrix} \lambda_A\lambda_B & \lambda_A^2 & \alpha\lambda_B^2 \\ 0 & \lambda_A(\lambda_A + \alpha\lambda_B) & 0 \\ 0 & 0 & \lambda_B(\lambda_A + \alpha\lambda_B) \end{bmatrix}.$$

We separate the Cdf of time to absorption in State 0s and in State 0u and we compute the eventual absorption probability as $t \to \infty$.

$$\begin{aligned} F_{0s}(\infty) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left[-\boldsymbol{Q}_u^{-1}\right] \begin{bmatrix} 0 \\ \lambda_B \\ 0 \end{bmatrix} \\ &= \frac{\lambda_A^2\lambda_B}{\lambda_A\lambda_B\,(\lambda_A + \alpha\lambda_B)} = \frac{\lambda_A}{\lambda_A + \alpha\lambda_B} \end{aligned}$$

$$\begin{aligned} F_{0u}(\infty) &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \left[-\boldsymbol{Q}_u^{-1}\right] \begin{bmatrix} 0 \\ 0 \\ \lambda_A \end{bmatrix} \\ &= \frac{\alpha\,\lambda_B^2\lambda_A}{\lambda_A\lambda_B\,(\lambda_A + \alpha\lambda_B)} = \frac{\alpha\,\lambda_B}{\lambda_A + \alpha\lambda_B}. \end{aligned}$$

Warm standby: Safety analysis - 3



we assume now that the system can recover from the *safe* failure condition and return to the fully operational State (11).

$$
\boldsymbol{Q} = \left[\begin{array}{cccc|c}
-(\lambda_A + \alpha\lambda_B) & \lambda_A & \alpha\lambda_B & 0 & 0 \\
0 & -\lambda_B & 0 & \lambda_B & 0 \\
0 & 0 & -\lambda_A & 0 & \lambda_A \\
\mu & 0 & 0 & -\mu & 0 \\
\hline
0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

Warm standby: Safety analysis - 4

The CTMC has a single absorbing state, and the expected time to the first unsafe failure can be computed by solving the linear system:

$$\begin{cases} -(\lambda_A + \alpha\,\lambda_B)\,\tau_{11} + \mu\,\tau_{0s} & = & 1 \\ \lambda_A\,\tau_{11} - \lambda_B\,\tau_{01} & = & 0 \\ \alpha\,\lambda_B\,\tau_{11} - \lambda_A\,\tau_{10} & = & 0 \\ \lambda_B\,\tau_{01} - \mu\,\tau_{0s} & = & 0 \end{cases}$$

$$E[T_a] = \tau_{11} + \tau_{01} + \tau_{10} + \tau_{0s}\,.$$

Expected first passage time

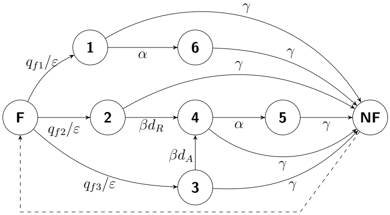
Evaluating the distribution and the moments of the first time that a CTMC jumps from a subset of initial states in Ω to another subset of final states in Ω is usually referred to as the *first passage time problem*

- The first example considers a survivability problem, where the metrics to study is the first passage time problem from a down state to the up state.
- The second example considers a safety problem where the metrics to study is the first passage time problem from a set of up states to a down state for a fire protection system in warm standby.

Survivability study of a power smart grid - 1

Survivability refers to the time-varying system behavior after a failure occurs.

In this example, we consider the survivability assessment of a smart grid distribution network.



The CTMC in Figure depicts the stages that the system goes through after a failure.

The initial state consists of a failure state. Then, based on manual and automated interventions, the system goes through different steps until reaching full recovery.

Survivability study of a power smart grid - 2

F	*Failure at section S*
1	*active & reactive power for* S^+ *OK*
2	*active power for* S^+ *OK, reactive power for* S^+ *not OK*
3	*active power for* S^+ *not OK*
4	*active & reactive power for* S^+ *OK due to DR and DG*
5	S^+ *fixed due to DR and DG*
6	S^+ *fixed*
NF	*No Failure - recovery completed*

q_{1f}	*probability enough active and reactive power for* S^+
q_{2f}	*probability enough active (but not reactive) power for* S^+
q_{3f}	*probability not enough active backup power for* S^+
d_R	*probability that demand response program for reactive power is effective*
d_A	*probability that demand response program for active power is effective*
α	*automatic repair rate*
β	*demand response rate*
γ	*manual repair rate*
ε	*circuit isolation time* $\varepsilon \ll 1$

Survivability study of a power smart grid - 3

The infinitesimal generator, where the rows are ordered from 1 to 6 being the last row the absorbing State NF, is:

$$\boldsymbol{Q} = \begin{bmatrix} -(\alpha+\gamma) & 0 & 0 & 0 & 0 & \alpha & \gamma \\ 0 & -(\beta d_R+\gamma) & 0 & \beta d_R & 0 & 0 & \gamma \\ 0 & 0 & -(\beta d_A+\gamma) & \beta d_A & 0 & 0 & \gamma \\ 0 & 0 & 0 & -(\alpha+\gamma) & \alpha & 0 & \gamma \\ 0 & 0 & 0 & 0 & -\gamma & 0 & \gamma \\ 0 & 0 & 0 & 0 & 0 & -\gamma & \gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Survivability study of a power smart grid - 4

The transient equation can be solved in closed form to give:

$$
\begin{aligned}
\pi_1(t) &= e^{-(\alpha+\gamma)\,t}\,\pi_1(0)\\
\pi_2(t) &= e^{-(\beta d_R+\gamma)\,t}\,\pi_2(0)\\
\pi_3(t) &= e^{-(\beta d_A+\gamma)\,t}\,\pi_3(0)\\
\pi_4(t) &= \frac{\beta d_R}{\alpha-\beta d_R}(e^{-\beta d_R\,t}-e^{-\alpha\,t})\,e^{-\gamma\,t}\,\pi_2(0) + \frac{\beta d_A}{\alpha-\beta d_A}(e^{-\beta d_A\,t}-e^{-\alpha\,t})\,e^{-\gamma\,t}\,\pi_3(0)\\
\pi_5(t) &= \frac{\alpha(1-e^{-\beta d_R\,t})-\beta d_R(1-e^{-\alpha\,t})}{\alpha-\beta d_R}\,e^{-\gamma\,t}\,\pi_2(0)\\
&\quad+\frac{\alpha(1-e^{-\beta d_A\,t})-\beta d_A(1-e^{-\alpha\,t})}{\alpha-\beta d_A}\,e^{-\gamma\,t}\,\pi_3(0)\\
\pi_6(t) &= (1-e^{-\alpha\,t})\,e^{-\gamma\,t}\,\pi_1(0)\\
\pi_{NF}(t) &= 1-e^{-\gamma\,t}
\end{aligned}
$$

Survivability study of a power smart grid - 5

Two reward rates are assigned in the Table to measure both the active and reactive energy not supplied per hours from the occurrence of the failure (State F) to the complete recovery (State NF).

State	1-3	4	5	6	NF
Active ENS/h	593.58	568.98	112.45	112.45	0
Reactive ENS/h	361.77	308.93	26.76	26.76	0

The expected accumulated reward up to time t is

$$E[Y(t)] = \sum_i r_i \int_0^t \pi_i(u)\, du \,.$$

Safety analysis of a fire-fighting pumping station - 1

Safety analysis and experience reveal fire to be one the most serious cause of accidents in industrial plants that store or handle a large amount of inflammable material.

The fire protecting system requires a high level of performance coupled with high reliability.

A safety analysis of a pumping station in a fire fighting system requires that the system provides a sufficient amount of flow (either water or foam) for a sufficient time upon demand.

During emergency operation no repair action can take place

A large petrochemical plant is divided in zones that may have a different request of flow rate and useful time to extinguish the fire

Safety analysis of a fire-fighting pumping station - 2

The following assumptions are made about the system operation:

1. To apply design diversity concepts the pumping system is formed by n_e pumps with electrical motor (EP) and n_d pumps with a diesel motor (DP).
2. The pumps are dormant in cold standby configuration until a demand arrives prescribing a given flow rate.
3. The minimum number of pumps is started upon demand to satisfy the requested flow rate, the other pumps being in (cold) standby.
4. Pumps are put into operation sequentially according to a prescribed order, first the electrical pumps and then the diesel pumps.
5. All the EP have the same failure rate λ_e and capacity r_e and all the DP have the same failure rate λ_d and capacity $r_d \geq r_e$.
6. The availability on demand is c_e for EP and c_d for DP.

Safety analysis of a fire-fighting pumping station - 3

The operation of the pumping station is considered successful if it provides the required flow for a sufficiently long time when a fire accident is detected and the protection system is demanded to operate.

We assume that upon demand the requested flow rate from the plant is q_{req}. Given that the failure rates are constant, the system operation can be modeled as a MRM.

In each state i the reward rate r_i is defined as the capacity delivered by the system in that state.

Safety analysis of a fire-fighting pumping station - 4

Given that $\pi_i(t)$ is the transient state probability of finding the system in state i at time t, two measures can be defined to characterize the system operation:

- The success probability $S(t)$ of correct operation in the interval $(0, t]$ conditioned on the arrival of a demand q_{req} at time $t = 0$:

$$S(t) = \sum_{i: r_i \geq q_{req}} \pi_i(t) \, .$$

- The total expected capacity (total accumulated reward) $Y(t)$ delivered in the interval $(0, t]$ conditioned on the arrival of a demand at time $t = 0$.

$$Y(t) = \int_0^t \sum_i^n r_i \, \pi_i(u) \, d\, u \, .$$

Safety analysis of a fire-fighting pumping station - 5

According to a previous study, an optimal configuration, with respect to installation costs and expected losses, for the plant under consideration is to configure the system with $n_e = 2$ EPs and $n_d = 1$ DP.

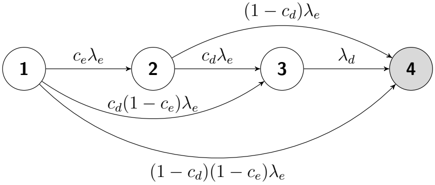
The initial system configuration depends on the flow demand q_{req}. We study three possible scenarios:

Case 1) -			q_{req}	$\leq$	r_e
Case 2) -	r_e	$\leq$	q_{req}	$\leq$	$2\,r_e$
Case 3) -	$2\,r_e$	$\leq$	q_{req}	$\leq$	$2\,r_e + r_d$

Fire-fighting pumping station: Case 1 - 6

State 1 - The first EP is in operation the other pumps are dormant ($r_1 = r_e$)
State 2 - The first EP has failed the second EP is in operation DP is dormant ($r_2 = r_e$)
State 3 - Both EPs have failed and DP is in operation ($r_3 = r_d$)
State 4 - Both the EPs and the DP have failed ($r_4 = 0$)

Upon demand, the system starts with the following initial probability vector:



$$\begin{aligned} \pi_1(0) &= c_e \\ \pi_2(0) &= c_e\,(1-c_e) \\ \pi_3(0) &= c_d\,(1-c_e)^2 \\ \pi_4(0) &= (1-c_e)^2\,(1-c_d)\,. \end{aligned}$$

The value $\pi_4(0) \neq 0$ indicates that there is a non-zero probability that the system does not start upon demand.

Fire-fighting pumping station: Case 1 - 7

The infinitesimal generator is:

$$
\boldsymbol{Q} = \begin{bmatrix} -\lambda_e & c_e\,\lambda_e & (1-c_e)\,c_d\,\lambda_e & (1-c_e)\,(1-c_d)\,\lambda_e \\ 0 & -\lambda_e & c_d\,\lambda_e & (1-c_d)\,\lambda_e \\ 0 & 0 & -\lambda_d & \lambda_d \\ 0 & 0 & 0 & 0 \end{bmatrix} .
$$

The operational states in which the provided flow rate satisfies the request are states 1, 2, and 3 so that the probability of safe operation and the total expected capacity are:

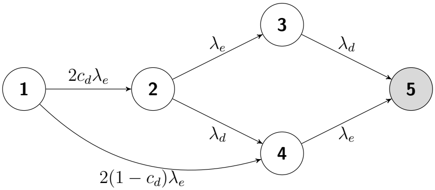
$$
\begin{aligned}
S_{Case1}(t) &= \pi_1(t) + \pi_2(t) + \pi_3(t) \\
Y_{Case1}(t) &= \int_0^t \sum_{i=1}^{3} r_i\,\pi_i(u)\,d\,u\,.
\end{aligned}
$$

Fire-fighting pumping station: Case 2 - 8

The requested instantaneous flow rate requires both EPs to be put into operation upon demand $q_{req2} = 2 \times r_e$.

State 1 -	Both EPs are in operation DP is dormant ($r_1 = 2\, r_e$)
State 2 -	One EP has failed the other EP and the DP are in operation ($r_2 = r_e + r_d$)
State 3 -	Both EPs have failed DP is in operation ($r_3 = r_d$)
State 4 -	One EP and DP have failed the other EP is in operation ($r_4 = r_e$)
State 5 -	Both the EPs and the DP have failed ($r_5 = 0$)

Upon demand, the initial probability vector has the following expression:



$$
\begin{aligned}
\pi_1(0) &= c_e^2 \\
\pi_2(0) &= 2\,(1-c_e)\,c_e\,c_d \\
\pi_3(0) &= c_d\,(1-c_e)^2 \\
\pi_4(0) &= 2\,c_e\,(1-c_e)\,(1-c_d) \\
\pi_5(0) &= (1-c_e)^2\,(1-c_d)\,.
\end{aligned}
$$

Fire-fighting pumping station: Case 2 - 9

The operational states in which the provided flow rate satisfies the request are states 1, 2, and possibly 3 if $r_d \geq q_{req}$.

Hence, the probability of safe operation and the total expected capacity are:

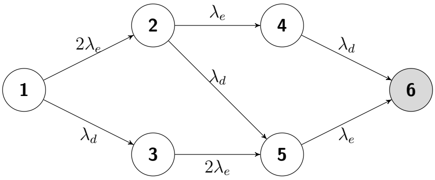
$$
\begin{aligned}
S_{Case2}(t) &= \pi_1(t) + \pi_2(t)(+\pi_3(t)) \\
Y_{Case2}(t) &= \int_0^t \sum_{i=1}^{4} r_i \, \pi_i(u) \, d\, u \,.
\end{aligned}
$$

Fire-fighting pumping station: Case 3 - 10

The requested instantaneous flow rate requires that all the three pumps are put in operation upon demand $q_{req3} = 2 \times r_e + r_d$.

State 1 -	Both the EPs and the DP are in operation ($r_1 = 2\, r_e + r_d$)
State 2 -	One EP has failed the other EP and the DP are in operation ($r_2 = r_e + r_d$)
State 3 -	The DP has failed both the EPs are in operation ($r_3 = 2\, r_e$)
State 4 -	Both the EPs have failed and the DP is in operation ($r_4 = r_d$)
State 5 -	One EP and the DP have failed the other EP is in operation ($r_5 = r_e$)
State 6 -	Both the EPs and the DP have failed ($r_6 = 0$)

Upon demand, the initial probability vector has the following expression:



$$
\begin{aligned}
\pi_1(0) &= c_e^2\, c_d \\
\pi_2(0) &= (1 - c_e)\, c_e\, c_d \\
\pi_3(0) &= c_e^2\, (1 - c_d) \\
\pi_4(0) &= c_d\, (1 - c_e)^2 \\
\pi_5(0) &= c_e\, (1 - c_e)\, (1 - c_d) \\
\pi_6(0) &= (1 - c_e)^2\, (1 - c_d)\,.
\end{aligned}
$$

Fire-fighting pumping station: Case 3 - 11

The probability of safe operation and the total expected capacity for Case 3 are:

$$
\begin{aligned}
S_{Case3}(t) &= \pi_1(t) \\
Y_{Case3}(t) &= \int_0^t \sum_{i=1}^{5} r_i \, \pi_i(u) \, d\,u \,.
\end{aligned}
$$

Fire-fighting pumping station: Results - 12

For the three cases ($i = 1, 2, 3$), we have calculated the non-success probability $(1 - S_{Case\ i}(t))$ and the relative flow rate difference $Y_{req\ i}(t)$:

$$\Delta\, Y_{Case\ i}(t) = \frac{Y_{Case\ i}(t) - Y_{req\ i}(t)}{Y_{req\ i}(t)} \quad \text{with} \quad Y_{req\ i}(t) = q_{req\ i} \times t\,.$$

In the computation, with the tool SHARPE, we have assumed the following numerical values

$$c_e = 0.99 \;;\; \lambda_e = 0.6\,10^{-5}\ f/h \;;\; c_d = 0.98 \;;\; \lambda_d = 0.3\,10^{-4}\ f/h$$

with two possible levels of the pump flow rates.

$$\begin{array}{llcl} \text{Level 1)} & r_e = 500\ m^3/h & ; & r_d = 500\ m^3/h \\ \text{Level 2)} & r_e = 500\ m^3/h & ; & r_d = 1000\ m^3/h \quad . \end{array}$$

Fire-fighting pumping station: Results - 13

Table: Non-success probability and flow rate difference for Level 1

t	Case 1		Case 2		Case 3	
(hr)	$1 - S_{C1}(t)$	$\Delta\ Y_{C1}(t)$	$1 - S_{C2}(t)$	$\Delta\ Y_{C2}(t)$	$1 - S_{C3}(t)$	$\Delta\ Y_{C3}(t)$
0	2.00E-06		4.96E-04		3.95E-02	
12	2.05E-06	-2.03E-06	5.05E-04	-2.51E-04	3.99E-02	-1.34E-02
24	2.13E-06	-2.07E-06	5.19E-04	-2.55E-04	4.05E-02	-1.35E-02
72	2.39E-06	-2.19E-06	5.64E-04	-2.66E-04	4.24E-02	-1.38E-02

Table: Non-success probability and flow rate difference for Level 2

t	Case 1		Case 2		Case 3	
(hr)	$1 - S_{C1}(t)$	$\Delta\ Y_{C1}(t)$	$1 - S_{C2}(t)$	$\Delta\ Y_{C2}(t)$	$1 - S_{C3}(t)$	$\Delta\ Y_{C3}(t)$
0	2.00E-06		4.96E-04		3.95E-02	
12	2.05E-06	9.65E-05	5.05E-04	9.53E-03	3.99E-02	-1.51E-02
24	2.13E-06	9.73E-05	5.19E-04	9.56E-03	4.05E-02	-1.52E-02
72	2.39E-06	9.99E-05	5.64E-04	9.68E-03	4.24E-02	-1.56E-02

CTMC with Self-Loops

CTMC with self-loops

Self-loops in a CTMC are possible and in some cases are very consistent with the nature of the system to be modelled.

A self-loop is a transition that exits from a state and returns to the same state.

Self-loops are not visible in the generator of a CTMC even if present. For this reason they are usually neglected without affecting the solution equations.

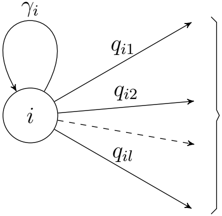
To make them visible we need to distinguish between the transition rate matrix and the infinitesimal generator.

The transition rate matrix groups all the transition rates including the self-loops on the diagonal; in the infinitesimal generator the diagonal is the opposite of the sum of all the rates (including the self-loop) outgoing from the state.

CTMC with self-loops

The Figure shows a self-loop emerging from state i of rate γ_i.

The transition rate matrix contains the value γ_i in the diagonal entry of row i, but in diagonal entry we have to add to γ_i minus the sum of all the rates outgoing from state i, including the self-loop.



$$
\begin{aligned}
q_i &= \gamma_i - (\gamma_i + \textstyle\sum_{j:j\neq i} q_{ij}) \\
&= \textstyle\sum_{j:j\neq i} q_{ij}
\end{aligned}
$$

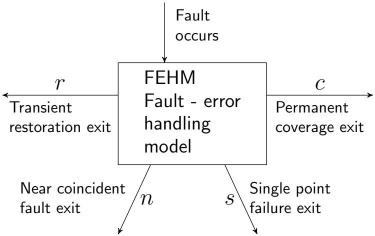
The final result is that the diagonal entry of the infinitesimal generator becomes minus the sum of the off-diagonal entries.

It turns out that the sojourn time in state i with self-loop is the same as the sojourn time in state i without self-loop.

Fault Error Handling Model (FEHM) - 1

Highly reliable systems that use extensive redundancy are highly reconfigurable and have complex recovery management techniques that are handled by a *Fault Error Handling Model (FEHM)*.

A schematic representation of a FEHM is given in Figure that allows for the modeling of permanent, intermittent, and transient faults, and models the on-line recovery procedure necessary for each type.



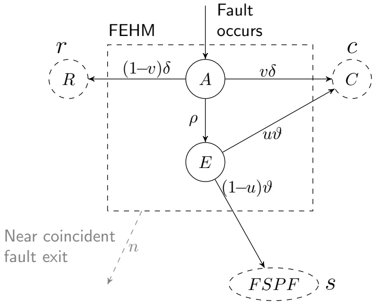
The FEHM has four possible exits for which a corresponding exit probability must be evaluate; it should be noted that:

$$r + c + s + n = 1$$

Fault Error Handling Model (FEHM) - 2

To evaluate the exit probabilities the internal structure of the FEHM must be made explicit.

A very simple CTMC model for a FEHM is represented in Figure where state A is the entry state as a fault occurs.



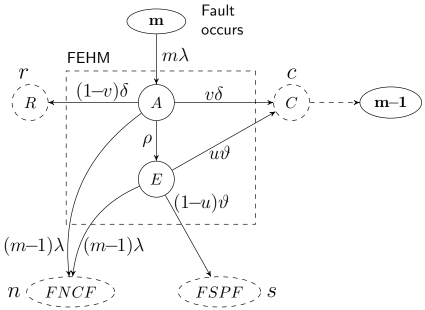
The following results are obtained (with $c + r + s = 1$):

$$c = \frac{v\delta + u\rho}{\delta + \rho} \quad ; \quad r = \frac{(1-v)\delta}{\delta + \rho} \quad ; \quad s = \frac{(1-u)\rho}{\delta + \rho}$$

Fault Error Handling Model (FEHM) - 3

Up to now the FEHM was considered in isolation. But if we now consider the FEHM integrated into a real system.

To be more concrete, suppose that the system is in a state with *m* redundant components in operation. As a failure occurs with rate $m\lambda$, state *A* of the FEHM is entered as shown in Figure.



During the handling of the fault in states *A* and *E*, $(m-1)$ redundant components are still working and they may fail at rate $(m-1)\lambda$ leading to the permanent failure state denoted by *FNCF*.

Fault Error Handling Model (FEHM) - 4

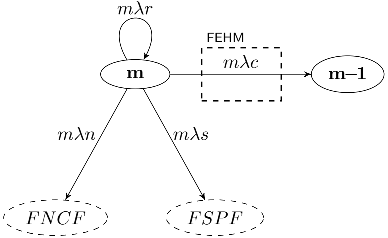
Computation of the exit probabilities provides the following results (with $c + r + s + n = 1$):

$$
\begin{aligned}
c &= \frac{((m-1)\lambda + \vartheta)v\delta + u\rho\vartheta}{((m-1)\lambda + \delta + \rho)((m-1)\lambda + \vartheta)} \\
r &= \frac{(1-v)\delta}{(m-1)\lambda + \delta + \rho} \\
s &= \frac{(1-u)\rho\vartheta}{((m-1)\lambda + \delta + \rho)((m-1)\lambda + \vartheta)} \\
n &= \frac{(m-1)\lambda((m-1)\lambda + \vartheta) + (m-1)\lambda\rho}{((m-1)\lambda + \delta + \rho)((m-1)\lambda + \vartheta)}
\end{aligned}
$$

Fault Error Handling Model (FEHM) - 5

Since the FEHM is composed of events that occur in rapid succession, once a fault has occurred, each FEHM can be replaced by an instantaneous branch point obtaining the *imperfect coverage* model of the Figure.

The transient restoration probability *r* gives rise to a self loop on state *m*.



The instantaneous imperfect coverage model is known to be conservative.

The uniformization of a CTMC - 1

Uniformization provides the most efficient numerical technique for solving a CTMC.

A simple argument for explaining how the uniformization works can be introduced resorting to the use of self-loops [*].

Consider a CTMC with infinitesimal generator $\boldsymbol{Q}$ and let q be at least equal to the maximum diagonal entry in absolute value

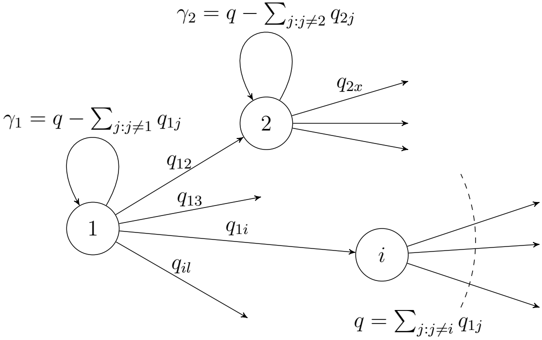
$$q \geq \max_i |q_{ii}|$$

$$\boldsymbol{Q} = \begin{bmatrix} -q_1 & q_{12} & \dots & q_{1,i} & \dots & q_{1,n} \\ q_{21} & -q_2 & \dots & q_{2,i} & \dots & q_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ q_{i,1} & q_{i,2} & \dots & -q_i & \dots & q_{i,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ q_{n,1} & q_{n2} & \dots & q_{n,i} & \dots & -q_n \end{bmatrix}$$

[*] R. Marie, Transient numerical solutions of stiff Markov chains, in Proceedings 20-th ISATA Symposium, 1989, pp. 255-270.

The uniformization of a CTMC - 2

In each state of the CTMC the outgoing rate $|q_{ii}|$ is less than q.



We add to each state j a self-loop with rate

$$\gamma_j = q - \sum_{k=1,n;k\neq j} q_{jk} = = q - |q_{jj}| \geq 0$$

The uniformization of a CTMC - 3

The CTMC with self-loops is the *uniformized* CTMC.

In the uniformized CTMC, all the departures out of any state occur with the same rate q and, hence, the sequence of departures forms a Poisson Process of rate q

Given that one transition is occurred, the CTMC jumps to the next state according to the conditional probability matrix $\boldsymbol{Q}^\star$

$$\boldsymbol{Q}^* = \begin{bmatrix} 1 - \frac{|q_{11}|}{q} & \frac{q_{12}}{q} & \dots & \frac{q_{1i}}{q} & \dots & \frac{q_{1n}}{q} \\ \frac{q_{21}}{q} & 1 - \frac{|q_{22}|}{q} & \dots & \frac{q_{2i}}{q} & \dots & \frac{q_{2n}}{q} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{q_{i1}}{q} & \frac{q_{i2}}{q} & \dots & 1 - \frac{|q_{ii}|}{q} & \dots & \frac{q_{in}}{q} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{q_{n1}}{q} & \frac{q_{n2}}{q} & \dots & \frac{q_{ni}}{q} & \dots & 1 - \frac{|q_{nn}|}{q} \end{bmatrix} = \boldsymbol{I} + \frac{\boldsymbol{Q}}{q} .$$

The uniformization of a CTMC - 4

$\boldsymbol{Q}^\star$ is a stochastic matrix with all entries $0 \leq q^\star_{ij} \leq 1$ and row sum equal to 1.

$\boldsymbol{Q}^\star$ represents the generator matrix of the Discrete Time Markov Chain (DTMC) embedded into the uniformized CTMC generated by matrix $\boldsymbol{Q}$.

The number $N(t)$ of transitions at time t, in the uniformized CTMC, is given by the number of events in a Poisson Process of rate q, hence:

$$P\{N(t) = k\} = \frac{(qt)^k}{k!} e^{-qt}$$

The uniformization of a CTMC - 5

The state probability vector after exactly ($j = 0, 1, 2, \ldots, k$) jumps is then:

$$
\begin{array}{ll}
\text{0 transitions} & \boldsymbol{\pi}(t|(N(t)=0) \;=\; \boldsymbol{\pi}(0)\, P\{N(t)=0\} \;=\; \boldsymbol{\pi}(0)\, e^{-qt} \\
\text{1 transition} & \boldsymbol{\pi}(t|(N(t)=1) \;=\; \boldsymbol{\pi}(0)\; \boldsymbol{Q}^{*}\; P\{N(t)=1\} \;=\; \boldsymbol{\pi}(0)\dfrac{q\,t}{1!}\, e^{-q\,t}\, \boldsymbol{Q}^{*} \\
\text{2 transitions} & \boldsymbol{\pi}(t|(N(t)=2) \;=\; \boldsymbol{\pi}(0)\; \boldsymbol{Q}^{*\,2} P\{N(t)=2\} \;=\; \boldsymbol{\pi}(0)\dfrac{(q\,t)^2}{2!}\, e^{-q\,t}\, \boldsymbol{Q}^{*\,2} \\
\ldots & \ldots \\
k \text{ transitions} & \boldsymbol{\pi}(t|(N(t)=k) \;=\; \boldsymbol{\pi}(0)\, \boldsymbol{Q}^{\star\,k}\, P\{N(t)=k\} \;=\; \boldsymbol{\pi}(0)\, \boldsymbol{Q}^{\star\,k}\, \dfrac{(qt)^k}{k!}\, e^{-qt}
\end{array}
$$

Summing up over k:

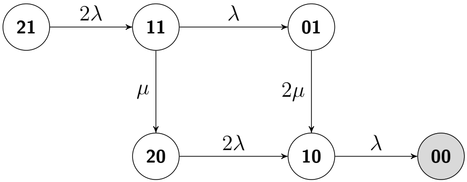
$$
\boldsymbol{\pi}(t) \;=\; \boldsymbol{\pi}(0)\, e^{-q\,t} \sum_{j=0}^{\infty} \frac{(q\,t)^j}{j!}\, \boldsymbol{Q}^{*j}
$$

Example: A redundant system with spare - 1

A system is composed by two active processors in parallel redundancy and one spare in cold standby.

The system is part of a satellite control system for which remote recovery is possible but not repair.

The system is working when at least one processor is working: Exhaustion of the components leads to a permanent failure.

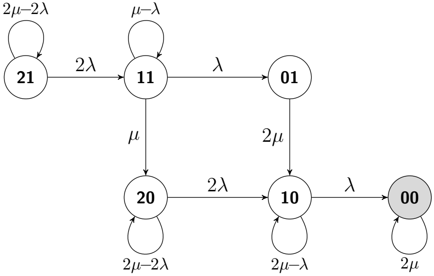


Example: A redundant system with spare - 2

Each state is labelled with tho indices (i, j) where the first index i denotes the number of active processors and j the number of active spares. The failure rate is λ and the recovery rate is μ.

The recovery rates are orders of magnitude larger than the failure rates and the largest diagonal entry in absolute value is $q = 2\mu$.

In the uniformized CTMC each state is added a self-loop so that the total exit rate is equal to 2μ.

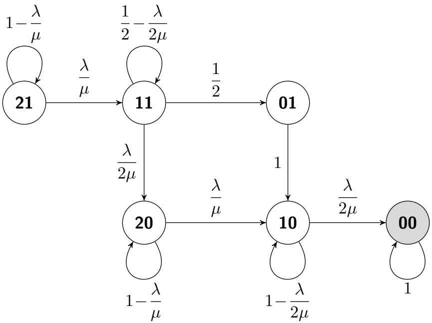


Example: A redundant system with spare - 3

We derive the embedded DTMC representing the conditional next state transition probability matrix

$$\boldsymbol{Q}^{\star} = \boldsymbol{I} + \frac{\boldsymbol{Q}}{q}$$

The labels are the probabilities of the DTMC



Transient Solution Methods

Transient Solution Methods

Transient solution methods can be categorized as fully symbolic, semi-symbolic or numerical.

Closed-form, fully symbolic solution is possible for either highly structured CTMCs (e.g., birth-death process) or very small CTMCs (as many examples in this Chapter).

In all the other cases, we must resort to numerical solution techniques.

In the following slides we illustrate the main solution techniques providing the appropriate references for implementation details.

Fully Symbolic and Semi-symbolic Methods

The convolution integration method and the Laplace transform method offer the possibility of a fully symbolic solution that has been used in many examples of this chapter.

Fully symbolic solution is applicable to CTMC with a very small number of states.

In a semi-symbolic (or semi-numerical) solution the entries in the $\boldsymbol{Q}$ matrix are numerical but the final solution $\boldsymbol{\pi}(t)$ is a symbolic function of the time.

The semi-symbolic method is simpler than the fully symbolic method and has been implemented in the SHARPE software package.

Transient Solution via Series Expansion - 1

The Taylor series expansion around $t = 0$ of the transition probability matrix $\boldsymbol{P}(t)$ is:

$$\boldsymbol{P}(t) = \boldsymbol{I} + \left.\frac{d\,\boldsymbol{P}(t)}{d\,t}\right|_{t=0} t + \frac{1}{2!}\left.\frac{d^2\,\boldsymbol{P}(t)}{d\,t^2}\right|_{t=0} t^2 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i\,!}\left.\frac{d^i\,\boldsymbol{P}(t)}{d\,t^i}\right|_{t=0} t^i\,.$$

Since

$$\frac{d\,\boldsymbol{P}(t)}{d\,t} = \boldsymbol{P}(t)\,\boldsymbol{Q} \quad \Longrightarrow \quad \left.\frac{d\,\boldsymbol{P}(t)}{d\,t}\right|_{t=0} = \boldsymbol{Q}$$

$$\frac{d^2\,\boldsymbol{P}(t)}{d\,t^2} = \frac{d}{d\,t}(\boldsymbol{P}(t)\,\boldsymbol{Q}) = \boldsymbol{P}(t)\,\boldsymbol{Q}^2 \quad \Longrightarrow \quad \left.\frac{d^2\,\boldsymbol{P}(t)}{d\,t^2}\right|_{t=0} = \boldsymbol{Q}^2$$

$$\ldots \qquad \ldots$$

$$\frac{d^i\,\boldsymbol{P}(t)}{d\,t^i} = \frac{d}{d\,t}(\boldsymbol{P}(t)\,\boldsymbol{Q}^{(i-1)}) = \boldsymbol{P}(t)\,\boldsymbol{Q}^i \quad \Longrightarrow \quad \left.\frac{d^i\,\boldsymbol{P}(t)}{d\,t^i}\right|_{t=0} = \boldsymbol{Q}^i$$

$$\ldots \qquad \ldots$$

$$\boldsymbol{P}(t) = \boldsymbol{I} + \boldsymbol{Q}\,t + \frac{1}{2}(\boldsymbol{Q}\,t)^2 + \ldots = \sum_{i=0}^{\infty} \frac{1}{i\,!}(\boldsymbol{Q}\,t)^i$$

Transient Solution via Series Expansion - 2

By analogy with scalar expansion, the above series is written in the compact form as a matrix exponential:

$$\boldsymbol{P}(t) = e^{\boldsymbol{Q}\,t}\,.$$

The above series can be utilized for implementing the numerical solution of a CTMC. There are, however, practical problems in the implementation of this approach.

1. $\boldsymbol{Q}$ has both negative and positive entries and hence the computation has both additions and subtractions (with poor numerical behavior);
2. raising the matrix $\boldsymbol{Q}$ to its powers is both costly and fills in the zeros in the matrix (that, in general, is very large and very sparse);
3. the infinite series needs to be truncated, and the approximation error is not known.

Transient Solution via Series Expansion - 3

A practical implementation, can however follow the following pattern.

Given that t_M is the final mission time of the transient analysis, we take a small time interval $h = t_M/n$ and we exploit the property.

$$\boldsymbol{\pi}(t+h) = \boldsymbol{\pi}(t) \cdot \boldsymbol{P}(h) = \boldsymbol{\pi}(t) \cdot e^{\boldsymbol{Q}\,h}\,.$$

Then, we adopt the following iterative procedure:

$$\begin{array}{lcl} \boldsymbol{\pi}(h) & = & \boldsymbol{\pi}(0) \cdot e^{\boldsymbol{Q}\,h} \\ \boldsymbol{\pi}(2\,h) & = & \boldsymbol{\pi}(0) \cdot e^{\boldsymbol{Q}\,2\,h} = \boldsymbol{\pi}(h) \cdot e^{\boldsymbol{Q}\,h} \\ \dots & & \dots \\ \boldsymbol{\pi}(n\,h) & = & \boldsymbol{\pi}((n-1)\,h) \cdot e^{\boldsymbol{Q}\,h} \\ \dots & & \dots \end{array}$$

With this scheme we need to compute the matrix exponential via the series expansion only once at the time point h.

Since h can be small the expansion uses few terms.

Transient Solution via Uniformization - 1

We search for the maximum diagonal entry in absolute value of the infinitesimal generator

$$q \;=\; \max_j \, |q_{jj}|$$

and we multiply both sides of the Kolmogorov Equation to get

$$\begin{aligned}
\boldsymbol{\pi}(t)\, e^{q\,t} &= \boldsymbol{\pi}(0) \cdot \; e^{q\,t}\, e^{\boldsymbol{Q}\,t} \\
&= \boldsymbol{\pi}(0) \cdot \; e^{[\boldsymbol{Q}/q+\boldsymbol{I}]q\,t} \\
&= \sum_{k=0}^{\infty} \pi(0)\, \frac{(qt)^k}{k!} \cdot (\boldsymbol{Q}^{\star})^k \,.
\end{aligned}$$

Finally,

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\, e^{-qt} \sum_{k=0}^{\infty} \frac{(qt)^k}{k!} (\boldsymbol{Q}^{\star})^k.$$

Matrix $\boldsymbol{Q}^{\star}$ is a DTMC matrix, with non negative entries and ≤ 1.

Transient Solution via Uniformization - 2

To avoid the problem of raising matrix $\boldsymbol{Q}$ to its powers, we rewrite the above Equation as

$$\boldsymbol{\pi}(t) = \sum_{k=0}^{\infty} \boldsymbol{\theta}(k) e^{-qt} \frac{(qt)^k}{k!} \,,$$

where $\boldsymbol{\theta}(0) = \boldsymbol{\pi}(0)$ and, recursively

$$\boldsymbol{\theta}(k) = \boldsymbol{\theta}(k-1)\, \boldsymbol{Q}^{\star}, \qquad k = 1, 2, \ldots \,.$$

The term $\boldsymbol{\theta}(k)$ can be interpreted as the kth step state probability vector of a DTMC with transition probability matrix $\boldsymbol{Q}^{\star}$.

The term $e^{-qt}(qt)^k/k!$ is the Poisson pmf with parameter qt.

Transient Solution via Uniformization - 3

In a numerical implementation the infinite series must be left/right-truncated between a minimal value k_m and a maximum value k_M.

$$\boldsymbol{\pi}(t) \approx e^{-qt} \sum_{k=k_m}^{k_M} \boldsymbol{\theta}(k) \frac{(qt)^k}{k!} \,. \qquad (1)$$

The values k_m and k_M can be determined from the specified truncation error tolerance ϵ by

$$\sum_{k=0}^{k_m-1} e^{-qt} \frac{(qt)^k}{k!} \le \frac{\epsilon}{2}, \qquad 1 - \sum_{k=0}^{k_M} e^{-qt} \frac{(qt)^k}{k!} \le \frac{\epsilon}{2} \,.$$

Transient Solution via Uniformization - 4

A simple way to estimate k_m and k_M is based on the fact that, as the parameter $q\,t$ is large enough, a Poisson distribution tends to be normally distributed with mean $q\,t$ and standard deviation $\sqrt{q\,t}$.

In a normal distribution $\mathcal{N} \sim (\mu, \sigma)$, the area outside the interval $\mu \pm 6\sigma$ is $1.2 \cdot 10^{-8}$.

Hence, we can assume:

$$k_m = q\,t - 6\,\sqrt{q\,t} \qquad\qquad k_M = q\,t + 6\,\sqrt{q\,t}$$

ODE-based Solution Methods - 1

The Kolmogorov Equation can be solved by resorting to standard techniques for ODEIVP.

ODE solution methods discretize the solution interval into a finite number of time intervals $\{t_1, t_2, ..., t_i, ..., t_n\}$.

Given a solution at t_i, the solution at $t_i + h$ $(= t_{i+1})$ is computed. Advancement in time is made with step size h, until the time at which the solution is desired (the *mission time*) is reached.

A method that only uses $(t_i, \boldsymbol{\pi}_i)$ to compute $\boldsymbol{\pi}_{i+1}$ is said to be an *explicit single step method*. A *multi-step method* uses approximations at several previous steps to compute the new approximation. *An implicit method* requires a value for $\boldsymbol{\pi}_{i+1}$ in computing $\boldsymbol{\pi}_{i+1}$.

ODE-based Solution Methods: TR-BDF2 - 2

A composite method that uses one step of TR (trapezoidal rule) and one step of BDF2 (second order backward difference equation [*].
The trapezoid rule applied to interval $(t_i, t_i + \gamma h_i]$ is:

$$\boldsymbol{\pi}_{i+\gamma} - \boldsymbol{\pi}_i = \gamma h_i \cdot \frac{\boldsymbol{\pi}_{i+\gamma}\, \boldsymbol{Q} + \boldsymbol{\pi}_i\, \boldsymbol{Q}}{2}$$

Application of the TR step is computationally the same as solving the first order linear algebraic system

$$\boldsymbol{\pi}_{i+\gamma}\,(\boldsymbol{I} - \frac{\gamma\, h_i}{2}\boldsymbol{Q}) = \boldsymbol{\pi}_i\,(\boldsymbol{I} + \frac{\gamma\, h_i}{2}\boldsymbol{Q})$$

After getting $\boldsymbol{\pi}_{i+\gamma}$, the TR-BDF2 method uses the 2^{nd} *order backward difference equations* (BDF2) to step from $t_i + \gamma h_i$ to t_{i+1}:

$$\boldsymbol{\pi}_{i+1}((2-\gamma)\,\boldsymbol{I} - (1-\gamma)\,h_i\,\boldsymbol{Q}) = \frac{1}{\gamma}\,\boldsymbol{\pi}_{i+\gamma} - \frac{(1-\gamma)^2}{\gamma}\,\boldsymbol{\pi}_i$$

ODE-based Solution Methods: TR-BDF2 - 3

Most implementations of ODE methods adjust the step-size at each step, based on the amount of error in the solution computed at the end of the previous step.

The TR-BDF2 method provides reasonable accuracy for error tolerances up to 10^{-8} and excellent stability for stiff Markov chains [*].

[*] A. Reibman and K. Trivedi, "Numerical transient analysis of Markov models," *Computers and Operations Research*, vol. 15, pp. 19–36, 1988.

ODE-based Solution Methods: Implicit Runge-Kutta - 4

Implicit Runge-Kutta methods of different orders of accuracy are possible as proposed in [**].

A third order method is given by:

$$\boldsymbol{\pi}_{i+1}(\boldsymbol{I} - \frac{2}{3}\,h\,\boldsymbol{Q} + \frac{1}{6}h^2\,\boldsymbol{Q}^2) = \boldsymbol{\pi}_i(\boldsymbol{I} + \frac{1}{3}h\boldsymbol{Q})\ .$$

Various possibilities exist for solving the above Equation.

- One possibility is to compute the matrix polynomial directly. It was found in various experiments that the fill-in of the squared generator matrix was reasonably low.
- The other option is to factorize the matrix polynomial. We then need to solve two successive linear algebraic systems.

[**] M. Malhotra, J. K. Muppala, and K. S. Trivedi, "Stiffness-tolerant methods for transient analysis of stiff markov chains," *Microelectronics and Reliability*, vol. 34, pp. 1825–1841, 1994.