February 2, 2015

Problems for Chapters 4 and 10 of *Advanced Mathematics for Applications*

The Fourier Transform

by Andrea Prosperetti

Notes: Some problems on the calculation of Fourier transforms by contour integration will be found in the problem set for chapter 17.

1 General

1. By suitably deforming the integration path calculate the Fourier transforms of the functions

$$u(x) = \cos ax^2\,, \qquad u(x) = \sin ax^2\,.$$

2. By suitably deforming the integration path calculate the Fourier transforms of the function

$$u(x) = e^{-x^2}\,, \qquad u(x) = \operatorname{erf} ax\,.$$

3. Use the Parseval relation to show that

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}\,.$$

4. Use the Parseval relation to show that

$$\int_0^\infty \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,\mathrm{d}x = \frac{\pi}{2(a+b)}\,.$$

5. Use the Parseval relation to show that

$$\int_0^\infty \frac{\sin ax\,\sin bx}{x^2}\,\mathrm{d}x = \frac{\pi}{2}\min(a,\,b)\,.$$

2 Exponential Fourier transform

1. By means of a Fourier transform in x solve, on the infinite line $-\infty < x < \infty$, the modified diffusion equation

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = -\alpha u$$

with D, α given positive constants. The initial condition is $u(x,0) = f(x)$ and u is bounded at infinity. Make sure that the solution satisfies the analog of (4.4.7) p. 95 for this case. Interpret the solution knowing that the equation represents the diffusion of heat in a medium in which there is an energy sink of strength αu per unit length and time.

2. By means of a Fourier transform in x solve the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the strip $-\infty < x < \infty$, $0 < y < L$ subject to $\partial u/\partial y|_{y=0} = f(x)$, $u(x, L) = 0$, $u \to 0$ for $|x| \to 0$. Compare with the solution found by using a Fourier series in y.

3. Solve the previous problem in the half plane $-\infty < x < \infty$, $0 < y < \infty$. The boundary condition is $u(x, y) \to 0$ for $|x|$ and y tending to infinity.

4. Solve the Poisson equation

$$\nabla^2 u = -4\pi\delta^{(3)}(\mathbf{x} - \mathbf{x}')$$

when the point $\mathbf{x}'$ is placed on the axis of an infinite cylinder on the surface of which $\mathbf{n} \cdot \boldsymbol{\nabla} u = 0$. This is the Green's function for the Neumann problem for the Poisson equation in which the right-hand side is only non-zero on the axis of the cylinder (cf. section 16.1 p. 400). Use Table 6.3 p. 148 for the appropriate form of the three-dimensional δ and refer to section 6.1 for the solution of the equation in the radial variable. The final answer contains an integral which cannot be evaluated in closed form.

5. For $-\infty < x < \infty$ and $t > 0$ solve the diffusion problem

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \lambda u = f(x, t)$$

in which D and λ are given positive constants and f is a known function. The initial condition is $u(x, 0) = u_0(x)$ and u is bounded at infinity for all $t > 0$.

6. For $-\infty < x < \infty$, $0 < y < \infty$ solve the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^4} - c^2 u = 0$$

subject to the boundary conditions $\partial u/\partial y|_{y=0} = -f(x)$ and boundedness at infinity.

7. For $-\infty < x < \infty$ and $t > 0$ solve the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0$$

The initial conditions are $u(x, 0) = u_0(x)$; u is bounded at infinity for all $t > 0$. (Use (10.7.15) p. 279 to invert the transform.)

8. For $-\infty < x < \infty$ and $t > 0$ express in terms of an integral the solution of the one-dimensional Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \gamma^2\frac{\partial^2 u}{\partial x^2} + c^2 u = 0$$

subject to general initial conditions $u(x, 0) = u_0(x)$, $\partial u/\partial t|_{t=0} = g(x)$.

9. For $-\infty < x < \infty$ and $t > 0$ express in terms of an integral the solution of the one-dimensional linearized Korteweg-deVries equation

$$\frac{\partial u}{\partial t} + a^2\frac{\partial u}{\partial x} + b^2\frac{\partial^3 u}{\partial x^3} = 0$$

subject to the general initial condition $u(x, 0) = u_0(x)$.

3 Sine and cosine transforms

1. By means of the Fourier transform in x solve, on the semi-infinite line $0 < x < \infty$, the modified diffusion equation
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = -\alpha u$$
with D, α given positive constants. The initial condition is $u(x,0) = 0$, $u(0,t) = f(t)$ and u is bounded at infinity. After finding the general solution consider the particular case
$$f(t) = \frac{a}{a^2 + (t-b)^2}$$
with a, b given positive constants. Find a closed-form approximate solution for this case assuming that $a \ll b$ and $a \ll \alpha^{-1}$. Is there a condition on x for the validity of this approximation? (The relation (4.6.10) p. 108 is useful to derive the approximation.)

2. By means of a Fourier transform solve, in the semi-infinite strip $0 < x < L$, $0 < y < \infty$, the Poisson equation
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x)$$
with f given, subject to
$$u(0,L) = u(L,y) = 0\,, \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$$
and boundedness at infinity. Is there a condition of f for such a solution to exist?

3. By means of a Fourier transform in y solve the two-dimensional Poisson equation
$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = -2\pi\delta(x-x')\delta(y-y')$$
in the semi-infinite strip $0 < x < L$, $0 < y < \infty$. The boundary conditions are $G = 0$ on the finite boundary and at infinity. Compare with the solution found by using a Fourier series in x. This is the Green's function for the Dirichlet problem for the Poisson equation in the semi-infinite strip (cf. section 16.1 p. 400).

4. Solve Laplace's equation
$$\nabla^2 u = 0$$
inside a semi-infinite cylinder of radius a. On the base of the cylinder $u = u_0$, a constant, while $u = 0$ on the lateral surface. The final answer contains an integral which cannot be evaluated in closed form.

5. For $x > 0$ and $t > 0$ solve the diffusion problem
$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} + \lambda u = f(x,t)$$
in which D and λ are given positive constants and f is a known function. The initial condition is $u(x,0) = u_0(x)$ and, for $x = 0$, $u(0,t) = g(t)$; u is bounded at infinity for all $t > 0$.

6. In $x < 0 < \infty$, use a Fourier transform to solve, for $t > 0$, the problem
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$
with $u(x,0) = 0$, $\partial u/\partial t|_{t=0} = 0$ and $u(0,t) = g(t)$.

7. In $0 < x < \infty$, use a Fourier transform to solve, for $t > 0$ the problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

with $u(x, 0) = 0$, $\partial u/\partial t|_{t=0} = 0$ and $\partial u/\partial x|_{x=0} = h(t)$.

8. In $0 < x < \infty$, use a Fourier transform to solve, for $t > 0$ the problem

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

with $u(x, 0) = 0$, $\partial u/\partial t|_{t=0} = 0$ and $\partial u/\partial x|_{x=0} = h(t)$.

9. In $0 < x < \infty$, $0 < y < \infty$ use a Fourier transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with u bounded at infinity, $\partial u/\partial x|_{x=0} = 0$ and $u(x, 0) = 0$ except for the range $0 < x < 1$ where $u(x, 0) = f(x)$.

10. By means of a double Fourier transform in x and y solve the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the quadrant $0 < x$, $y < \infty$ subject to the conditions of boundedness at infinity and

$$u(x, 0) = f(x)\,, \qquad u(0, y) = g(y)\,.$$

11. By means of a double Fourier transform in x and y solve the two-dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in the quadrant $0 < x$, $y < \infty$ subject to the conditions of boundedness at infinity and

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = f(x)\,, \qquad u(0, y) = g(y)\,.$$

12. Use a suitable combination of Fourier transforms to solve the two-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

in the range $-\infty < x < \infty$, $0 < y < \infty$, subject to the initial condition $u(x, y, 0) = f(x, y)$ given and $u(x, 0, t) = 0$.

4 Integral equations

1. Solve the integral equation

$$(\exp b^2)\; u(x) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left[-(x - y)^2\right] u(y)\, dy = (\cosh b^2)\, \cos 2bx\,,$$

where b is a real constant.

2. Solve the integral equation
$$\int_{-\infty}^{\infty} e^{-\lambda|x-\xi|} u(\xi)\,\mathrm{d}\xi \;=\; f(x)$$
where $\mathrm{Re}\,\lambda > 0$.

3. Solve the integral equation
$$\int_{-\infty}^{\infty} \sin(\lambda|x-\xi|)\, u(\xi)\,\mathrm{d}\xi \;=\; f(x)\,.$$

4. Solve the integral equation
$$\int_{-\infty}^{\infty} K_0(|x-\xi|)\, u(\xi)\,\mathrm{d}\xi \;=\; f(x)\,,$$
where K_0 is the modified Bessel function of the second kind.

5. Show that
$$f(x) \;=\; \int_{-\infty}^{\infty} e^{-|x-y|} u(y)\mathrm{d}y\,, \qquad -\infty < x < \infty\,,$$
in which f is a given function, has the solution $u(x) = \frac{1}{2}[f(x) - f''(x)]$ provided f, f' are suitably behaved as $|x| \to \infty$.

6. Solve Fox's integral equation
$$u(x) \;=\; f(x) + \lambda\sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin xy\, u(y)\,\mathrm{d}y\,, \qquad 0 < x < \infty\,.$$
What are the values of λ for which a solution exits only when the given function f satisfies suitable restrictions? [Hint: Take the Fourier sine transform and use its properties.]

5 Integral asymptotics

1. Make the change of variable $y = \xi^2$ in the integral defining the complementary error function
$$\mathrm{erfc}\, x \;=\; \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-\xi^2}\,\mathrm{d}\xi$$
and derive an asymptotic expansion of $\mathrm{erfc}\, x$ for $x \to \infty$ by repeated integration by parts.

2. Find the leading order term as $\epsilon \to 0$ of the integral
$$\int_0^{\infty} (1+\epsilon x)^{-1} e^{-x}\,\mathrm{d}x$$
by reducing it to a Laplace-type integral with a suitable change of variable.

3. Find the leading order term as $k \to \infty$ of the integral
$$\int_0^{\infty} e^{-k\,\sinh^2 x}\,\mathrm{d}x\ .$$

4. Find the leading order term as $k \to \infty$ of the integral
$$\int_0^{\infty} \frac{e^{-kx^2}}{\sqrt{\sinh x}}\,\mathrm{d}x\ .$$

5. Show that

$$\int_0^\infty \log\left(\frac{x}{1-e^{-x}}\right)\frac{e^{-kx}}{x}\,\mathrm{d}x \;\to\; \frac{1}{2k}$$

for $k \to \infty$.

6. Show that

$$\int_0^1 \sqrt{x}\,e^{ikx}\,\mathrm{d}x \;\to\; \frac{\sqrt{\pi}}{k}\,e^{i\pi/4}$$

for $k \to \infty$.

7. Find the leading order behavior of the integrals

$$\int_0^1 (\tan x)\,e^{ikx^4}\,\mathrm{d}x\;, \qquad \int_{1/2}^2 (1+x)e^{ik(x^3/3-x)}\,\mathrm{d}x$$

for $k \to \infty$.

8. Apply the method of stationary phase to the integral representation (12.2.25) p. 308 of the Bessel function $J_n(x)$ to obtain the asymptotic result (12.2.29) p. 309 for $x \to \infty$.

9. Apply the method of stationary phase to the integral representation (4.3.6) p. 94 of the Airy function $\mathrm{Ai}\,(x)$ to show that, for $x \to -\infty$,

$$\mathrm{Ai}\,(x) \to \frac{|x|^{-1/4}}{\sqrt{\pi}}\sin\left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right)\,.$$

Does the method work for $x \to \infty$?

10. For $-\infty < x < \infty$ and $t > 0$ a general form of the solution of the one-dimensional Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \gamma^2\frac{\partial^2 u}{\partial x^2} + c^2 u \;=\; 0$$

is

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U_+(k)e^{i(\omega t-kx)}\,\mathrm{d}k \;+\; \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U_+(k)e^{-i(\omega t+kx)}\,\mathrm{d}k$$

where $\omega = \sqrt{\gamma^2k^2+c^2}$. Obtain an asymptotic approximation for u for large x and t. (You may wanto to refer to section 6.8 p. 166 for a similar calculation.)

11. For $-\infty < x < \infty$ and $t > 0$ express in terms of an integral the solution of the one-dimensional linearized Korteweg-deVries equation

$$\frac{\partial u}{\partial t} + a^2\frac{\partial u}{\partial x} + b^2\frac{\partial^3 u}{\partial x^3} \;=\; 0$$

subject to the general initial condition $u(x,0) = u_0(x)$. Apply the method of stationary phase to find the limit form for large x and t.