Adaptive Wireless Communications

MIMO Channels and Networks

Selected Solutions to Problems

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1 History

No problems.

2 Notational and Mathematical Preliminaries

Problems

2.1 Evaluate the following expressions.

(a) $\Re\sqrt{e^{-i\,\pi}}$

$$\Re\sqrt{e^{-i\,\pi}} = \Re\sqrt{-1} = \Re\left\{i\right\} = 0$$

(b) $\log_4(1+i)$

$$\begin{aligned}\log_4(1+i) &= \frac{1}{2}\log_2\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)\\ &= \frac{1}{2}\times\log_2\left(\sqrt{2}\right)+\frac{1}{2}\times\log_2\left(e^{i\frac{\pi}{4}}\right)\\ &= \frac{1}{4}+i\left(\frac{\pi}{8}+\pi\,m\right)\log_2\left(e\right) \quad \forall\, m\in\mathbb{Z}\end{aligned}$$

(c) $\int_{-\infty}^{\infty} dx\,\delta(x-1)\,\frac{\cosh^2[\pi\,(x-1)]}{\sqrt{2-x^2}}$

$$\int_{-\infty}^{\infty} dx\,\delta(x-1)\,\frac{\cosh^2[\pi\,(x-1)]}{\sqrt{2-x^2}} = \left.\frac{\cosh^2[\pi\,(x-1)]}{\sqrt{2-x^2}}\right|_{x=1} = 1$$

(d) $\left|\frac{1}{a}\mathbf{I}_4\right|$

$$\left|\frac{1}{a}\mathbf{I}_4\right| = \frac{1}{a^4}\left|\mathbf{I}_4\right| = a^{-4}$$

(e) $\Gamma(2)$

$$\Gamma(2) = (2-1)\,! = 1$$

2.2 For complex vectors $\mathbf{a}$ and $\mathbf{b}$, evaluate the following expressions.

(a) $\text{rank}\{\mathbf{a}\,\mathbf{b}^\dagger\}$

The SVD of the matrix $\mathbf{a}\,\mathbf{b}^\dagger$ with a single nonzero singular value, so the matrix is rank 1. The associated lefthand and righthand singular vectors are proportional to $\mathbf{a}$ and $\mathbf{b}$ respectively.

(b) $[\mathbf{I} - \mathbf{a}(\mathbf{a}^\dagger\mathbf{a})^{-1}\mathbf{a}^\dagger]\,\mathbf{a}$

$$\begin{aligned}[\mathbf{I} - \mathbf{a}(\mathbf{a}^\dagger\mathbf{a})^{-1}\mathbf{a}^\dagger]\mathbf{a} &= \mathbf{a} - \mathbf{a}(\mathbf{a}^\dagger\mathbf{a})^{-1}(\mathbf{a}^\dagger\mathbf{a})\\ &= \mathbf{a} - \mathbf{a}\\ &= \mathbf{0}\end{aligned}$$

(c) $[\mathbf{I} - \mathbf{a}(\mathbf{a}^\dagger\mathbf{a})^{-1}\mathbf{a}^\dagger]\,\mathbf{b}(\mathbf{b}^\dagger\mathbf{b})^{-1}\mathbf{b}^\dagger\,\mathbf{a}$

$$\begin{aligned}[\mathbf{I} - \mathbf{a}(\mathbf{a}^\dagger\mathbf{a})^{-1}\mathbf{a}^\dagger]\,\mathbf{b}(\mathbf{b}^\dagger\mathbf{b})^{-1}\mathbf{b}^\dagger\mathbf{a} &= \left[\mathbf{I} - \frac{\mathbf{a}\,\mathbf{a}^\dagger}{\|\mathbf{a}\|^2}\right]\frac{\mathbf{b}\,\mathbf{b}^\dagger}{\|\mathbf{b}\|^2}\,\mathbf{a}\\ &= \frac{\|\mathbf{a}\|^2\,(\mathbf{b}^\dagger\mathbf{a})\,\mathbf{b} - \|\mathbf{a}^\dagger\mathbf{b}\|^2\,\mathbf{a}}{\|\mathbf{a}\|^2\,\|\mathbf{b}\|^2}\end{aligned}$$

(d) $\left(\mathbf{I} + \mathbf{a}\mathbf{a}^\dagger\right)^{-1}\,\mathbf{b}$ if $\|\mathbf{a}^\dagger\mathbf{b}\| = 0$

From $\left\|\mathbf{a}^\dagger\mathbf{b}\right\| = 0$, we can conclude that $\mathbf{a}^\dagger\mathbf{b} = 0$. By using the matrix inversion formula, we have

$$\begin{aligned}(\mathbf{I} + \mathbf{a}\mathbf{a}^\dagger)^{-1}\mathbf{b} &= \mathbf{b} - \frac{\mathbf{a}\mathbf{a}^\dagger}{1 + \mathbf{a}^\dagger\mathbf{a}}\mathbf{b}\\ &= \mathbf{b}\end{aligned}$$

(e) $\left(\mathbf{I} + \mathbf{a}\mathbf{a}^\dagger\right)^{-1}\,\mathbf{b}$ if $\|\mathbf{a}^\dagger\mathbf{b}\| = 1/2$

From $\left\|\mathbf{a}^\dagger\mathbf{b}\right\| = 0$, we can conclude that $\mathbf{a}^\dagger\mathbf{b} = e^{i\phi}/2$, for some arbitrary phase ϕ. By using the matrix inversion formula, we have

$$\begin{aligned}(\mathbf{I} + \mathbf{a}\mathbf{a}^\dagger)^{-1}\mathbf{b} &= \mathbf{b} - \frac{\mathbf{a}\mathbf{a}^\dagger}{1 + \mathbf{a}^\dagger\mathbf{a}}\mathbf{b}\\ &= \mathbf{b} - \frac{e^{i\phi}\mathbf{a}}{2(1 + \mathbf{a}^\dagger\mathbf{a})}\end{aligned}$$

(f) $\log_2|\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger|$ if $\|\mathbf{a}\| = 1$

$$\begin{aligned}\log_2|\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger| &= \log_2|1 + \mathbf{a}^\dagger\mathbf{a}|\\ &= \log_2(1 + \|\mathbf{a}\|^2)\\ &= \log_2(1 + 1) = 1\end{aligned}$$

2.3 For unit-norm complex vectors $\mathbf{a}$ and $\mathbf{b}$, evaluate the following expressions.

(a) $\lambda_m\{\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\}$ if $\|\mathbf{a}^\dagger\mathbf{b}\|^2 = 1/2$

$$\lambda_m\{\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\} = 1 + \lambda_m\{\mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\}$$

$$\begin{aligned}
\lambda_m\{\mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\} &= \left\{\frac{\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \pm \sqrt{\left(\|\mathbf{a}\|^2 - \|\mathbf{b}\|^2\right)^2 + 4\|\mathbf{a}^\dagger\,\mathbf{b}\|^2}}{2}, 0, \cdots, 0\right\} \\
&= \left\{\frac{1 + 1 \pm \sqrt{(1-1)^2 + 4 \times \left(\frac{1}{2}\right)^2}}{2}, 0, \cdots, 0\right\} \\
&= \left\{\frac{2 \pm 1}{2}, 0, \cdots, 0\right\} \\
\lambda_m\{\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\} &= \left\{\frac{5}{2}, \frac{3}{2}, 1, \cdots 1\right\}
\end{aligned}$$

(b) $\mathrm{tr}\{\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\}$
If N is the dimension of the vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathrm{tr}\{\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger\} = \sum \lambda_m = \frac{5}{2} + \frac{3}{2} + (N-2) = N + 2$$

(c) $|\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger|$

$$|\mathbf{I} + \mathbf{a}\,\mathbf{a}^\dagger + \mathbf{b}\,\mathbf{b}^\dagger| = \prod_m \lambda_m = \frac{5}{2} \cdot \frac{3}{2} \cdot 1 \cdot \cdots \cdot 1 = \frac{15}{4}$$

2.4 For matrices $\mathbf{A} \in \mathbb{C}^{m,p}$ and $\mathbf{B} \in \mathbb{C}^{m,q}$, show that

$$|\mathbf{I} + \mathbf{A}\,\mathbf{A}^\dagger + \mathbf{B}\,\mathbf{B}^\dagger| \geq |\mathbf{I} + \mathbf{A}\,\mathbf{A}^\dagger|\,.$$

Assume:

$$\mathbf{M} = (\mathbf{I} + \mathbf{A}\mathbf{A}^\dagger)$$

Note that $\mathbf{M}^\dagger = \mathbf{M}$ and $\lambda_m\{\mathbf{M}\} \geq 1$. Showing the result is the same as

$$\begin{gathered}
|\mathbf{I} + \mathbf{A}\,\mathbf{A}^\dagger + \mathbf{B}\,\mathbf{B}^\dagger| \geq |\mathbf{M}| \\
|\mathbf{I} + \mathbf{A}\,\mathbf{A}^\dagger + \mathbf{B}\,\mathbf{B}^\dagger||\mathbf{M}|^{-1} \geq 1
\end{gathered}$$

This can be demonstrated by

$$
\begin{aligned}
|\mathbf{I} + \mathbf{A}\,\mathbf{A}^\dagger + \mathbf{B}\,\mathbf{B}^\dagger||\mathbf{M}|^{-1} &= |\mathbf{M}^{-1/2}(\mathbf{M} + \mathbf{B}\,\mathbf{B}^\dagger)\mathbf{M}^{-1/2}| \\
&= |(\mathbf{I} + \mathbf{M}^{-1/2}\,\mathbf{B}\,\mathbf{B}^\dagger\,\mathbf{M}^{-1/2})| \\
&= \prod_m \lambda_m\{\mathbf{I} + \mathbf{M}^{-1/2}\,\mathbf{B}\,\mathbf{B}^\dagger\,\mathbf{M}^{-1/2}\} \\
&= \prod_m (1 + \lambda_m\{\mathbf{M}^{-1/2}\,\mathbf{B}\,\mathbf{B}^\dagger\,\mathbf{M}^{-1/2}\}) \\
\geq 1
\end{aligned}
$$

Because the matrix is Hermitian, $\lambda_m\{\mathbf{M}^{-1/2}\,\mathbf{B}\,\mathbf{B}^\dagger\,\mathbf{M}^{-1/2}\} \geq 0$.

2.5 Evaluate the following Wirtinger derivatives (where z^* is interpreted as the doppelganger variable for the conjugation of z).

(a) $\frac{\partial}{\partial z^*} \sum_{m=0}^{\infty} \frac{z^m}{m^m}$

$$
\frac{\partial}{\partial z^*} \sum_{m=0}^{\infty} \frac{z^m}{m^m} = \sum_{m=0}^{\infty} \frac{\partial}{\partial z^*} \frac{z^m}{m^m} = \sum_{m=0}^{\infty} \frac{z^{m-1}}{m^{m-1}} \frac{\partial z}{\partial z^*} = 0
$$

(b) $\frac{\partial}{\partial \mathbf{z}^*} \mathbf{z}^\dagger\,\mathbf{z}$

$$
\frac{\partial}{\partial \mathbf{z}^*} \mathbf{z}^\dagger\,\mathbf{z} = \mathbf{z}^T
$$

(c) $\frac{\partial}{\partial \mathbf{z}^\dagger} \mathbf{z}^\dagger\,\mathbf{z}$

$$
\frac{\partial}{\partial \mathbf{z}^\dagger} \mathbf{z}^\dagger\,\mathbf{z} = \mathbf{z}
$$

(d) $\frac{\partial}{\partial \mathbf{z}^\dagger} \frac{\mathbf{z}^\dagger\,\mathbf{A}\,\mathbf{z}}{\mathbf{z}^\dagger\,\mathbf{B}\,\mathbf{z}}$

$$
\frac{\partial}{\partial \mathbf{z}^\dagger} \left(\frac{\mathbf{z}^\dagger\,\mathbf{A}\,\mathbf{z}}{\mathbf{z}^\dagger\,\mathbf{B}\,\mathbf{z}} \right) = \frac{\mathbf{A}\,\mathbf{z}}{\mathbf{z}^\dagger\,\mathbf{B}\,\mathbf{z}} - \frac{\mathbf{z}^\dagger\,\mathbf{A}\,\mathbf{z}}{(\mathbf{z}^\dagger\,\mathbf{B}\,\mathbf{z})^2} \mathbf{B}\,\mathbf{z}
$$

(e) $\frac{\partial}{\partial \mathbf{z}^*} \mathbf{z}^\dagger\,\mathbf{A}$

2.6 Evaluate the following integrals under the assumption that the closed contour encloses a radius of 10 of the origin.

(a) $\oint dz\, \frac{1}{(z-1)^2\, z}$

The poles are 1, 1 and 0. Hence,

$$
\mathrm{res}(f, 0) = \left. \frac{1}{(z-1)^2} \right|_{z=0} = 1
$$

$$
\mathrm{res}(f, 1) = \left. \frac{d}{dz} \frac{1}{z} \right|_{z=1} = -1
$$

Therefore, the integral is

$$2\pi(1-1)i = 0$$

(b) $\oint dz\, \frac{1}{(z-20)^2\, z}$
The poles are 20,20 and 0. Only the pole at the origin is enclosed by our loop. Hence,

$$\mathrm{res}(f,0) = \left.\frac{1}{(z-20)^2}\right|_{z=0} = \frac{1}{20^2}$$

Therefore, the integral is

$$\frac{2\pi\, i}{400}$$

(c) $\oint dz\, \frac{(z-2)(z-3)}{(z-1)^2\, z}$
The poles are 1,1 and 0. Hence,

$$\mathrm{res}(f,0) = \left.\frac{(z-2)(z-3)}{(z-1)^2}\right|_{z=0} = 6$$

$$\mathrm{res}(f,1) = \left.\frac{d}{dz}\frac{(z-2)(z-3)}{z}\right|_{z=1} = -5$$

Therefore, the integral is

$$2\pi(6-5)i = 2\pi i$$

(d) $\oint dz\, \frac{z}{(z^2-1)}$
The poles are +1 and -1. Hence,

$$\mathrm{res}(f,-1) = \left.\frac{z}{z-1}\right|_{z=-1} = \frac{1}{2}$$

$$\mathrm{res}(f,-1) = \left.\frac{z}{z+1}\right|_{z=1} = \frac{1}{2}$$

Therefore, the integral is

$$2\pi\left(\frac{1}{2}+\frac{1}{2}\right)i = 2\pi i$$

(e) $\oint dz\, \frac{e^z}{(z-1)}$
The simple pole is at z=1. Hence the residue is

$$\mathrm{res}(f,1) = e^z|_{z=1} = e$$

Therefore, the integral is

$$2\pi\, i\, e$$

2.7 Evaluate the following integrals where $\mathcal{V}$ indicates the entire volume spanned by the variables of integration.

(a) For real variables x and y
$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (x^2 + y^2)\, e^{-x^2}\, e^{-y^2}$

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (x^2 + y^2)\, e^{-x^2}\, e^{-y^2} \\
=&\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (x^2 + y^2)\, e^{-\left(x^2+y^2\right)}|_{x=r\cos\theta, y=r\sin\theta} \\
=&\int_{0}^{2\pi} d\theta \int_{0}^{\infty} dr\, r^3 e^{-r^2} \\
=&\pi
\end{aligned}$$

(b) For complex variable z
$\int_{\mathcal{V}} d\Omega_z \|z\|^2\, e^{-\|z\|^2}$

$$\int_{\mathcal{V}} d\Omega_z \|z\|^2\, e^{-\|z\|^2}|_{z=re^{i\theta}} = \int_{0}^{2\pi} d\theta \int_{0}^{\infty} dr\, r^3 e^{-r^2} = \pi$$

(c) For the complex n-vector $\mathbf{z}$
$\int_{\mathcal{V}} d\Omega_{\mathbf{z}} \|\mathbf{z}\|^2\, e^{-\|\mathbf{z}\|^2}$

$$\begin{aligned}
\int_{\mathcal{V}} d\Omega_{\mathbf{z}} \|\mathbf{z}\|^2\, e^{-\|\mathbf{z}\|^2} &= \int dV_{2n}\, r^2\, e^{-r^2} \\
&= \int dr \frac{\pi^n}{\Gamma(n)}\, r^{2n-1}\, r^2\, e^{-r^2} \\
&= \frac{\pi^n \Gamma(1+n)}{\Gamma(n)} \\
&= n\, \pi^n\,,
\end{aligned}$$

where V_m is the volume of a unit hypersphere of radius 1 in m real dimensions.

2.8 Evaluate Gauss hypergeometric function expressions in terms of common functions.

(a) ${}_2F_1(1, 2, 4; 1)$
(b) ${}_2F_1(1, 1, 2; -1)$
(c) ${}_2F_1(1/2, 1/2, 3/2; -3)$
(d) ${}_2F_1(-2, 1/2, 1/2; 1/2)$

2.9 By using the calculus of variation, find the shortest distance between a point on the zenith of a sphere (the north pole) and a point on the equator.

3 Probability and Statistics

Problems

3.1 Evaluate the variance of the log-normal distribution.

We know if Y is log-normal, then $X = \log Y$ is normal with $\mu = 0$ and $\sigma = 1$

Since Y is log-normal, we can write the distribution

$$f_Y(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\{-\frac{(\log y - \mu)^2}{2\sigma^2}\}$$

First, find the $\langle Y \rangle$. By definition,

$$\begin{aligned}
\langle Y \rangle &= \int_0^\infty dy\, y \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} \\
&= \int_0^\infty dy\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} \\
&= \int_{-\infty}^\infty d(e^x)\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} \qquad \text{(by replacing } \log y \text{ with } x) \\
&= \int_{-\infty}^\infty dx\, e^x\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\} \\
&= e^{\mu + \frac{1}{2}\sigma^2}
\end{aligned}$$

Second, find the second moment $\langle Y^2 \rangle$. Again, by definition

$$\begin{aligned}
\langle Y^2 \rangle &= \int_0^\infty dy\, y \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} \\
&= \int_{-\infty}^\infty dx\, e^x\, e^x\, \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} \qquad \text{(by replacing } \log y \text{ with } x) \\
&= e^{2\mu + 2\sigma^2}
\end{aligned}$$

Finally, $\mathrm{Var}(Y) = \langle Y^2 \rangle - \langle Y \rangle^2$

$$\begin{aligned}
\mathrm{Var}(Y) &= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \\
&= (e^{\sigma^2} - 1)\, e^{2\mu + \sigma^2}
\end{aligned}$$

3.2 If the real variables X and Y with values x and y are given by $Y = X^2$, evaluate the probability density for Y given the density for X for the cases

(a) In general

Using the formula

$$p_Y(y) = \sum_j \frac{p_X(f_j^{-1}(y))}{\left\| \frac{\partial}{\partial x} f(x)|_{x=f_j^{-1}(y)} \right\|}$$

x and y are real. $y = x^2$, so $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$

$$\begin{aligned} p_{\mathbf{Y}}(y) &= \frac{p_X(x_1)}{|2x_1|} + \frac{p_x(x_2)}{|2x_2|} \\ &= \frac{p_X(x_1) + p_X(x_2)}{2\sqrt{y}} \\ &= \frac{p_X(\sqrt{y}) + p_X(-\sqrt{y})}{2\sqrt{y}} \end{aligned}$$

(b) If X is given by a Rayleigh distribution

If X is drawn from a *Rayleigh* distributed

$$p_X(x) = \begin{cases} \frac{2x}{\sigma^2} e^{-\frac{x^2}{\sigma^2}}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} p_Y(y) &= \frac{\frac{2\sqrt{y}}{\sigma^2} e^{-\frac{y}{\sigma^2}}}{2\sqrt{y}} \quad \text{(using above equation but only using one root here)} \\ &= \frac{1}{\sigma^2} e^{-\frac{y}{\sigma^2}} \end{aligned}$$

which is related to the χ^2 distribution.

3.3 Evaluate the characteristic function of the sum of variables drawn from a real Gaussian distribution and a Rayleigh distribution.

3.4 Evaluate the first four central moments of a random variable that is characterized by unit variance and is uniform over phase.

$$\mu_m = \int dx \, (x - \langle X \rangle)^m \, p_X(x)$$

$$\mu_1 = \int dx \, (x - \langle X \rangle) \, p_X(x) = \int dx \, x \, p_X(x) - \langle X \rangle \int dx \, p_X(x) = \langle X \rangle - \langle X \rangle = 0$$

$$\mu_2 = \int dx \, (x - \langle X \rangle)^2 \, p_X(x) = \sigma^2 = 1$$

$$\mu_3 = \int dx \, (x - \langle X \rangle)^3 \, p_X(x) = 0 \, ,$$

because the integrand is odd about $\langle X \rangle$ and decays quickly to zero as $(x - \langle X \rangle)$ goes to either $\pm\infty$.

$$\mu_4 = \int dx \, (x - \langle X \rangle)^4 \, p_X(x) = 3\sigma^4$$

3.5 For an n-vector with norm-square of q, randomly drawn from a circularly symmetric complex Gaussian distribution with a covariance matrix proportional to the identity matrix, what is the probability that after projecting onto a subspace of rank $3n/4$ that at least half of norm-squared of the vector $q/2$ remains, assuming that n is given by

The distribution that characterizes the fraction of the vector norm squared retained of a complex Gaussian random vector of size k+j when projected on a subspace of rank k is the beta distribution whose CDF is given by (Eq. 3.71):

$$P_\beta(x_o; j, k) = \frac{\Gamma(j+k)}{\Gamma(j)\Gamma(k)} B(x_o; j, k)$$

where $B(x_o; j, k)$ is the incomplete beta function:

$$B(z; x, y) = \frac{z^x}{x} {}_2F_1(x, 1-y; x+1; z)$$

(a) 4
The rank of the Subspace is 3, the size of the Gaussian vector is 4.

$$P = 1 - P_\beta(0.5; 3, 1) = 0.875$$

(b) 8
The rank of the Subspace is 6, the size of the Gaussian vector is 8.

$$P = 1 - P_\beta(0.5; 6, 2) = 0.9375$$

(c) 12
The rank of the Subspace is 9, the size of the Gaussian vector is 12.

$$P = 1 - P_\beta(0.5; 9, 3) = 0.9673$$

3.6 For large Wishart matrices, constructed by the outer product $\mathbf{G}\,\mathbf{G}^\dagger$ where the matrix $\mathbf{G} \in \mathbb{C}^{m \times n}$ contains entries drawn independently and randomly from a complex circular Gaussian distribution, evaluate the approximate peak-to-average eigenvalue ratio under assumptions of

(a) $n/m = 1$

$$b_r = \left(\sqrt{r}+1\right)^2 = \left(\sqrt{m/n}+1\right)^2 = \left(\sqrt{1}+1\right)^2 = 4$$

(b) $n/m = 2$

$$b_r = \left(\sqrt{r}+1\right)^2 = \left(\sqrt{m/n}+1\right)^2 = \left(\sqrt{\frac{1}{2}}+1\right)^2 = \frac{3+2\sqrt{2}}{2}$$

(c) $n/m = 4$

$$b_r = \left(\sqrt{r}+1\right)^2 = \left(\sqrt{m/n}+1\right)^2 = \left(\sqrt{\frac{1}{4}}+1\right)^2 = \frac{9}{4}$$

(d) $n/m = 16$

$$b_r = \left(\sqrt{r}+1\right)^2 = \left(\sqrt{m/n}+1\right)^2 = \left(\sqrt{\frac{1}{16}}+1\right)^2 = \frac{25}{16}$$

3.7 For m independent observations of complex variable Z with value z, given by $Z = a\,e^{i\theta} + N$, where a is an unknown deterministic real amplitude and θ is an unknown deterministic real phase with additive circularly symmetric complex Gaussian noise N with value n and variance σ_n^2, evaluate the minimum phase estimation variance for an unbiased estimator.

$$\mathbf{z} = a\,e^{i\,\theta}\,\mathbf{1} + \mathbf{n}$$
$$\{\mathbf{z}\}_k \sim \mathcal{CN}(a\,e^{i\theta}, \sigma^2)$$

Because the distribution is Gaussian and we assume that the noise is not dependent upon the parameters of interest, such that the derivative of the covariance matrix zero, the Fisher information matrix is given by

$$\{\mathbf{J}\}_{\alpha,\beta} = 2\,\Re\left\{\frac{\partial \boldsymbol{\mu}^\dagger}{\partial \alpha}\,\mathbf{R}^{-1}\,\frac{\partial \boldsymbol{\mu}}{\partial \beta}\right\}$$
$$\mathbf{R} = \sigma^2\,\mathbf{I}_m$$

where α and β could be either the parameter a or θ, and $\boldsymbol{\mu} = a\,e^{i\,\theta}\,\mathbf{1}$ is the mean vector.

$$\frac{\partial \boldsymbol{\mu}}{\partial a} = e^{i\,\theta}\,\mathbf{1}$$
$$\frac{\partial \boldsymbol{\mu}^\dagger}{\partial a} = e^{-i\,\theta}\,\mathbf{1}^T$$
$$\frac{\partial \boldsymbol{\mu}}{\partial \theta} = a\,i\,e^{i\,\theta}\,\mathbf{1}$$
$$\frac{\partial \boldsymbol{\mu}^\dagger}{\partial \theta} = -a\,i\,e^{-i\,\theta}\,\mathbf{1}^T$$

Consequently, the Fisher information matrix is given by

$$\begin{aligned}\mathbf{J} &= \frac{m}{\sigma^2}\,2\,\Re\begin{pmatrix} 1 & a\,i \\ -a\,i & a^2 \end{pmatrix} \\ &= \frac{2\,m}{\sigma^2}\begin{pmatrix} 1 & 0 \\ 0 & a^2 \end{pmatrix}.\end{aligned}$$

The variance of estimation is greater than is greater than inverse of the Fisher information matrix. Consequently, the variance of the estimators are given by

$$\begin{aligned}\mathrm{Var}\,\{\hat{a}\} &\geq \frac{\sigma^2}{2\,m} \\ \mathrm{Var}\,\left\{\hat{\theta}\right\} &\geq \frac{\sigma^2}{2\,m\,a^2}\,.\end{aligned}$$

3.8 Let $h(t)$ be the impulse response of a linear-time-invariant system such that $h(t)$ is square integrable. Let the input to this system be a stationary random process $X(t)$. Show that the autocorrelation function of the output process $Y(t)$ is given by

$$R_{YY}(\tau) = h * \overleftarrow{h} * R_{XX}(\tau) \tag{3.1}$$

and the power spectral density of $Y(t)$ is

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{3.2}$$

First, by definition $Y(t_1) = X(t_1) * h(t_1)$ and $Y(t_2) = X(t_2) * h(t_2)$

$$\begin{aligned}
R_{YY}(\tau) &= \langle Y(t+\tau)\, Y^*(t)\rangle = \langle Y(t)\, Y^*(t-\tau)\rangle \\
&= \langle Y(t_1)\, Y^*(t_2)\rangle \; ; \quad t_2 = t_1 - \tau \\
&= \left\langle \int d\gamma\, X(t_1-\gamma)\, h(\gamma) \int d\eta\, X^*(t_2-\eta)\, h^*(\eta) \right\rangle \\
&= \int d\eta \int d\gamma\, \langle X(t_1-\gamma)\, X^*(t_1-\tau-\eta)\rangle\; h(\gamma)\; h^*(\eta) \\
&= \int d\eta \int d\gamma\, \langle X(t)\, X^*(t+\gamma-\tau-\eta)\rangle\; h(\gamma)\; h^*(\eta) \\
&= \int d\eta \int d\gamma\, R_{XX}(\tau-\gamma+\eta)\, h(\gamma)\; h^*(\eta) \\
&= \int d\eta\, \{h * R_{XX}\}(\tau+\eta)\, h^*(\eta) \\
&= \int d\eta'\, \{h * R_{XX}\}(\tau-\eta')\, h^*(\eta') \\
&= h * \overleftarrow{h} * R_{XX}(\tau)\,.
\end{aligned}$$

By using the above result,

$$\begin{aligned}
S_Y(f) &= \int d\tau\, e^{-i\,2\pi\,\tau\,f}\, R_{YY}(\tau) \\
&= \int d\tau\, e^{-i\,2\pi\,\tau\,f} \int d\gamma \int d\eta\, R_{XX}(\tau-\gamma+\eta)\, h(\gamma)\; h^*(\eta) \\
&= \int d\gamma \int d\eta\; h(\gamma)\; h^*(\eta) \int d\tau\, e^{-i\,2\pi\,\tau\,f}\, R_{XX}(\tau-\gamma+\eta) \\
&= \int d\gamma \int d\eta\; h(\gamma)\; h^*(\eta)\, e^{-i\,2\pi\,[\gamma-\eta]\,f} \int d\tau'\, e^{-i\,2\pi\,\tau'\,f}\, R_{XX}(\tau') \\
&= \int d\gamma\, e^{-i\,2\pi\,\gamma\,f}\; h(\gamma) \int d\eta\, e^{i\,2\pi\,\eta\,f}\; h^*(\eta)\, S_X(f) \\
&= \int d\gamma\, e^{-i\,2\pi\,\gamma\,f}\; h(\gamma) \left(\int d\eta\, e^{-i\,2\pi\,\eta\,f}\; h(\eta)\right)^*\, S_X(f) \\
&= \|H(f)\|^2\, S_X(f)\,.
\end{aligned}$$

3.9 Let $Y = X_1^2 + X_2^2 + \cdots + X_K^2$ where X_i are independent, identically distributed Gaussian random variables with zero mean and unit variance. Show that Y follows a χ^2 distribution with K degrees of freedom.

First, derive the χ-*square* distribution with one degree of freedom, that is, $\mathbf{Y} = \mathbf{X^2}$, where X is *gaussian* distribution with zero mean and unit variance.

Using the formula

$$p_y(\mathbf{y}) = \sum_{\mathbf{j}} \frac{\mathbf{p_x}(\mathbf{f_j^{-1}}(\mathbf{y}))}{\left\| \frac{\partial}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x})|_{\mathbf{x}=\mathbf{f_j^{-1}}(\mathbf{y})} \right\|}$$

We can show that

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} \\ f_Y(y) &= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{2\pi y}} \exp\left\{-\frac{y}{2}\right\}, \qquad\qquad y \geq 0 \end{aligned}$$

The *characteristic* function of $\mathbf{Y}$ can be shown as

$$\begin{aligned} F(\varphi) &= \int_{-\infty}^{\infty} e^{j\varphi y} f_Y(y)\, dy \\ &= (1 - 2j\varphi)^{-\frac{1}{2}} \end{aligned}$$

Now If we define

$$Y = \sum_{i=1}^{n} X_i^2$$

where $X_i's$ are *iid*. The *characteristic* function can be shown as

$$F(\varphi) = (1 - 2j\varphi)^{-\frac{n}{2}}$$

If we take inverse *fourier transform*, we can get the *pdf* of Y

$$f_Y(y) = \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{1}{2}n)} y^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}, \text{for } y \geq 0$$

which is the *pdf* of a χ-*square* distribution with n degrees of freedom.

3.10 Suppose that there are n unit-power transmitters distributed independently and identically with uniform probability in a disk of radius R with polar coordinates $(r_1, \theta_1), (r_2, \theta_2), \cdots (r_n, \theta_n)$. Let the aggregate signal received at the center of the circle be given by

$$\mathcal{I} = \sum_{k=1}^{n} r_k^{-\alpha} \tag{3.3}$$

where $\alpha > 2$.

(a) Show that the mean signal power at the center of the disk $\langle \mathcal{I} \rangle$ is infinite.

(b) Show that the signal power at the center of the disk $\mathcal{I}$ is finite with probability 1.

3.11 Use Equation (3.5) to derive Equation (3.59) from Equation (3.53).

$$p_y(y) = \frac{p_x(f^{-1}(y))}{\left\| \frac{\partial}{\partial x} f(x)\big|_{x=f^{-1}(y)} \right\|} \tag{3.5}$$

$$p^C_{\chi^2}(q; n, \sigma^2, v^C) = \frac{1}{\sigma^2} e^{-(q+v^C)} \left(\frac{q}{v^C}\right)^{\frac{n-1}{2}} I_{n-1}\left(\frac{2\sqrt{v^C q}}{\sigma^2}\right) \tag{3.53}$$

$$p_{Rice}(y) = \frac{2y}{\sigma^2} I_0\left(\frac{2ay}{\sigma^2}\right) e^{\frac{-(y^2+a^2)}{\sigma^2}}, y \geq 0\,, \tag{3.59}$$

where v^C is the *complex noncentrality parameter*

$$v^C = \sum_{m=1}^{N} \|\mu_m\|^2$$

In our case, we want to derive the *pdf* of *Rician* distribution. Denotes Z is a *Rician* distributed random variable and Y is a *Gaussian* distributed random variable, that is, $Z = \sqrt{Y}$. And $v^C = |\mu|^2$. Thus, (3.53) and (3.59) reduce to

$$p^C_{\chi^2}(y; 1, \sigma^2, \|\mu\|^2) = \frac{1}{\sigma^2} e^{-(y+\|\mu\|^2)} \left(\frac{y}{\|\mu\|^2}\right)^0 I_0\left(\frac{2\sqrt{|\mu|^2 m}}{\sigma^2}\right) \tag{3.53*}$$

$$p_{Rice}(z) = \frac{2z}{\sigma^2} I_0\left(\frac{2\,\|\mu\|\, z}{\sigma^2}\right) e^{\frac{-(m^2+|\mu|^2)}{\sigma^2}}; \quad z \geq 0\,. \tag{3.59*}$$

By using above equations, we can write

$$\begin{aligned} p_z(z) &= \frac{p^C_{\chi^2}(z^2)}{\left\| \frac{\partial}{\partial y}\sqrt{y}\,\Big|_{y=f^{-1}(z)} \right\|} \\ &= \frac{p^C_{\chi^2}(z^2)}{\frac{1}{2}\frac{1}{z}} \\ &= \frac{2z}{\sigma^2} e^{-\frac{z^2+\|\mu\|^2}{\sigma^2}} I_0\left(\frac{2\,z\,\|\mu\|}{\sigma^2}\right). \end{aligned}$$

4 Wireless Communications Fundamentals

Problems

4.1 In considering various constellations, when in the presence of additive Gaussian noise, the largest probability of confusing one point with another point on the constellation is driven by the distance between the closest points. Compared to a BPSK constellation, find the relative average power required to hold this minimum amplitude distance equal for the following constellations:

(a) QPSK
$(2/\sqrt{2})^2 = 2$

(b) 8-PSK
$(2/(2\cos[\pi/8]))^2 = \csc^2(\pi/8) \approx 6.8$

(c) 16-QAM
$2\,\frac{1}{16}\sum_{m=-1,n=-1}^{m=2,n=2} \|2[(m-1/2)+(n-1/2)i]\|^2 = 20$

(d) 64-QAM
$2\,\frac{1}{64}\sum_{m=-3,n=-3}^{m=4,n=4} \|2[(m-1/2)+(n-1/2)i]\|^2 = 84$

(e) 256-QAM
$2\,\frac{1}{256}\sum_{m=-7,n=-7}^{m=8,n=8} \|2[(m-1/2)+(n-1/2)i]\|^2 = 340$

4.2 Under the assumption of 30 C temperature measured at the receiver find the observed noise power for the following parameters

(a) noise figure of 2 dB, bandwidth of 10 kHz
The observed thermal noise power P_n is bounded by

$$P_n = f_n\, k_B\, T_K\, B$$

where $k_B \approx 1.38 \cdot 10^{-23}$ W/K/Hz, and $T_K = 30 + 273K = 303K$

$$\begin{aligned} P_n &= 1.58 \cdot 1.38 \cdot 10^{-23} \cdot 303 \cdot 10000 \\ &= 1.58 \cdot 4.18 \cdot 10^{-17} \\ &= 2 - 163.79\,(\text{dBW}) = -161.79\,(\text{dBW}) \end{aligned}$$

(b) noise figure of 6 dB, bandwidth of 10 MHz
Similarly, the observed noise power for noise figure of 6 dB and bandwidth

10 MHz

$$\begin{aligned} P_n &= 4 \cdot 1.38 \cdot 10^{-23} \cdot 303 \times 10 \cdot 10^6 \\ &= 4 \cdot 4.18 \cdot 10^{-14} \\ &= 6 - 133.79\,(\text{dBW}) \\ &= -127.79\,(\text{dBW}) \end{aligned}$$

4.3 In considering a two-stage superheterodyne downconversion to complex baseband for a system with a carrier frequency of 1 GHz and a bandwidth of 40 MHz, find IF frequency ranges such that undesirable images are suppressed by more than the square of the relative filter sidelobe levels under assumption of the same filter being used at each stage.

4.4 By assuming that nodes in a wireless network communicate directly with their nearest neighbor, evaluate the scaling law for networks that are constrained to

(a) a linear geometry

(b) a three-dimensional geometry

4.5 Consider the following constellation diagram for a QPSK system with the constellation points assigned to 2-bit sequences. If possible, find an alternative assignment of bits that leads to a lower average probability of bit error.

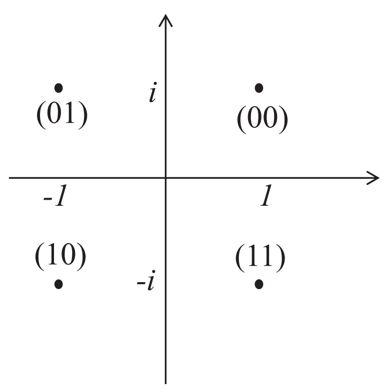

Figure 4.16 QPSK bit assignments.

4.6 Consider the following equation

$$Z = S + N\,.$$

Let N be distributed according to a zero mean, circularly symmetric, Gaussian random variable with variance $\frac{1}{2}\sigma^2$ per real dimension and is independent of S.

The random variable Z is used to estimate S such that the probability of error in making the estimation is minimized.

(a) Suppose that $S = \pm V$ with equal probability. Find the probability of error in terms of the Q function. Recall that the Q function is the integral of the tail of a standard Gaussian probability density function, i.e.

$$Q(t) = \int_t^\infty dx \, \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \, .$$

(b) Suppose that $S = \pm U + \pm U \, i$. Find the probability of error.

(c) How should U and V be related such that the probabilities of error in the previous two parts are equal? (This question really is about the SNR requirement for a QPSK system and BPSK system to have the same probability of symbol error.)

(a) When $S = \pm V$, the pdf of Z when $S = +V$ is circular Gaussian with mean at +V and when $S = -V$, it is the same but with mean at -V. Hence the decision is made in favor of +V when $Z > 0$ and in favor of $-V$ if $Z < 0$. Therefore,

$$P_e = \frac{1}{2} Q\left(\frac{0+V}{\sigma}\right) + \frac{1}{2}\left\{1 - Q\left(\frac{0-V}{\sigma}\right)\right\} = Q\left(\frac{V}{\sigma}\right)$$

(b) In this case, a given point on the constellation can be confused with three other points. As example $S = -U - i\,U$, could be confused if either the real or imaginary portion fluctuates past the origin. Consequently, both must be satisfied

$$P_e' = 1 - \left[1 - Q\left(\frac{U}{\sigma}\right)\right]^2$$

(c) The setting the probability of symbol error the same the implicit solution is given by

$$\begin{aligned} P_e &= P_e' \\ Q\left(\frac{V}{\sigma}\right) &= 1 - \left[1 - Q\left(\frac{U}{\sigma}\right)\right]^2 \\ \left(\frac{V}{\sigma}\right)^2 &= \left(Q^{-1}\left[1 - \left[1 - Q\left(\frac{U}{\sigma}\right)\right]^2\right]\right)^2 , \end{aligned}$$

where $Q^{-1}(\cdot)$ is the inverse Q-function.

4.7 Suppose that $\mathbf{c}_1, \mathbf{c}_2, \cdots, \mathbf{c}_M$ are $M \times 1$ vectors and $\mathbf{c}_j^\dagger \, \mathbf{c}_j = 1$ for all j.

$$\mathbf{z} = \sum_{j=1}^{M} s_j \mathbf{c}_j + \mathbf{n} \, ,$$

where the a_i terms take on values of ± 1 with equal probability and $\mathbf{n}$ contains

Independent, identically distributed Gaussian random variables with zero mean and variance 0.01. Let an estimate of a_1 be given by $\hat{a}_1 = \text{sign}\left(\mathbf{c}_1^\dagger \mathbf{y}\right)$.

(a) Suppose that $\mathbf{c}_j^\dagger \mathbf{c}_1 = 0$ for $j \neq 1$. Find the probability that $\hat{a}_1 \neq a_1$.

(b) Suppose that the entries of the vectors $\mathbf{c}_j$ are i.i.d. random variables taking values of $\pm\frac{1}{\sqrt{M}}$ with equal probability. Find the probability that $\hat{a}_1 \neq a_1$.

4.8 Consider a network comprising interferers distributed according to a Poisson point process with density of interferers ρ and subject to the standard inverse-power-law path-loss model with path-loss exponent $\alpha > 2$. Consider a link of length r in this network between a receiver that is not part of the process and an additional transmitter at a distance r away. Assuming that the signals are subject to Nakagami fading with shape parameter μ equaling a positive integer, find the cumulative distribution function (CDF) of the signal-to-interference ratio of the link in question. Hint: The upper incomplete gamma function $\Gamma(s, x)$ for positive integers s can be expressed as follows

$$\Gamma(s, x) = (s-1)!\, e^{-x} \sum_{k=0}^{s-1} \frac{x^k}{k!}. \tag{4.1}$$

This problem was inspired by Reference [148].

5 Simple Channels

Problems

5.1 For a satellite in geosynchronous orbit about the earth centered over the continental United States,

(a) find the antenna gain required to cover the continental United States well (approximate 3 dB northeast corner of Maine and southwest corner of California, about 5000 km).

(b) evaluate the approximate effective area assuming a carrier frequency of 10 GHz.

5.2 In a line-of-sight environment without scatterers, find the largest achievable data rate between two short dipoles with the same orientation, separated by 1 km, transmitting 1 W, operating at a carrier frequency of 1 GHz, with a receiver at temperature of 300 K.

5.3 Consider Figure 5.14, which is a block diagram of a Tomlinson-Harashima precoder developed in the 1970s to mitigate intersymbol-interference. The principles of its operation are very similar to the Costa precoding described in this chapter.

(a) Suppose that the box marked $f(., V)$ is removed, i.e., $x[k]$ equals $m[k]$ with the output of the filter $g[k]$ subtracted out. Find $g[k]$ such that $y[k] = x[k]$.

(b) Suppose that $|m[k]| \leq M$. What is the largest possible value that $x[k]$ can take, assuming that the box marked $f(., V)$ is still not present?

(c) Please specify $f(., V)$ such that $|x[k]| \leq V \; \forall k$ and show that $y[k] = x[k]$.

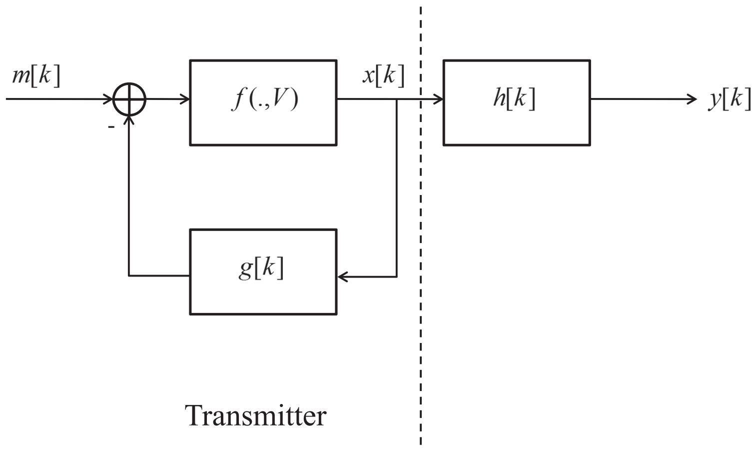

Figure 5.14 Tomlinson-Harashima precoder.

5.4 By noting that at low spectral efficiency the best case $E_b/N_0 \approx -1.59$ dB, evaluate the minimum received energy required to decode 1000 bits at a temperature of 300 K.

we know $N_0 = K_B T$ and $\frac{E_b}{N_0} \approx -1.59$ dB ≈ 0.69.

$$\begin{aligned}
\frac{E_b}{N_0} &\approx 0.69 \\
E_b &\approx 0.69\, N_0 = 0.69\, k_B\, T \\
&= 0.69\, 1.38 \cdot 10^{-23}\,(\mathrm{J/K})\ 300(\mathrm{K}) = 2.86 \cdot 10^{-21}\,(\mathrm{J}) \\
1000\, E_b &\approx 2.86 \cdot 10^{-18}\,(\mathrm{J})
\end{aligned}$$

5.5 Evaluate the differential entropy for a real Rayleigh random variable.

6 Antenna Arrays

Problems

6.1 Considering plane wave approximation for the reception of a narrowband signal on a continuous linear antenna array in a plane with a source located along boresight of the array, evaluate the root-mean-square error as a function of range R, length L, and signal wavelength.

6.2 Construct unit-normalized steering vectors as a function of azimuthal angle ϕ and angle from zenith θ for the following geometries:

(a) An 8-element square with elements at ± 1 wavelength and on point half way in between each corner along the periphery of the square.

(b) An 11-element spiral that begins at the origin and ends at 2 wavelengths along the $\{\mathbf{x}\}_1$ axis that follows the polar form in which radius of the n^{th} element follows the form $r_n = a(n-1)$ and angle of the n^{th} element follows the form $\phi_n = b(n-1)$, where a and b are undetermined coefficients.

6.3 For a four-element linear regular array with 1 wavelength spacing that incorporates the array element amplitude pattern $a(\theta)$

$$a(\theta) = 2\cos[\sin(\theta)\pi/2]\,,$$

where the angle θ is measured from boresight of the array:

(a) Formulate an unnormalized steering vector in a plane

$$\{\mathbf{v}(\theta)\}_m = a(\theta)\, e^{i\frac{2\pi}{\lambda}(m-1)d\sin\theta}$$

(b) Assuming the array is pointed at boresight, evaluate ratio of power beam pattern of this array to an 8-element array with isotropic elements and half wavelength spacing

$$\begin{aligned} P_{\theta_0}(\theta) &= \|\mathbf{v}^\dagger(\theta)\mathbf{v}(\theta_0)\|^2 \\ &= \frac{a^2(\theta)}{n_r^2}\left\|\sum_{m=0}^{n_r-1}\cos\left(\frac{2\pi\, m\, d\,[\sin(\theta)-\sin(\theta_0)]}{\lambda}\right)\right\|^2 \\ &\quad + \frac{a^2(\theta)}{n_r^2}\left\|\sum_{m=0}^{n_r-1}\sin\left(\frac{2\pi\, m\, d\,[\sin(\theta)-\sin(\theta_0)]}{\lambda}\right)\right\|^2 \end{aligned}$$

4-elment array $\theta_0 = 0$ $\lambda = d$ $n_r = 4$
8-elment array $\theta_0 = 0$ $\lambda = d/2$ $n_r = 8$

$$Ratio = \frac{\frac{1}{4}\|\sum_{m=0}^{3}\cos(\sin(\theta)\frac{\pi}{2})\cos(2\pi m\sin(\theta))\|^2 + \frac{1}{4}\|\sum_{m=0}^{3}\cos(\sin(\theta)\frac{\pi}{2})\sin(2\pi m\sin(\theta))\|^2}{\frac{1}{64}\|\sum_{0}^{7}\cos(\pi m\sin(\theta))\|^2 + \frac{1}{64}\|\sum_{m=0}^{7}\sin(\pi m\sin(\theta))\|^2}$$

(c) Assuming the array is pointed at $\theta = \pi/4$, evaluate the ratio of power beam pattern of this array to an 8-element array with isotropic elements and half-wavelength spacing.

$$Ratio = \frac{\frac{1}{4}\|\sum_{m=0}^{3}\cos(\sin(\theta)\frac{\pi}{2})\cos(2\pi m(\sin(\theta)-\frac{\sqrt{2}}{2}))\|^2 + \frac{1}{4}\|\sum_{m=0}^{3}\cos(\sin(\theta)\frac{\pi}{2})\sin(2\pi m\sin(\theta)-\frac{\sqrt{2}}{2}))\|^2}{\frac{1}{64}\|\sum_{0}^{7}\cos(\pi m(\sin(\theta)-\frac{\sqrt{2}}{2}))\|^2 + \frac{1}{64}\|\sum_{m=0}^{7}\sin(\pi m(\sin(\theta)-\frac{\sqrt{2}}{2}))\|^2}$$

6.4 For the continuous array construction discussed in Section 6.3.3 that exists over the spatial domain of $0 \geq x \geq L$, find the normalized power beam pattern under the assumption the receive array uses the following tapering or windowing functions:

(a) Triangular –

$$w(x) = \begin{cases} \frac{2x}{L} & ; 0 \geq x \geq L/2 \\ 2 - \frac{2x}{L} & ; L/2 > x \geq L \end{cases}$$

(b) Hamming –

$$w(x) = 0.54 - 0.46\cos\left(\frac{2\pi x}{L}\right)$$

6.5 Consider the linear sparse array problem with randomly moving elements that have uniform probability density, assuming 32 isotropic antennas; find an the aperture in terms of wavelengths such that the peak sidelobe is no worse than 5 dB 90% of the time.

$$P_r(r < \eta) = [1 - e^{-n_r\eta^2}]e^{-\left[4\pi\sqrt{n_r\eta^2}e^{-n_r\eta^2}\right]\sqrt{\frac{L^2}{12\pi}}}$$

The probability is 0.9 and the value of the sidelobe power is given to be less than 5dB relative to the mainlobe. The ratio $\eta^2 = \text{undb}(-5) \approx 0.32$. By numerical search, the solution is $L \approx 401.5$.

6.6 Consider the linear sparse array design problem assuming 32 isotropic antennas; find an the aperture in terms of wavelengths such that a designer would likely find an array with peak sidelobe is no worse than 5 dB after 10 random array evaluations.

6.7 By assuming that a source is in the plane spanned by $\{\mathbf{x}\}_1$ and $\{\mathbf{x}\}_2$, construct the unnormalized steering vector for an array of three phase centers with half-wavelength spacing along $\{\mathbf{x}\}_2$ axis, assuming that the elements are constructed with small electric dipoles and that

(a) the array elements and single source are vertically (along $\{\mathbf{x}\}_3$) polarized.

(b) the array elements are horizontally polarized along the $\{\mathbf{x}\}_2$ axis and the single source is horizontally polarized (in the $\{\mathbf{x}\}_1$-$\{\mathbf{x}\}_2$ plane) and perpendicular to the direction of propagation.

(c) the array is phased to point at source, find the ratio of received power for the horizontally polarized to vertically polarized systems as a function of angle.

6.8 By assuming that a source is in the plane spanned by $\{\mathbf{x}\}_1$ and $\{\mathbf{x}\}_2$, construct the unnormalized steering vector for an array of three phase centers with half wavelength spacing along $\{\mathbf{x}\}_2$ axis, assuming that the elements are constructed with small electric dipoles, that at each phase center there is an electric dipole along each axis ($\{\mathbf{x}\}_1$, $\{\mathbf{x}\}_2$, and $\{\mathbf{x}\}_3$) and that

(a) source is vertically (along $\{\mathbf{x}\}_3$) polarized.

(b) source has arbitrary polarization.

(c) the array is phased to point at source, find the ratio of received power for the arbitrarily polarized to vertically polarized sources as a function of angle.

7 Angle-of-Arrival Estimation

Problems

7.1 Considering a 5-element regular linear array with half-wavelength spacing and isotropic antennas, under the assumption of receive SNR per sample per antenna of 10 dB and 10 samples independently drawn from a complex Gaussian distribution for each source, evaluate and plot the pseudo-spectrum as a function of direction parameter $u = \sin\phi$ for angle ϕ, where $\phi = 0$ is along boresight,

(a) for beamscan and MVDR with a single source at $\sin\phi = -0.3$

(b) for beamscan and MVDR with sources at $\sin\phi = -0.3, 0.4$

(c) for beamscan and MVDR with sources at $\sin\phi = -0.4, -0.3, 0.4$

(d) for beamscan and MVDR with sources at $\sin\phi = -0.8$, -0.4, -0.3, 0.0, 0.3, 0.4, 0.8

7.2 Considering a 5-element regular linear array with half-wavelength spacing and isotropic antennas, under the assumption of receive SNR per sample per antenna of 10 dB where all sources coherently transmit the same 10 samples drawn from a complex Gaussian distribution, evaluate and plot the psuedospectrum as a function of direction parameter $u = \sin\phi$ for angle ϕ, where $\phi = 0$ is along boresight,

(a) for beamscan and MVDR with a single source at $\sin\phi = -0.3$

(b) for beamscan and MVDR with sources at $\sin\phi = -0.3, 0.4$

(c) for beamscan and MVDR with sources at $\sin\phi = -0.4, -0.3, 0.4$

(d) for beamscan and MVDR with sources at $\sin\phi = -0.8$, -0.4, -0.3, 0.0, 0.3, 0.4, 0.8

7.3 Considering the best unbiased angle-estimator variance for a linear array with mean-squared antenna position σ_y^2, with receive SNR per sample per antenna P and direction parameter $u = \sin\phi$ for angle ϕ,

(a) evaluate the ratio of best variance of direction parameter u estimation for a single transmitted sequence that is drawn from a Gaussian distribution relative to a known sequence

(b) discuss the ratio in the regime of small SNR but large number of samples

7.4 Considering the best unbiased angle-estimator variance for a linear array with mean-squared antenna position σ_y^2, with receive energy per sample per antenna P and direction parameter $u = \sin\phi$ for angle ϕ,

(a) evaluate best unbiased angle-estimator variance for angle estimation ϕ for a single transmitted sequence that is drawn from a Gaussian distribution

(b) discuss the variance as ϕ approaches end fire of the array.

7.5 Show that the following relationship between Marcum Q functions and Bessel functions is true,

$$Q_M(\sqrt{2a}, \sqrt{2b}) + Q_M(\sqrt{2b}, \sqrt{2a}) = 1 + e^{-(a+b)}\, I_0(2\sqrt{a\, b})\,.$$

7.6 Considering the problem of angle estimation based upon a single observation of a narrowband signal of wavelength λ for the antenna array with phase centers at positions $\{0, 1, 3, 5, 8\}\lambda/2$ along the $\{\mathbf{x}\}_2$ axis, find the approximate probability of confusing a sidelobe for a mainlobe as a function of per receive antenna SNR.

7.7 Considering the problem of angle estimation based upon a single observation of a narrowband signal of wavelength λ for the antenna array with phase centers at positions $\{0, 1, 3, 5, 8\}\lambda/2$ along the $\{\mathbf{x}\}_2$ axis, evaluate the variance bound using the method of intervals, keeping only the dominant sidelobe as a function of per receive antenna SNR.

7.8 Show for Equation (7.134) that the values for the parameters α and β given in Equations (7.140) and (7.139)

(a) decorrelate the variables $\underline{\mathbf{y}}_1$ and $\underline{\mathbf{y}}_2$ found in Equation (7.129)

(b) produce the variances presented in Equation (7.141)

7.9 For a single source known to be in $\{\mathbf{x}\}_1 - \{\mathbf{x}\}_2$ plane, observed by vector sensor, evaluate the Cramer-Rao angle-estimation bound as a function of integrated SNR and polarization.

8 MIMO Channel

Problems

8.1 Develop the result in Equation (8.48) for the case in which $n_t > n_r$.

8.2 Show that for diagonal real matrix $\mathbf{D}$, under the constraint $\mathrm{tr}\{\mathbf{D}\} = k$, the form

$$\left\langle \log |\mathbf{I} + \mathbf{G}\,\mathbf{D}\,\mathbf{G}^{\dagger}| \right\rangle \tag{8.1}$$

is maximized when random matrix $\mathbf{G} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$.

8.3 Under the assumption of an uninformed transmitter, 4×4 MIMO system and an i.i.d. complex Gaussian channel, find the probability that at least 50% of the energy remains after mitigating

(a) 1

(b) 2

(c) 3

strong interferers.

8.4 Evaluate the capacity in Equation (8.52) if the interference, noise, and channel are all not frequency selective.

8.5 Considering the received signal projected onto a subspace orthogonal to a known interference that is presented in Equation (8.5)

(a) Express the projection in terms of the least-squares channel estimation

(b) Express the average fractional signal loss due to the temporal mitigation in terms of the number of samples

8.6 Reevaluate the relations in Equation (8.48) under the assumption that the interference of rank n_i is much larger than the signal SNR (INR $\gg$ SNR).

8.7 Evaluate the informed to uninformed capacity ratio c_{IT}/c_{UT} in the limit of an infinite number of transmit and receive antennas (as discussed in Section 8.7) and in the limit of low SNR as a function of the ratio of receive to transmit antennas κ.

8.8 For a frequency-selective 2×2 MIMO channel that is characterized with reasonable accuracy by 2 frequency bins with channel values

$$\breve{\mathbf{H}} = \begin{pmatrix} 1 & -3 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -1 \end{pmatrix}, \tag{8.2}$$

evaluate the informed transmitter capacity expressed in Equation (8.55) as a function of per receive antenna SNR.

8.9 Consider the outage capacities (at 90% probability of closing) presented in Equation (8.103) for a 4×4 flat fading channel matrix characterized by the exponential shaping model introduced in Equation (8.83) Assume that the transmit and receive shaping parameters have the same value α. As a function of per receive antenna SNR (-10 dB to 20 dB), numerically evaluate and plot the ratio of the outage capacity (at 90% probability of closing) for the values of exponential shaping parameter α:

(a) $\alpha = 0.25$
(b) $\alpha = 0.5$
(c) $\alpha = 0.75$

to the outage capacity assuming that $\alpha = 1$.

9 Spatially Adaptive Receivers

Problems

9.1 At high SNR, compare the symbol error performance ML and MAP decoding for an unencoded QPSK constellation under the assumption that points on the constellations of $\{\pm 1, \pm 1\}$ have

(a) equal symbol probability: $p_{\{\pm 1, \pm 1\}} = 1/4$,

(b) symbol probabilities defined by

$$\begin{aligned} p_{\{1,\pm 1\}} &= 2/6 \\ p_{\{-1,\pm 1\}} &= 1/6\,. \end{aligned}$$

9.2 At SNR of 20 dB per receive antenna (can assume high SNR), compare the symbol error performance for a MMSE and MI beamformer for an unencoded QPSK constellation with equal probabilities for each symbol. Assume a four-antenna receiver in a line-of-sight environment in the far field with a signal of interest, and a single interferer of arbitrary power, all with known channels. Assume that the normalized inner product between the array responses of the signal of interest and the inner product is $1/\sqrt{2}$.

9.3 Develop the estimators expressed in Equations (9.25) and (9.45)

9.4 By employing the Wirtinger calculus, show that Equation (9.74) is the least squared error solution for the estimator of $\mathbf{X}$.

9.5 Evaluate the least-squares error beamformer which minimizes the Frobenius norm squared of the error matrix $\mathbf{E}$ defined by

$$\mathbf{E} = \mathbf{W}^\dagger \mathbf{Z} - \mathbf{X}\,, \tag{9.1}$$

and show that it provides the same solution as the approximate MMSE beamformer found in Equation (9.74)

9.6 Extend the result in Equation (9.125) to include external Gaussian interference. Show that performance is still bounded by the uninformed transmitter capacity.

9.7 Show that the LMS beamformer solution converges to the MMSE solution in the limit of a large number of samples.

9.8 For a four-antenna receiver observing a known signal with 0 dB SNR per receive antenna in a block-fading i.i.d. Gaussian channel that is static for at least 50 samples over which the beamformers are estimated, numerically evaluate the

average (over many channel draws) estimated signal error as a function of samples 1 to 50 for
(a) RLS
(b) LMS
(c) estimated MMSE using blocks of 10 samples,
where the RLS and LMS have no knowledge of the channel at the first sample.

9.9 For a four-antenna receiver observing a known signal with 0 dB SNR per receive antenna with a 10 dB INR per receive antenna interferer in a block-fading i.i.d. Gaussian channel that is static for at least 50 samples over which the beamformers are estimated, numerically evaluate the average (over many channel draws) estimated signal error as a function of samples 1 to 50 for
(a) RLS
(b) LMS
(c) estimated MMSE using blocks of 10 samples
where the RLS and LMS have no knowledge of the channel at the first sample.

9.10 For a 10-antenna receiver observing a known signal with 0 dB SNR per receive antenna in a block-fading i.i.d. Gaussian channel that is static for the period of observation over which the beamformers are estimated, numerically evaluate the average (over many channel draws) estimated signal error using the estimated MMSE beamformer of the forming

$$\mathbf{w} = \left(\frac{\mathbf{Z}\,\mathbf{Z}^\dagger}{n_s} + \epsilon\,\mathbf{I} \right)^{-1} \frac{\mathbf{Z}\,\mathbf{X}^\dagger}{n_s}\,, \tag{9.2}$$

using blocks of five samples as a function of diagonal loading for the form described in Equation (9.178).

10 Dispersive and Doubly Dispersive Channels

Problems

10.1 Consider a static single SISO channel that is represented by

$$\tilde{h}(\tau) = a\,\delta(\tau - T)\,, \tag{10.1}$$

where the notation in Section 10.1 is employed. For a complex signal bandwidth $B = 1/T_s$, evaluate the discrete channel representation

(a) $T = 0$

(b) $= T = T_s/2$

(c) $= T = T_s/4$

10.2 The model frequency shifting (displayed in Equation (10.8)) is commonly used in analyses rather than the more accurate time-dilation approach (displayed in Equation (10.9)) A significant issue in practical systems is that the time dilation eventually causes a chip slip that cannot be corrected by a simple phase correction. For a filtered BPSK signal with a carrier frequency of 1 GHz and a bandwidth of 1 MHz with a relative fractional frequency error of 10^{-6} between the transmitter and receiver, evaluate the expected receiver loss because of chip misalignment as a function of time since a perfect synchronization.

10.3 By using the notation in Section 10.2, consider a loop in parameter space for delay and Doppler channel operators on a signal. Starting at some point, and moving operators through the space of delay and Doppler along some path, and then returning to the original point, the signal should be unaffected, because the effect should only be determined by the parameters' values. However, evaluate an effect on some signal $s(t)$ of the following sequence of operators $\mathcal{T}_d\,\mathcal{F}_\epsilon\,\mathcal{T}_{-d}\,\mathcal{F}_{-\epsilon}$ and evaluate the error.

10.4 Consider the eigenvalues of the observed space-time covariance matrix that is observing critically sampled signals. For a 10 receive antenna array and a single transmit antenna with a line-of-sight channel (equal channel responses across receive antennas), evaluate the eigenvalue distribution of the receive space-time covariance matrix of a 0 dB SNR per receive antenna assuming unit variance per antenna noise. Evaluate the eigenvalues under the assumption that the space-time covariance matrix includes

(a) 1 (spatial-only)

(b) 2

(c) 4

delay samples at Nyquist spacing.

10.5 Consider the eigenvalues of the observed space-time covariance matrix with two delays. The signal and noise are strongly filtered so that they significantly oversampled (that is, the sampling rate is large compared to the Nyquist sample rate). For a 10 receive antenna array and a single transmit antenna with a line-of-sight channel (equal channel responses across receive antennas), evaluate the eigenvalue distribution of the receive space-time covariance matrix of a 0 dB SNR per receive antenna assuming unit variance per antenna noise in the region of spectral support. Evaluate the eigenvalues approximately under the assumption that signal and noise are temporally oversampled significantly.

10.6 Consider the signal $s(t)$ and the SISO doubly dispersive channel characterized by time-varying channel $h_t(t, \tau)$ and the delay-frequency channel $h_D(f_D, \tau)$; develop the form of a bound on the average squared error in using $h_D(f_D, \tau)$ form under the assumption of a bounded temporal T and bounded spectral B signal.

10.7 Develop the results in Section 10.6.1 using discrete rather than integral Fourier transforms.

10.8 Develop the Doppler-frequency analysis dual of Equation (10.7).

10.9 Consider a simple discrete-time channel whose impulse response is given by the following

$$h(m) = \delta_{m,0} + \frac{1}{2}\delta_{m,1} + \frac{1}{4}\delta_{m,2}\,,$$

where $\delta_{m,k}$ is the Kronecker delta function defined in Section 2.1.3. Using a computer, generate 10,000 orthogonal-frequency-division-multiplexing (OFDM) symbols using 16 carriers with each carrier transmitting a BPSK symbol taking values of ± 1 with equal probability. Using the inverse fast Fourier transform (IFFT), convert each OFDM symbol into its time-domain representation. Create two vectors to store the time-domain samples, $\mathbf{s}_{zp}$ and $\mathbf{s}_{cp}$. The vector $\mathbf{s}_{zp}$ should contain the time-domain representation of the OFDM symbols with 3 samples of zero padding between consecutive symbols, and $\mathbf{s}_{cp}$ should contain a 3-sample cyclic prefix. Convolve $\mathbf{s}_{zp}$ and $\mathbf{s}_{cp}$ with the channel impulse response $h(m)$ and add pseudorandom complex Gaussian noise of mean zero and variance 0.09. per sample to the results of the convolutions. A convenient way to do this is to represent the impulse response in a vector $\mathbf{h}$, create a vector of noise samples $\mathbf{n}$, and write

$$\mathbf{z}_{zp} = \mathbf{h}^T\,\mathbf{s}_{zp} + \mathbf{n}$$
$$\mathbf{z}_{cp} = \mathbf{h}^T\,\mathbf{s}_{cp} + \mathbf{n}\,.$$

Therefore, $\mathbf{z}_{zp}$ and $\mathbf{z}_{cp}$ simulate received samples in an OFDM system with zero padding and a cyclic prefix respectively. Decode the OFDM symbols by selecting appropriate portions of $\mathbf{z}_{zp}$ and $\mathbf{z}_{cp}$, using a fast Fourier transform (FFT) to convert the time-domain samples into frequency domain values and checking the sign of the frequency domain-values to decode the BPSK data. Estimate the bit error rate by comparing the decoded data to the transmitted data for both the zero-padding and cyclic prefix schemes. This computer exercise is intended to illustrate the effects of using a cyclic-prefix versus simple zero padding for an OFDM system.

11 Space-Time Coding

Problems

11.1 Using Monte Carlo simulations to generate channel matrices $\mathbf{H}$, please plot the empirical outage probability versus r for a system with two transmit and receive antennas, $n_r = n_t = 2$, utilizing Equation (11.11). You may use and SNR of 10 dB.

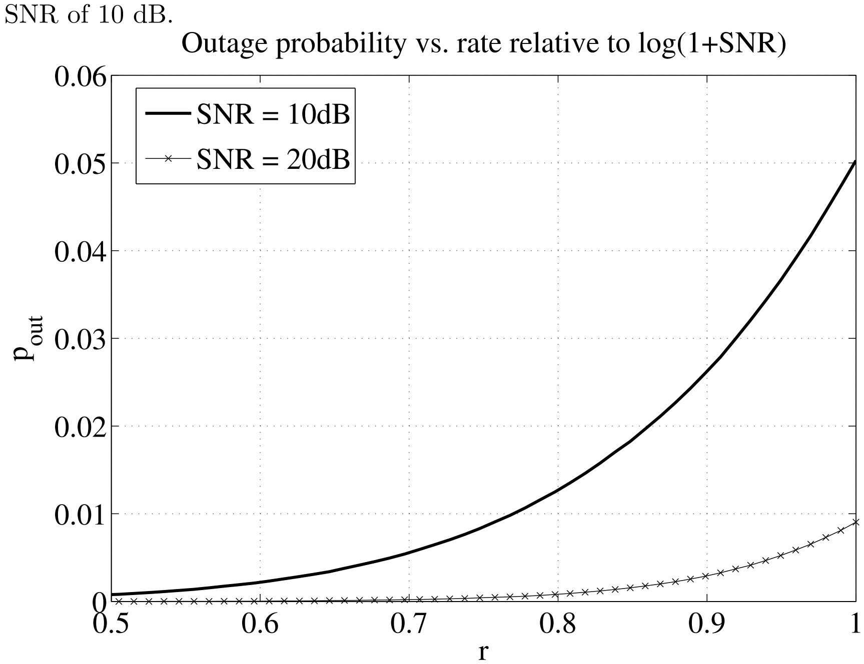

Figure Sol. 11.1 Outage probability vs r, where $p_{\text{out}} = \Pr(c < r\,\log_2(1 + \text{SNR}))$

In Figure Figure Sol. 11.1, the outage probability vs r is plotted from numerical simulations, where the outage probability is $p_{\text{out}} = \Pr(c < r\,\log_2(1 + SNR))$. This figure serves to illustrate how we can increase reliability at the expense of the data rate.

11.2 Evaluate the diversity order of a SIMO system in a Rayleigh-faded additive-white Gaussian noise channel when the receiver uses a spatial matched-filter receiver described in Section 9.2.1.

In a SIMO system, one can perform the analog of maximal ratio transmission by optimally combining the signals received on the multiple antennas of the receiver. This process is known as maximal ratio combining or spatial matched filtering. Consider a system with $n_t = 1$ and $n_r \geq 1$. The matrices in (11.13) are then

$$\mathbf{H} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n_r} \end{pmatrix}$$

$$\mathbf{C} = x$$

$$\mathbf{N} = \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_{n_r} \end{pmatrix}$$

The receiver optimally combines the signals on each antenna using the following weight vector to construct an estimate of x denoted by $\hat{x}$:

$$\mathbf{x} = \mathbf{w}^\dagger \mathbf{Z}.$$

Consider the following choice of $\mathbf{w}$:

$$\mathbf{w} = \frac{1}{||\mathbf{H}||}\mathbf{H}^\dagger, \tag{11.1}$$

which is unit norm and thus does not increase the transmit power.

This weight vector yields

$$\hat{x} = ||\mathbf{H}||x + \tilde{n}.$$

Note that since $\mathbf{w}$ is unit norm, the variance of $\tilde{n}$ equals the variance of a single entry of $\mathbf{N}$. Following the analysis of the previous seciton, the probability of error of this system can be expressed as

$$\begin{aligned} \Pr\left\{||\mathbf{H}||^2 \leq \frac{1}{\mathrm{SNR}}\right\} &= \frac{1}{\Gamma(n_r)}\gamma\left(n_r, \frac{1}{2\,\mathrm{SNR}}\right) \\ &\approx \frac{1}{n_r}\left(\frac{1}{2\,\mathrm{SNR}}\right)^{n_r} + o\left(\frac{1}{\mathrm{SNR}^{n_r}}\right). \end{aligned}$$

implying that a diversity order of n_r is achievable using maximal ratio combining scheme.

11.3 Consider a MIMO system with $n_t = n_r = 2$ antennas at the transmitter and receiver. Assume that the Alamouti scheme is used to encode transmissions and let h_{jk} denote the channel coefficient between the j-th transmitter antenna and the k-th receiver antenna. Additionally, let z_{jk} be the sampled received signal on the j-th antenna at the k-th time slot and write the following,

$$\hat{s}_1 = h_{11}^* z_{11} + h_{12}^\dagger z_{12}^* + h_{21}^* z_{21} + h_{22}^\dagger z_{22}^*$$
$$\hat{s}_2 = h_{12}^* z_{11} - h_{11}^\dagger z_{12}^* + h_{22}^* z_{21} - h_{21}^\dagger z_{22}^* .$$

By comparing the equations above with that of a SIMO system with four receiver antennas, show that a diversity order of 4 is achievable with the Alamouti scheme and two receiver antennas.

11.4 Consider a space-time coding system with $n_t = 2$ transmit antennas, $n_r = 2$ receive antennas and and coding preformed over $n_s = 2$ symbol times. Let the codewords be as follows.

$$\mathbf{C}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \mathbf{C}_2 = \begin{pmatrix} 1 & -j \\ j & j \end{pmatrix},$$
$$\mathbf{C}_3 = \begin{pmatrix} 1 & 1+j \\ 1 & 1-j \end{pmatrix}, \mathbf{C}_4 = \begin{pmatrix} 1-j & -j \\ 1-j & j \end{pmatrix}. \tag{11.2}$$

Using the determinant criteria, find the maximum diversity gain achievable using this space-time code.

The last sentence of the question should read: Using the rank and determinant criteria, find the maximum diversity and coding gain achievable with the above code. It is evident that the codeword difference matrix formed by codeword pairs ($\mathbf{C}_1$, $\mathbf{C}_3$), ($\mathbf{C}_3$, $\mathbf{C}_4$), and ($\mathbf{C}_2$, $\mathbf{C}_4$) have unit rank. Hence, the diversity order of this code is 2. The coding gain for these matrices is unity and is the minimum amongst all pairs of codewords. Hence, the coding gain is 1.

11.5 Using the constellation diagram in Figure 11.3 and the space-time trellis code given in Figure 11.5, please list the transmitted symbols from each transmit antenna due to the following sequence of bits. 10 11 11 01 10. You should start at state zero.

We prove this using the method introduced in [305] which is the original source of this real orthogonal design. Consider two distinct sequences of transmit symbols $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_4$ and $\breve{s}_1, \breve{s}_2, \cdots, \breve{s}_4$.

Let the code-words associated with these sequences respectively be denoted by $\tilde{\mathbf{C}}$ and $\breve{\mathbf{C}}$. Observe that $\tilde{\mathbf{C}} - \breve{\mathbf{C}}$ satisfies the structure imposed by the orthogonal matrix in (11.28), which we repeat here for convenience, and represent as the product of a scale factor and an orthonormal matrix

$$\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ -s_2 & s_1 & -s_4 & s_3 \\ -s_3 & s_4 & s_1 & -s_2 \\ -s_4 & -s_3 & s_2 & s_1 \end{pmatrix} = \sqrt{s_1^2 + s_2^2 + s_3^2 + s_4^2}\, \mathbf{O} \tag{11.3}$$

where $\mathbf{O}$ is an orthonormal.

The rank criteria requires that $\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)^\dagger$ be full rank, i.e. its determinant is non-zero. Since the codeword difference matrix conforms to the structure imposed by (11.3), with the j-th entry given by $\tilde{s}_j - \breve{s}_j$, the determinant of the matrix $\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)^{\dagger}$ can be written as

$$\begin{aligned}\left|\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)\left(\tilde{\mathbf{C}} - \breve{\mathbf{C}}\right)^{\dagger}\right| &= \sum_{j=1}^{4} ||\tilde{s}_j - \breve{s}_j||^2 \left|\mathbf{O}\mathbf{O}^{\dagger}\right| \\ &= \sum_{j=1}^{n} ||\tilde{s}_j - \breve{s}_j||^2 \end{aligned} \tag{11.4}$$

Since the sequence of symbols $\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_4$ and $\breve{s}_1, \breve{s}_2, \cdots, \breve{s}_4$ are distinct, at least one of the $\tilde{s}_j$ must differ from $\breve{s}_j$, implying that the previous equation is greater than zero. Thus, the rank criteria is satisfied and the real orthogonal design provides full diversity.

11.6 Use the determinant criteria to compute the diversity gain of the Alamouti code with n_R receiver antennas.

The problem should read: Use the rank and determinant criteria to determine the diversity and coding gain of the Alamouti code with n_r receiver antennas. The rank of the matrix $\mathbf{A}_{k\ell}$ for the Alamouti code is 2. Hence the maximum diversity gain is $2\,n_r$. The maximum coding gain on the other hand is given by the square root of the determinant of the codeword difference matrices of the Alamouti code, which is dependent on the specific values assigned to the codewords since the Alamouti scheme does not specify the code-words exactly. For instance, if the symbols of the Alamouti code are $\pm 1/\sqrt{2} \pm j/\sqrt{2}$ which could represent a Quadrature-Phase-Shift-Keyeing system, then the coding gain would be given by

$$\sqrt{\frac{1}{2}\left|(1+j) - (-1+j)\right|^2 + \frac{1}{2}\left|(1+j) - (1+j)\right|^2} = 1. \tag{11.5}$$

Note that the above expression is obtained in the case where the first symbol of the two codewords are adjacent constellation points, and the second symbol is identical for both codewords.

11.7 Perform a Monte Carlo simulation of an Alamouti space-time coding system with Quadrature-Phase-Shift-Keyeing (QPSK) symbols and single-antenna receivers. Show that the diversity order is what you expect by plotting the logarithm of the error probability at high SNR.

11.8 Show that the 4×4 space-time block code described by the real orthogonal generator matrix in (11.28) has full diversity.

11.9 Explain why the requirement that all antennas are used for any given thread in the universal space-time code framework described in Section 10.7 results in full diversity gain.

12 2×2 Network

Problems

12.1 Consider the 2×2 MIMO interference channel with Han-Kobayashi encoding as described in Section 12.2.2. For each of the possible choices of decoding orders, please write the inequalities that govern the common data rates.

12.2 Consider an extension of the Han-Kobayashi scheme for a network with three transmitter and receiver pairs. How many streams should each transmitter employ to achieve all possible combinations of partial interference cancelation. Also, qualitatively describe the achievable rate region for this type of network.

i. A private stream to be decoded at $R1$.
ii. A common stream to be decoded at $R1, R2, R3$.
iii. A common stream to be decoded at $R1, R2$.
iv. A common stream to be decoded at $R1, R3$.

Hence, each transmitter employs 4 streams.

12.3 For the 2×2 MIMO interference channel as described in Section 12.3.2, show that receiver 1 is able to decode both $\mathbf{s}_1$ and $\mathbf{s}_2$ with arbitrarily low probability of error if (12.42) is satisfied.

12.4 Consider the 2×2 MIMO cognitive radio channel described in Section 12.4, but now assume that the primary transmitter does not have channel-state information and hence transmits equal power and independent data on each of its antennas. Derive an expression that the transmit covariance matrix of the secondary link (that is, $\mathbf{K}_2$) needs to satisfy in order not to interfere with the primary link.

In order not to interferer with the transmit streams of the primary link, the secondary link will have to transmit in a subspace such that the signal arrives in an orthogonal subspace to that occupied by the cognitive transmitter, at the cognitive receiver. Since the primary transmitter is transmitting data on all its antennas, this amounts to requiring that the interfering signal from the cognitive link equal zero at the primary link, i.e.

$$\mathbf{H}_{21}\,\mathbf{s}_2 = \mathbf{0}$$

If $n_{t2} \leq n_{r1}$, then, the above equation cannot be satisfied unless $\mathbf{s}_1 = 0$. $n_{t2} >$

n_{r1}, however, the equation is satisfied if

$$\mathbf{s}_2 = \left(\mathbf{I} - \mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\mathbf{H}_{21}\right)\tilde{\mathbf{s}}_2$$

where $\tilde{\mathbf{s}}_2 \in \mathbb{C}^{n_{t2}\times 1}$ represents transmitted symbols from the cognitive transmitter. The matrix

$$\left(\mathbf{I} - \mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\mathbf{H}_{21}\right)$$

projects the transmitted signal vector of the cognitive transmitter such that it arrives at the primary receiver in a subspace orthogonal to that in which the transmitted signals from the primary transmitter is. If $\tilde{\mathbf{s}}_2$ comprise i.i.d. entries, then

$$\mathbf{K}_2 = \left(\mathbf{I} - \mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\mathbf{H}_{21}\right)\left(\mathbf{I} - \mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\mathbf{H}_{21}\right)^{\dagger}.$$

12.5 Consider the 2×2 MIMO cognitive radio channel described in Problem 12.4, but assume that the cognitive transmitter knows the transmit signal of the primary transmitter $\mathbf{s}_1$. Show how this information could be used by the cognitive link to increase its data rate compared to the previous problem.

Suppose that $n_{t2} = n_{r1}$. As observed in the previous problem, without knowledge of $\mathbf{s}_1$, the cognitive transmitter, transmitter 2 cannot transmit a non-zero symbol of its own without causing interference to the legacy link. But with knowledge of $\mathbf{s}_1$, the cognitive transmitter can increase the received signal energy at the legacy receiver, relative to its own interference plus noise. Suppose that the cognitive receiver uses a gaussian code and transmits a vector $\mathbf{s}_c$ with i.i.d. symbols of unit variance. Suppose that the cognitive transmitter then transmits the following vector

$$\mathbf{s}_2 = \mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\left(\left(\sqrt{\frac{P_c+\sigma^2}{\sigma^2}}-1\right)\mathbf{H}_{11}\,\mathbf{s}_1 + \sqrt{P_c}\,\mathbf{s}_c\right) \tag{12.1}$$

Using this transmitted signal vector in (12.50), we see

$$\begin{aligned}\mathbf{z}_1 &= \mathbf{H}_{11}\mathbf{s}_1 + \\ &\quad \mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\left(\mathbf{H}_{21}\,\mathbf{H}_{21}^{\dagger}\right)^{-1}\left(\left(\sqrt{\frac{P_c+\sigma^2}{\sigma^2}}-1\right)\mathbf{H}_{11}\,\mathbf{s}_1 + \sqrt{P_c}\,\mathbf{s}_c\right) + \mathbf{n}_1 \\ \mathbf{z}_1 &= \sqrt{\frac{P_c+\sigma^2}{\sigma^2}}\mathbf{H}_{11}\mathbf{s}_1 + \sqrt{P_c}\mathbf{s}_c + \mathbf{n}_1 \end{aligned} \tag{12.2}$$

Since $\mathbf{s}_c$ comprises i.i.d., unit variance gaussian entries, it appears to the legacy receiver as noise, thereby increasing the effective noise variance to $P_c + \sigma^2$. The signal power received at the legacy receiver has also been increased accordingly. Hence the data rate of the legacy receiver is not changed. Note here that we have made the assumption that the legacy receiver can operate without changing its behaviour with the increased signal and noise power. This may not be feasible in practice if the cognitive transmitter begins its transmissions after the legacy link has already been established. However, if the cognitive transmitter is able to transmit during the startup phase of the legacy link, it could artificially increase the signal and noise powers seen in the channel estimation phase of the legacy link.

Another approach that could be used is for the cognitive transmitter to perform dirty-paper coding discussed in Section 5.3.4. to precompensate for the interference caused by the legacy transmitter, as observed at the legacy receiver. This strategy could be combined with the strategy described in the previous problem to increase the rate of the cognitive link, without impacting the legacy link.

The assumption that the cognitive link may non-causally know the transmit vector of the legacy transmitter $\mathbf{s}_1$ (and then perform some sophisticated processing on the signal in a manner synchronous to the legacy link), is very strong and mostly serves as an upper bound to what could be achievable in the cognitive radio channel. These techniques and others are analyzed in more detail from an information theoretic perspective in [80], and related works.

12.6 Suppose that a 2×2 MIMO system has n_{t1} and n_{t2} antennas at transmitters 1 and 2 and n_{r1} and n_{r2} antennas at receivers 1 and 2 respectively. Assume that $1 < n_{t1} < n_{r2}$, $n_{t2} = 1$ and $n_{r1} > 1$. Find the transmit covariance matrix $\mathbf{T}_1$ of transmitter 1 which minimizes the interference caused on receiver 2 assuming transmitter 1 and receiver 1 have full channel-state information (that is, they know all the channel matrices in the network) and receiver 2 knows only the channel vector between transmitter 2 and itself.

12.7 Consider a 2×2 MIMO channel with a legacy link and a cognitive link. Under what conditions on the relative numbers of antennas at all nodes can the cognitive link operate without disrupting the legacy link? Assume that the legacy link does not change its behavior in response to the presence of the legacy link.

12.8 Consider a legacy 2×2 link for which each transmitter has a single antenna and each receiver has two antennas. It is assumed that the receivers use zero-forcing to cancel the interference from their respective undesired transmitters. Assume that a cognitive link with 2 transmitter antennas wishes to operate in the same frequency band as this existing 2×2 link in a manner such that the existing links are completely unaffected, that is neither their communication rates nor their behavior changes. Show that it is possible for the cognitive link to operate with non-zero rate by appropriately phasing transmit signals. You may make reasonable assumptions on the realizations of the various channel matrices.

Let the channels between the j-th legacy transmitter and the k-th legacy receiver be denoted by h_{jk} and the channels between the j-th antenna of the cognitive transmitter to the ℓ-th antenna of the k-th cognitive receiver be denoted by $h_{k\ell}^{\{j\}}$. Lets denote the channel between the j-th legacy transmitter and the k-th legacy receiver by $\mathbf{h}_{jk} \in \mathbb{C}$ and the channels between the cognitive transmitter and the j-th legacy receiver by $\mathbf{H}_j \in \mathbb{C}^{2\times2}$. Let the received signal vectors at legacy receivers 1 and 2 be respectively denoted by $\mathbf{y}_1 \in \mathbb{C}^{2\times1}$ and $\mathbf{y}_2 \in \mathbb{C}^{2\times1}$, respectively which are given by

$$\mathbf{y}_1 = \mathbf{h}_{11}\, s_1 + \mathbf{h}_{21}\, s_2 + \mathbf{H}_1\, \mathbf{s}_c + \mathbf{n}_1 \tag{12.3}$$

where s_1, s_2, and $\mathbf{s}_c \in \mathbb{C}^{2\times1}$ are the transmitted symbols from legacy transmitter 1, legacy transmitter 2 and the cognitive transmitter, $\mathbf{n} \in \mathbb{C}^{2\times1}$ contains i.i.d. noise samples of variance σ^2. Similarly, we could write a vector $\mathbf{y}_2$ of received samples at the antennas of legacy receiver 2:

$$\mathbf{y}_2 = \mathbf{h}_{12}\, s_1 + \mathbf{h}_{22}\, s_2 + \mathbf{H}_2\, \mathbf{s}_c + \mathbf{n}_2 \tag{12.4}$$

The number of degrees of freedom available at the cognitive transmitter is insufficient for a general set of channel coefficients. If the channel coefficients happen have the following property:

$$\mathbf{h}_{12} = \mathbf{H}_2\, \mathbf{H}_1^{-1}\, \mathbf{h}_{21}\,, \tag{12.5}$$

however, the cognitive transmitter can encode its signals such that they arrive at each of the primary receivers in the same subspace that the signal from the other legacy transmitter does. Here, we have assumed that the channel matrix $\mathbf{H}_1$ is invertible. For instance, suppose that the cognitive transmitter transmits the symbol s_c but precodes its transmissions such that the vector of signals transmitted from its two antennas is

$$\mathbf{H}_1^{-1}\, \mathbf{h}_{21}\, s_c. \tag{12.6}$$

The received signal vectors at the legacy receivers are then

$$\mathbf{y}_1 = \mathbf{h}_{11}\, s_1 + \mathbf{h}_{21}\, s_2 + \mathbf{h}_{21}\, s_c + \mathbf{n}_1\,. \tag{12.7}$$

Thus, the signal from the cognitive transmitter will be forced to zero by the zero-forcing receiver at legacy receiver 1. Similarly, at legacy receiver 2, we have

$$\mathbf{y}_2 = \mathbf{h}_{12}\, s_1 + \mathbf{h}_{22}\, s_2 + \mathbf{h}_{12}\, s_c + \mathbf{n}_2\,. \tag{12.8}$$

Note that while the assumptions about the channel coefficients are rather strong dhere, a similar approach is used in interference-alignment whereby all transmitters pre-code their signals to align interference at unintended targets.

13 Cellular Networks

Problems

13.1 Consider a multiple-access channel with three transmitters. Show that the sum capacity of this channel is achievable using TDMA and provide expressions for the fraction of time used by each of the tree transmitters.
Let the $1-\alpha_2-\alpha_3$, α_2 and α_3 be the fraction of time assigned to users 1,2, and 3. Suppose that their values are

$$\begin{aligned}
\alpha_2 &= \frac{P_3}{P_1+P_2+P_3}\\
\alpha_3 &= \frac{P_2}{P_1+P_2+P_3}
\end{aligned} \tag{13.1}$$

Then we have the following

$$\begin{aligned}
R_1 &< (1-\alpha_2-\alpha_3)\log_2\left(1+\frac{P_1}{(1-\alpha_2-\alpha_3)\sigma^2}\right)\\
&= (1-\alpha_2-\alpha_3)\log_2\left(1+\frac{P_1+P_2+P_3}{\sigma^2}\right)\\
R_2 &< \alpha_2\log_2\left(1+\frac{P_2}{\alpha_2\sigma^2}\right) = \alpha_2\log_2\left(1+\frac{P_1+P_2+P_3}{\sigma^2}\right)\\
R_3 &< \alpha_2\log_2\left(1+\frac{P_3}{\alpha_3\sigma^2}\right) = \alpha_2\log_2\left(1+\frac{P_1+P_2+P_3}{\sigma^2}\right)\\
R_1+R_2+R_3 &< \log_2\left(1+\frac{P_1+P_2+P_3}{\sigma^2}\right)
\end{aligned}$$

Note that the RHS above is the sum capacity since it is an upper bound on the capacity based on pooling the powers of all the transmitters.

13.2 Show that the sum capacity of the 2 user broadcast channel with additive Gaussian noise with power constraint $P = P_1 + P_2$ and channel coefficients of h_1 and h_2 with $||h_1|| < ||h_2||$ is given by Equation (13.14). You may wish to use the optimization techniques described in Section 2.12.

13.3 Consider a narrow-band broadcast channel with transmit power P, and a Poisson distributed number of receivers distributed i.i.d. with uniform probability in a circle of radius R, and mean number of receivers μ. Assuming the inverse-power-law path-loss model and independent Rayleigh fading between all nodes, derive the CDF of the sum capacity of the broadcast channel. Note that the sum capacity of a broadcast channel with K users and narrow-band fading coefficients $h_1, h_2, \cdots, h_K$ is

$$\max_{||h_j||} \log_2\left(1 + \frac{P\,||h_k||^2}{\sigma^2}\right) \tag{13.2}$$

First, we observe that for a n independent, identically distributed random variables $X_1, X_2, \cdots X_n$ with CDF $P_X(x)$, the CDF of the maximum of these random variables $X_{\max}$ is

$$P_{X_{\max}}(x) = (P_X(x))^n \tag{13.3}$$

Next, we find the CDF of the random variable representing the received power of one interferer from our model. From Equation (13.72), we have

$$P_p(p) = 1 - \mathcal{L}\{p_G(g)\}\left(\frac{p}{P}\right) \tag{13.4}$$

where $p_G(g)$ is given by

$$p_G(g) = \begin{cases} 0, & \text{for } R^\alpha < g \\ \frac{2\,g^{\frac{2}{\alpha}-1}}{\alpha\,R^2}, & \text{for } 0 \le g \le R^\alpha. \end{cases} \tag{13.5}$$

and $\mathcal{L}(\cdot)$ is the Laplace transform operator. Taking the laplace transform of p_G, we find that

$$\mathcal{L}\{p_G(g)\}(\tau) = \frac{2\,\tau^{-\frac{2}{\alpha}}}{\alpha\,R^2}\Gamma\left(\frac{2}{\alpha}, R^\alpha \tau\right) \tag{13.6}$$

and substituting into $P_p(p)$ yields

$$P_p(p) = 1 - \frac{2\left(\frac{p}{P}\right)^{-\frac{2}{\alpha}}}{\alpha\,R^2}\Gamma\left(\frac{2}{\alpha}, R^\alpha\left(\frac{p}{P}\right)\right) \tag{13.7}$$

substituting into (13.3), yields the CDF of the largest received power from n users.

$$P_{p_{\max}}(p) = \left(1 - \frac{2\left(\frac{p}{P}\right)^{-\frac{2}{\alpha}}}{\alpha\,R^2}\Gamma\left(\frac{2}{\alpha}, R^\alpha\left(\frac{p}{P}\right)\right)\right)^n. \tag{13.8}$$

Given that the sum capacity with n users is $\log_2(1 + p_{\max}/\sigma^2)$, the CDF of the sum capacity conditioned on n users is then

$$\Pr(c_{\text{sum}} \le x|n) = \left(1 - \frac{2\left(\frac{(2^x-1)\,\sigma^2}{P}\right)^{-\frac{2}{\alpha}}}{\alpha\,R^2}\Gamma\left(\frac{2}{\alpha}, R^\alpha\left(\frac{(2^x-1)\,\sigma^2}{P}\right)\right)\right)^n \tag{13.9}$$

Since n is a Poisson distributed random variable with mean $\pi\rho R^2$, we can remove the conditioning by multiplying by the pmf of n and summing over all n

$$\Pr(c_{\text{sum}} \le x) = \sum_{n=0}^{\infty}\left(1 - \frac{2\left(\frac{(2^x-1)\,\sigma^2}{P}\right)^{-\frac{2}{\alpha}}}{\alpha\,R^2}\Gamma\left(\frac{2}{\alpha}, R^{\alpha}\left(\frac{(2^x-1)\,\sigma^2}{P}\right)\right)\right)^n \times \frac{\left(\pi\,\rho\,R^2\right)^n}{n!}e^{-\pi\,\rho\,R^2}$$
$$= \sum_{n=0}^{\infty}\left(\pi\,\rho\,R^2 - \frac{2\,\pi\rho}{\alpha}\left(\frac{(2^x-1)\,\sigma^2}{P}\right)^{-\frac{2}{\alpha}}\Gamma\left(\frac{2}{\alpha}, R^{\alpha}\left(\frac{(2^x-1)\,\sigma^2}{P}\right)\right)\right)^n \times \frac{1}{n!}e^{-\pi\,\rho\,R^2}$$
$$= \exp\left(1 - \frac{2\,\pi\rho}{\alpha}\left(\frac{(2^x-1)\,\sigma^2}{P}\right)^{-\frac{2}{\alpha}}\Gamma\left(\frac{2}{\alpha}, R^{\alpha}\left(\frac{(2^x-1)\,\sigma^2}{P}\right)\right)\right) \tag{13.10}$$

13.4 Consider a cellular network approximated by a circular cell of radius R_c. Assume that there is a Poisson distributed number of single-antenna wireless nodes in the cell with mean number of nodes equal to $\rho_w\,\pi\,R_c^2$, independent Rayleigh fading between all nodes, and a transmit power budget of P per wireless node. Assume that the base station is connected to an infrastructure link with capacity $B\log_2(1+g\,P_b/\sigma_b^2)$, where g is a unit mean exponential random variable. Suppose that the wireless nodes have a bandwidth of B, find the probability that the sum capacity of the multiple-access channel formed by the wireless nodes and the base station at the origin exceeds the capacity of the infrastructure link.

$$\Phi_T(s) = e^{\rho\,A_R\,(z-1)}\Big|_{z=\Phi_p(s)}$$
$$= exp\left(\rho\,\pi\,R^2\left(\Phi_p(s)-1\right)\right) \tag{13.11}$$

where $\Phi_p(s)$ is given by

$$\Phi_P(s) = \left(\frac{2\,s^{2/\alpha}\,P^{2/\alpha}}{\alpha\,R^2}\Gamma\left(-\frac{2}{\alpha}\right)\Gamma\left(\frac{2+\alpha}{\alpha}\right) + {}_2F_1\left(1, -\frac{2}{\alpha}, \frac{-2+\alpha}{\alpha}, -R^{-\alpha}\,s\,P\right)\right).$$

Now the sum capacity of the multiple-access channel (MAC) is given by the log of one plus the ratio of the total received interference power at the receiver to the noise variance. In other words, if T is the sum of the received signals at the receiver of the MAC then, the sum capacity is $B\log_2(1+T/\sigma^2)$, where σ^2 is the variance of the noise at the receiver. Conditioned on the total received power T on the infrastructure link, the probability that the capacity of the MAC exceeds the capacity of the infrastructure link (which we term the uplink infrastructure outage probability) is

$$\Pr\left\{ B\log_2\left(1+g\frac{P_b}{\sigma_b^2}\right) < B\log_2(1+T/\sigma^2)\,\middle|\, T\right\} = \Pr\left\{ g < T\frac{\sigma_b^2}{P_b\sigma^2}\,\middle|\, T\right\}$$
$$= \exp\left(-T\frac{\sigma_b^2}{P_b\sigma^2}\right) \tag{13.12}$$

Removing the conditioning on T yields

$$\Pr\left\{ B\log_2\left(1+g\frac{P_b}{\sigma_b^2}\right) < B\log_2(1+T/\sigma^2)\right\} \quad = \left\langle \exp\left(-T\frac{\sigma_b^2}{P_b\sigma^2}\right)\right\rangle_T$$
$$= \Phi_T\left(\frac{\sigma_b^2}{P_b\sigma^2}\right) \tag{13.13}$$

13.5 Consider the network model of Problem 13.4 but assume that the average received power from the n interferers in the network are i.i.d. random variables taking the values of P_1 or P_2 with probabilities q and $1-q$ respectively.

a) Find the asymptotic spectral efficiency when the number of streams $M=1$, and the number of receiver antennas n_r and interferers n go to infinity with a constant ratio a. Hint: You will find the formula for the roots of a cubic polynomial useful in this question.

b) Explain a context where this network model could be useful.

c) Write a computer simulation to verify your answer to part a. with $P_1=1$, $P_2=0.5$ and $q=0.9$.

a) Since the probability of receiving power P_1 is q and $P)2$ is 1-w, as $n, n_r \to \infty$ such that $n/n_r = a > 0$, the empirical distribution function of the received interference powers converges to a function with two steps at P_2 and P_1, where we have assumed that $P_1 > P_2$. The first step has height q and the second has height $1-q$. Hence the fixed-point equation becomes

$$-\sigma^2 + 1 = \beta\, a \int_0^\infty \frac{x}{1+x\,\beta}\left(q\,\delta(x-P_1) + (1-q)\delta(x-P_2)\right)\,, \tag{13.14}$$

where $\delta(\cdot)$ denotes the Dirac measure. Evaluating the integral results in

$$-\sigma^2 + 1 = \beta\, a\,\frac{q\,P_1}{1+P_1\,\beta} + \beta\, a\,\frac{(1-q)\,P_2}{1+P_2\,\beta}\,. \tag{13.15}$$

After cross multiplying, we express this equation using the following polynomial convention

$$T_1\beta^3 + T_2\beta^2 + T_3\beta - 1 = 0 \tag{13.16}$$

where

$$T_1 = \sigma^2 P_1 P_2$$
$$T_2 = (1-q)aP_1P_2\gamma^2 + aqP_1P_2 + (\sigma^2 - P_1)P_2 + \sigma^2 P_1$$
$$T_3 = -P_2 - P_1 + \sigma^2 + (1-q)aP_2 + qaP_1$$

Additionally, we define the following to simplify notation:

$$T_4 = 2T_2^3 - 9T_1T_2T_3 - 27T_1^2$$
$$T_5 = \sqrt{T_4^2 - 4(T_2^2 - 3T_1T_3)^3}$$

The solution to this is found using the standard formula for the roots of a cubic polynomial, which yields:

$$\beta = -\frac{T_2}{3T_1} + \frac{1 - j\sqrt{3}}{6T_1}\left(\frac{T_4 + T_5}{2}\right)^{\frac{1}{3}} + \frac{1 + j\sqrt{3}}{6T_1}\left(\frac{T_4 - T_5}{2}\right)^{\frac{1}{3}} \tag{13.17}$$

b) One context in which this could be useful is in networks with two classes of users. One which is only allowed (or capable of) single-rank transmissions and another which is capable of spatial multiplexing.

13.6 Plot and qualitatively describe the capacity region of the vector multiple-access channel with 2 receivers. Explain how it differs from multiple-access channel capacity region for single antenna systems.

13.7 Find the CDF of the SINR for a multiantenna receiver in a circular cell communicating with a single antenna transmitter in the presence of equal-transmit power, single-antenna interferers distributed according to a Poisson point process on the plane, assuming that the receiver only knows the spatial covariance matrix of the interferers in the circular cell as in Section 13.3.5. In other words, the out-of-cell interference is modeled as white noise here.

13.8 Consider a hexagonal-cell system with no out-of-cell interference, the power control algorithm of Equation (13.119), with $P_{max} = \infty$ and single antenna transmitters with n_r antennas at the base station. You may approximate the hexagonal cell by a circle.

(a) Find the CDF of the spectral efficiency of a random link in this system.

(b) Using the previous result and the fact that the integral of one minus the CDF is equals the mean of a random variable, compute the mean spectral efficiency of this system.

(c) Assuming that a reuse factor of K is required for the assumption of no out-of-cell interference to hold, compare the mean area spectral efficiency using the asymptotic analysis of and a reuse factor of 1 in (13.143) with your answer in the previous section. Make sure you take into account the penalty on the area spectral efficiency due to the reuse factor of K.

14 Ad Hoc Networks

Problems

14.1 Using the Ozgur hierarchical cooperation scheme, compare the bound on the network throughput capacity given by Equation (14.6), for one and two levels of hierarchy. In particular, use the bound to estimate the number of nodes required in the network for the throughput capacity with two levels of hierarchy to exceed the throughput capacity with one level of hierarchy. Your answer will illustrate a weakness in using capacity scaling laws to estimate the performance of practical wireless networks.

14.2 Consider a square wireless network of fixed area which is divided into n uniform squares. In each of these squares, place a wireless node with uniform probability as shown in Figure 14.10 which illustrates a network with $n = 36$ nodes. For simplicity, assume that the total interference in the network is proportional to $n^{\alpha/2}$ and signal power decays with distance according to the inverse-power-law model with path-loss exponent $\alpha > 2$. Using these simplifying assumptions, show that a multi-hop protocol can achieve a per-link throughput capacity of the order $\sqrt{n}$ as $n \to \infty$.

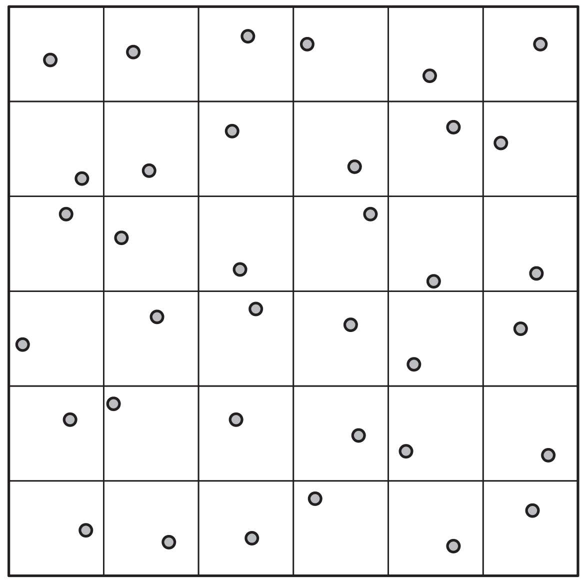

Figure 14.10 Illustration of a square network with $n = 36$ nodes.

14.3 The upper regularized gamma function is defined as the ratio of an upper incomplete gamma function and a gamma function (see Section 2.14.1) as follows

$$Q(\nu, x) = \frac{\Gamma(\nu, x)}{\Gamma(\nu)} .$$

When the first parameter $\nu = L$ is an integer, the upper regularized gamma function simplifies to a sum of elementary functions as follows,

$$Q(s, x) = \sum_{k=0}^{L-1} \frac{x^k}{k!} e^{-x} .$$

Additionally, the following is known about the upper regularized gamma function for a positive real number q and integer L [364],

$$\lim_{L\to\infty} Q(L, q\,L) = \begin{cases} 0, & \text{if } q \geq 1 \\ 1, & \text{if } q < 1 . \end{cases}$$

Consider a homogenous Poisson network with a multi-antenna receiver, and single-antenna transmitters, with i.i.d. Rayleigh fading between all antennas.

(a) Ignoring the contribution of the noise and using the above properties of the upper regularized incomplete gamma function, show that the SIR converges in probability to a non-random limit if the number of antennas at the receiver is increased linearly with node density.

(b) The result above suggests that it may be possible to scale ad-hoc wireless networks by increasing the number of receiver antennas with node density. Discuss the feasibility of doing so.

(a) Comparing the CDF of the SINR for a linear MMSE multiantenna receiver in a homogenous Poisson network with i.i.d. Rayleigh fading between antennas given by Equation (14.33). Setting the noise to zero, we have the CDF of the SIR as follows

$$\Pr(\text{SIR} \leq \beta) = 1 - \sum_{k=0}^{n_r-1} \frac{\left(\rho K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right)^k}{k!} \exp\left(-\rho K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right) \tag{14.1}$$

where the parameter $K_\alpha = \frac{2\pi\Gamma(2/\alpha)\Gamma(1-2/\alpha)}{\alpha}$, and the node density is ρ. Let $\rho = n_r\, \rho_c$, where ρ_c is a positive constant and n_r is the number of receiver antennas. Thus, the density of nodes is scaled linearly with the number of antennas. Substituting into $\Pr(\text{SIR} \leq \beta)$ we have

$$\begin{aligned}
\Pr(\text{SIR} \leq \beta) &= 1 - \sum_{k=0}^{n_r-1} \frac{\left(n_r\, \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right)^k}{k!} \exp\left(-n_r\, \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right) \\
&= Q\left(n_r, n_r\, \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right) \\
\lim_{n_r\to\infty} \Pr(SIR \leq \beta) &= \lim_{n_r\to\infty} Q\left(n_r, n_r\, \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2\right) \\
&= \begin{cases} 0, & \text{if } \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2 \geq 1 \\ 1, & \text{if } \rho_c K_\alpha \beta^{\frac{2}{\alpha}} r_1^2 < 1. \end{cases} \\
&= \begin{cases} 0, & \text{if } \beta \geq \left(\frac{1}{\rho_c\, K_\alpha\, r_1^2}\right)^{\frac{\alpha}{2}} \\ 1, & \text{if } \beta < \left(\frac{1}{\rho_c\, K_\alpha\, r_1^2}\right)^{\frac{\alpha}{2}}. \end{cases}
\end{aligned} \tag{14.2}$$

Since the RHS of the previous expression is a step function at $\beta \geq \left(\frac{1}{\rho_c\, K_\alpha\, r_1^2}\right)^{\frac{\alpha}{2}}$, the SIR converges in probability to a constant implying that it converges in probability as well.

(b) The result in the previous part suggests that it may be able to maintain a non-zero SIR by increasing the number of antennas with user density. While this approach seems attractive in theory, it could have limited utility in practice, because the result above is derived assuming independent fading between antennas. As the number of antennas increases, if the receiver is limited in the volume it occupies, this assumption will become increasingly difficult to satisfy. Additionally, with large numbers of antennas, the parameters required to compute the MMSE receiver, namely the covariance matrix of the interferers and the channel coefficients between the target transmitter and the receiver, would require longer and longer training times.

14.4 Prove that the second term on the left-hand side of Equation (14.15) goes to zero as $n/n_r \to \infty$.

Suppose that the maximum power level is P_M. Then, we have the second term on the left-hand side of Equation (14.15) can be written as follows

$$\begin{aligned}\int_0^\infty d\tau\,\frac{\tau^{-\frac{2}{\alpha}}}{1+\tau\,\beta}\int_{\tau/b}^\infty dx\,f_j(x)\,x^{\frac{2}{\alpha}} &\leq \int_0^{b\,P_M} d\tau\,\frac{\tau^{-\frac{2}{\alpha}}}{1+\tau\,\beta}\int_{\tau/b}^\infty dx\,f_j(x)\,P_M^{\frac{2}{\alpha}}\\ &= \int_0^{b\,P_M} d\tau\,\frac{\tau^{-\frac{2}{\alpha}}}{1+\tau\,\beta}\,P_M^{\frac{2}{\alpha}}\int_{\tau/b}^\infty dx\,f_j(x)\\ &\leq \int_0^{b\,P_M} d\tau\,\frac{\tau^{-\frac{2}{\alpha}}}{1+\tau\,\beta}\,P_M^{\frac{2}{\alpha}}\int_{\tau/b}^\infty dx\,f_j(x)\\ &\leq \int_0^{b\,P_M} d\tau\,\frac{\tau^{-\frac{2}{\alpha}}}{1+\tau\,\beta}\,P_M^{\frac{2}{\alpha}}\,. \end{aligned} \tag{14.3}$$

Since $b \to 0$ as $n_r/n \to 0$, the expression above goes to zero.

14.5 Consider an ad-hoc network with interferers distributed according to a homogenous Poisson point process on the plane with single antenna transmitters and a multi-antenna receiver with n_r antennas at the origin. Assume that the receiver uses a zero-forcing algorithm and cancels the interference due to the $n_r - 1$ closest interferers to it. Assuming Rayleigh fading across all antennas and noting that the unitary transformation of a matrix with i.i.d. circularly symmetric Gaussian random variables does not change the statistical properties of the matrix, find the CDF of the SINR for this system.

14.6 Derive the integer-relaxed optimum number of streams for the multi-stream transmissions in an ad hoc wireless network with multi-antenna MMSE receivers given in Equation (14.24).

14.7 Consider a circular network of radius R with a multi-antenna receiver with n_r antennas at the origin. Let n interferers be independently distributed in the circular network such that the distribution of their distances from the origin is uniform, that is the probability density function of the distance of an interferer, r, from the origin is

$$p(r) = \begin{cases} \frac{1}{R}, & \text{if } 0 \leq x \leq R, \\ 0, & \text{otherwise}\,. \end{cases} \tag{14.4}$$

Assume that the noise power is equal to $P\,N^{-\alpha}$ at each antenna of the representative receiver and all nodes transmit with equal power in the network,, with the standard inverse power-law path loss, and i.i.d., unit variance fading between all antennas in the network. Show that $\beta_N = N^{-\alpha}$ SINR converges with probability 1 to a limit β as $n, n_r, R \to \infty$ such that n/n_r equals a positive constant c, and $n = \rho\,R$ with $\rho > 0$ equal to a nominal density. Find an implicit expression analogous to (14.15) that β needs to satisfy in this case. Inspired by results in Reference [122].

14.8 Consider an ad hoc wireless network with a representative receiver at the origin and a representative transmitter at a fixed distance r_T from the origin. Assume that this link operates in the presence of interferers who are modeled according to a homogenous Poisson point process on the plane with density

ρ. Assume that all users have single antennas and the channel between any wireless node and any other wireless node follows a block fading model with n_c coherence bands, and M frequency sub-channels. The channels between all transmitter-receiver pairs on the N_c coherence bands are i.i.d. Rayleigh random variables. Figure 14.11 illustrates the channel strengths between a transmitter-receiver pair with $N_c = 6$ coherence bands and $M = 24$ subchannels, with four subchannels per coherence band. Each transmitter picks one subchannel out of the strongest coherence band between itself and its target receiver, on which to transmit. Compute the CDF of the SINR of the representative link. You may wish to use the result on the antenna selection receiver to solve this problem. This problem is inspired by the results in [128].

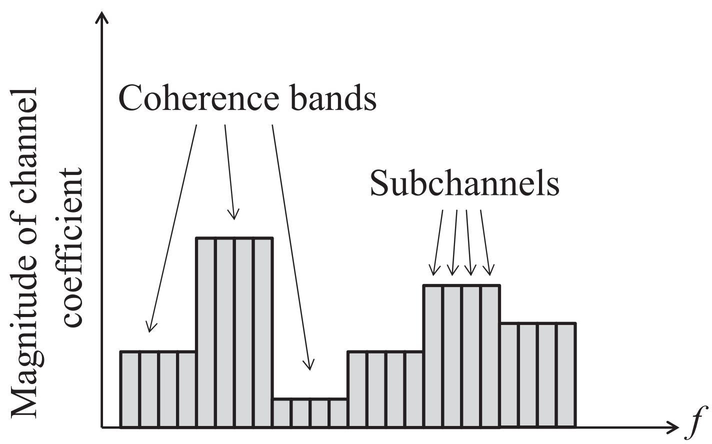

Figure 14.10 Illustration of a block-frequency fading channel with six coherence-bands and twenty four subchannels.

Each transmitter (representative transmitter and the interferers) picks one channel out of the strongest coherence band between themselves and their target receivers on which to transmit. Thus, the density of interferers in the channel selected by the representative transmitter is ρ/M. Additionally, the representative transmitter picks the coherence band that is strongest amongst n_c independently, Rayleigh faded coherence bands. Thus the problem is analogous to the antenna-selection receiver with $1/M$th the density of interferers, resulting in the following CDF of the SINR:

$$\Pr\{\text{SINR} \le x | r_1\} =$$

$$\sum_{k=0}^{n_c} \binom{n_c}{k} (-1)^k \exp\left(-k\,\frac{x\, r_1^\alpha}{P_1}\,\sigma^2 - \pi\,\frac{\rho}{M}\,\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)\left(k\frac{x}{P_1}\right)^{\frac{2}{\alpha}} r_1^2\right)$$

14.9 Consider the antenna selection receiver in a spatially distributed network whose SINR CDF conditioned on the link-length r_1 is given by Equation (14.37).

Suppose that r_1 is distributed according to a nearest-neighbor Poisson distribution whereby r_1 is the distance between a reference point such as the origin, to the nearest point of a homogenous Poisson point process with intensity ρ_c. Note that the PDF of r_1 is given by Equation (13.32). Please find a closed-form expression for the CDF of the SINR (without the conditioning on r_1). This problem is inspired by the results in [128].

Supposed that the interferers are distributed according to a Poisson point process on the plane with uniform density ρ. Substituting (14.36) into (14.37), we have the CDF of the SINR given by

$$\Pr\{\mathrm{SINR} \le x | r_1\} = \sum_{k=0}^{n_r} \binom{n_r}{k} (-1)^k \exp\left(-k\,\frac{x\,r_1^{\alpha}}{P_1}\,\sigma^2 - \pi\,\rho\,\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)\left(k\frac{x}{P_1}\right)^{\frac{2}{\alpha}} r_1^2\right)$$

Multiplying by the PDF of r_1 in (13.32) and integrating:

$$\Pr\{\mathrm{SINR} \le x | r_1\} = \int_0^{\infty} dr_1 \sum_{k=0}^{n_r} \binom{n_r}{k} (-1)^k \exp\left(-k\,\frac{x\,r_1^{\alpha}}{P_1}\,\sigma^2 - \pi\,\rho\,\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)\left(k\frac{x}{P_1}\right)^{\frac{2}{\alpha}} r_1^2\right) 2\,\pi\rho_c r_1 e^{-\pi\rho_c r_1^2}$$

Now, if we assume that the thermal noise is negligible, i.e. $\sigma \to 0$, we have

$$\Pr\{\mathrm{SINR} \le x | r_1\} = \int_0^{\infty} dr_1 \sum_{k=0}^{n_r} \binom{n_r}{k} (-1)^k \exp\left(-\pi\,\rho\,\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right)\left(k\frac{x}{P_1}\right)^{\frac{2}{\alpha}} r_1^2\right) \times 2\,\pi\rho_c r_1 e^{-\pi\rho_c r_1^2}$$

$$= \sum_{k=0}^{n_r} \binom{n_r}{k} (-1)^k \frac{2\,\pi\rho_c}{\pi\,\rho\,\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1-\frac{2}{\alpha}\right) + \pi\rho_c}$$

15 Medium-Access-Control Protocols

Problems

15.1 Derive the throughput capacity of unslotted ALOHA under the assumption that the packet duration is a constant.

We assume that the packet arrivals in the network follow a Poisson point process with arrival rate G, packets are of length d and that propagation delays are negligible.

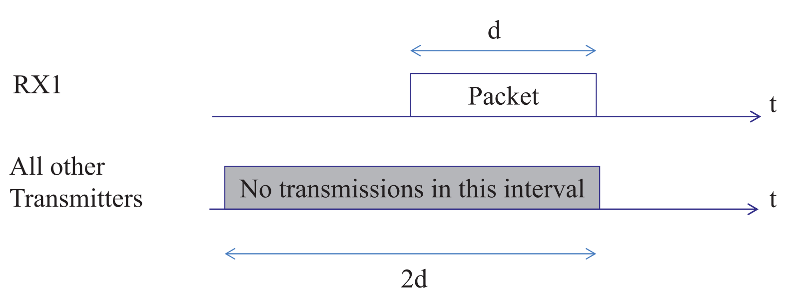

Figure 15.1 Illustration of unslotted ALOHA with constant packet length of d.

In Figure 15.1, we observe that when a receiver RX1 is receiving a packet, there cannot be other packet arrivals in the network in the duration that this packet is being received at RX1, or in the immediate interval of length d before the reception of the packet begins. Thus, there can be no packet arrivals in a window of duration d to ensure not collision. The probability of success is therefore equal to the probability of no arrivals in a Poisson point process of rate G in a window of length $2d$, which is given by

$$p_s = e^{-2\,d\,G}\,.$$

The throughput is thus

$$d\,G\,e^{-2\,d\,G}$$

To maximize the throughput, we take the derivative and set it to zero:

$$\begin{aligned} \frac{\partial}{\partial G} d\,G e^{-2\,d\,G} &= d\,e^{-2\,d\,G} - 2\,d^2\,G\,e^{-2\,d\,G} = 0 \\ e^{-2\,d\,G} - 2\,d\,G\,e^{-2\,d\,G} &= 0 \end{aligned} \tag{15.1}$$

Thus, the value of G that maximizes the throughput is $1/(2d)$. Substituting into (15.1) yields the maximum throughput of the unslotted ALOHA protocol of

$$\frac{1}{2\,e}.$$

15.2 Construct a different scenario where the CSMA/CA protocol fails and results in a collision during transmission of the data packet. Your answer should include a timing diagram as well as a figure which describes the relative positions of nodes and/or obstacles.

15.3 Modify the Ward protocol described in Section 15.5.5 such that it applies to ad hoc wireless networks, that is networks with one-to-one links. You should construct a timing diagram for a scenario where a link is successfully established.

15.4 For the Carrier-Sense-Multiple-Access system described in Section 15.3, construct a scenario where a packet collision occurs. Discuss the role of propagation delays and the amount of time required for sensing the medium in the probability of a collision.

15.5 Qualitatively explain why a randomized sensing duration could result in lower probability of collision in Carrier-Sense-Multiple-Access systems compared to a fixed sensing duration.

15.6 Assuming that channels are static, construct a scenario where the SPACE-MAC protocol described in Section 15.5.3 results in a collision during the data packet transmission.

Suppose that node 1 wishes to transmit to node 2, and node 3 wishes to transmit to node 4. Furthermore, assume tat nodes 3 and 4 are hidden from node 1, and node 2 is hidden from node 4. The timing diagram illustrated in Fig. 15.2 depicts an RTS/CTS sequence that results in a packet collision at node 2 during the data transmission. Since nodes 3 and 4 are hidden from node 1, they do not receive the RTS1 packet that initiates the link between nodes 1 and 2. During the transmission of node 2's CTS packet, node 3 happens to transmit its RTS packet and thus does not hear node 2's CTS packet even though it is not hidden from node 3. Thus nodes 3 and 4 are unaware of the ongoing link between nodes 1 and 2 and hence collide during the data transmission phase of their link.

15.7 Construct a timing diagram for a simple interference-alignment protocol with three transmit and receive pairs, each with three antennas. You may use the system described in Section 14.4. Your may assume channel reciprocity between all antennas, and only consider a case where links are successfully established. Your answer should indicate when all necessary channel estimations are performed and in which packets channel parameters that cannot be estimated are exchanged.

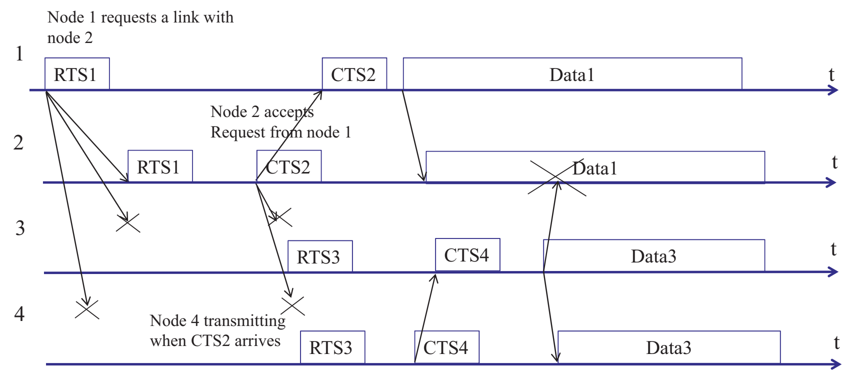

Figure 15.2 Timing diagram for SPACE-MAC protocol resulting in collision.

We assume that all nodes in the network have carrier frequency synchronization and are synchronized at the frame level. These can be accomplished if nodes have GPS disciplined receivers which can derive a high frequency reference (e.g. a 10 MHz signal) and a pulse-per-second reference from the Global Positioning System (GPS) satellites. We assumed that the transactions involved in establishing a link are divided into three phases. In the first phase, which we call the contention phase, nodes in the network transmit request to send (RTS) packets where the probability that an RTS packet is transmitted at any given time is low, reducing the probability of collisions. The RTS packets contain the ID numbers of the transmitters and their target receivers as well as a pseudorandom training sequence that will enable all receivers in the network to perform channel estimation. After three RTS packets are received the system can move to the second phase. Note that any other nodes which are in the network do not get to transmit in this round. The contention phase is depicted in Figure 15.3. Note that at the conclusion of the contention phase, all receivers will have knowledge of the channel state information between each transmitter and themselves.

In the second phase, the three receivers that are the targets of the RTS packets in the contention phase respond with CTS packets. Note that these can be transmitted in rapid sequence, with the convention that the receivers transmit in the order that their corresponding transmitters transmitted their RTS packets in the contention phase. The CTS packet of the i-th receiver contains the ID number of that receiver, and the channel state information between each transmitter and receiver i (estimated in the previous phase). Thus, at the conclusion of this phase, all nodes in the network will know the channel-state information between all transmitters and all receivers. This phase is depicted in Figure 15.4.

Finally, in the data phase, all three transmitters can transmit their data packet. The first transmitter to send an RTS packet in the contention phase gets to transmit 2 data streams as depicted in Figure 15.5.

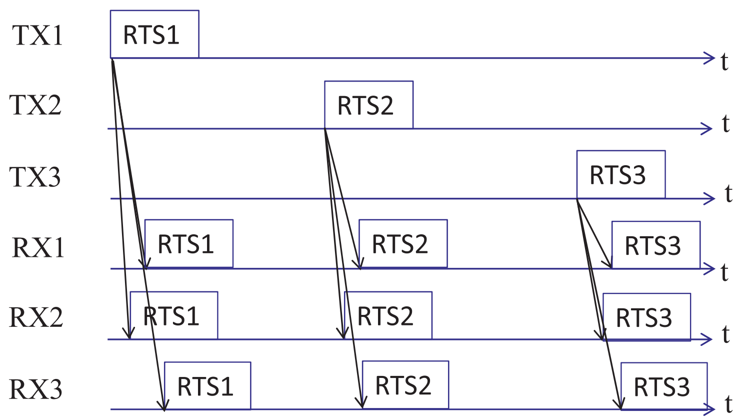

Figure 15.3 Timing diagram for phase-1 of interference alignment protocol.

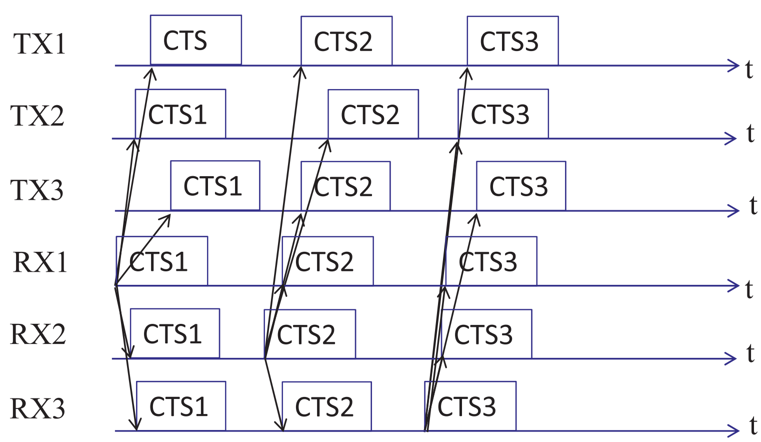

Figure 15.4 Timing diagram for phase-2 of interference alignment protocol.

15.8 Assuming devices that can simultaneously transmit and receive signals in different frequency bands, consider the following communications protocol which is a variant of busy-tone protocols (see for instance Reference [20]). The avaliable bandwidth B is divided into n_c data subchannels and n_c busy-tone channels where each data-channel has a corresponding busy-tone channel at significantly different frequency range to enable simultaneous transmissions and receptions. A node that is receiving a data in the k-th data channel simultaneously transmits a noise-like busy-tone signal in the k-th busy-tone channel. Any node that wishes to transmit can only do so in a data channel in which the average received energy is below a threshold.

(a) Describe how this protocol alleviates the hidden node problem.

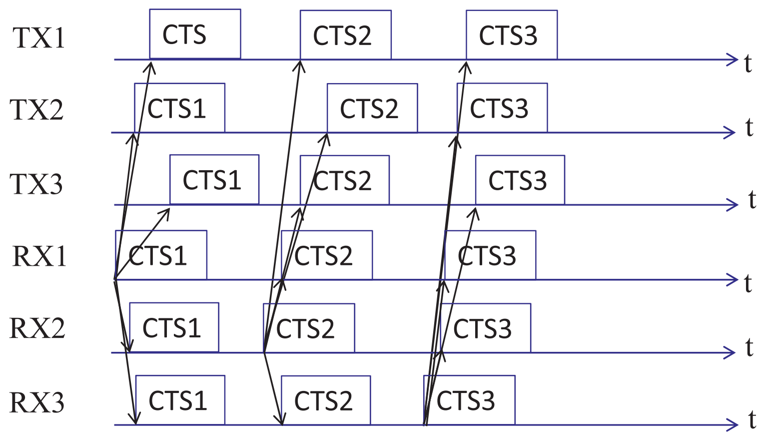

Figure 15.5 Timing diagram for phase-3 of interference alignment protocol.

(b) Describe how this protocol alleviates the exposed node problem.

(c) Suppose that the busy-tone channel occupies a very narrow range of frequencies. Qualitatively describe a failure mode of the protocol that results in a collision in a data channel when multiple receivers are successfully receiving data in a given channel, but a new transmitter believes that the data channel is available and starts transmitting. Hint: the narrow bandwidth of the busy tone channel causes this problem.

(a) The hidden-node problem occurs when a transmitter is not aware of the presence of a receiver which is receiving data. The transmitter could then transmit a signal and cause a collision on at the receiver. Assume that receives have to transmit busy tone signals with sufficient time to allow for propagation delays before they begin receiving data.The hidden node problem can be alleviated using this technique if each transmitter is required to sense the busy-tone channel associated with the channel on which it wishes to transmit. Suppose that a transmitter wishes to transmit on channel k. It must first sense the amount of energy it receives in busy-tone channel k. If the energy in busy-tone channel k exceeds a certain threshold, then the transmitter senses that a receiver is currently receiving data in its vicinity and is not allowed to transmit.

(b) The exposed-node problem occurs when there is a node (Node 2, say) that can potentially successfully receive data from a transmitter (Node 1), but Node 1 does not initiate a transmission to Node 2 because it detects an ongoing transmission in its vicinity which may be too weak from disrupting reception at Node 2. Thus the channel is used less efficiently as it would have been possible for Node 1 to transmit data to Node 2.

This protocol can help alleviate the exposed node problem by requiring potential transmitters to avoid transmitting in a given channel, only if the signal power it observes in the associated busy-tone channel is lower than a threshold. This requirement reduces the probability that an available channel is believed by a transmitter to be unavailable.

Note that the exposed node problem still exists in this protocol. Consider a situation were there are multiple receivers receiving data in a given channel, channel k say. Suppose that the system is such that none of these receivers would experience a collision if a given transmitter transmits in channel k, e.g. if the receivers are far away from the transmitter in concern. However, the aggregate busy tone signal could still exceed the threshold, which would prevent from that transmitter from transmitting.

(c) If the busy-tone channel is assigned a narrow bandwidth, there could be two possible types of collisions resulting from insufficient bandwidth. Firstly, if the bandwidths of the busy-tone channels are small, there may be insufficient frequency diversity so that the channel between an active receiver and a potential transmitter may be strong in a given data channel, but may be weak in the corresponding busy-tone channel.

Another failure mode can occur if there are multiple active receivers using a particular channel. Despite the fact that the busy tone signals are designed to be noise like in this example, if their bandwidth is narrow, the signals could add constructively or destructively, where the latter case could give a potential transmitter the false impression that a channel is not busy. Note that in the extreme case, the busy-tone signal is simply a pure cosine. With multiple busy tones transmitted at the same frequency, the busy tone signals could add constructively or destructively.

16 Cognitive Radios

Problems

16.1 Consider a single observation at an n_r receiver with known noise covariance matrix $\mathbf{R} = \mathbf{I}$. Evaluate the probability of detection and probability of false alarm under the assumption of a single transmitter in a Gaussian channel with a constant modulus transmit signal as a function of SNR per receive antenna.

16.2 For a single-antenna energy detection approach under the assumptions of circular complex Gaussian noise and a signal with SNR of 0 dB, numerically find the minimum number of samples required to achieve a false-alarm rate of no larger than 10^{-6} and a probability detection of no less than 0.9, for the models of

(a) Gaussian signal in Gaussian noise of known variance

(b) Unknown deterministic signal in Gaussian noise of known variance

16.3 Considering a single-antenna, new-energy detector, by assuming 10 independent observations, for probability of false alarm of 10^{-5} numerically find the required SNR for the probability of detection to be at least 0.9.

16.4 Extend the evaluation of probability of detection and false alarm for the single-antenna, new-energy detector found in Equation (16.28) to include unequal numbers of observations for the old and new variance estimates.

16.5 Under the assumption of a four-antenna receiver and a single transmitter received at 0 dB SNR per antenna, numerically evaluate the receiver operating curves for the multiple-antenna, new-energy detectors defined in Equations (16.43) and (16.46) under the assumption of

(a) $n_s = 4$

(b) $n_s = 8$

(c) $n_s = 32$

independent samples.

16.6 Under the assumption of a four-antenna receiver and a single transmitter with 10 samples received at 0 dB SNR per antenna, consider a modification of that form in Equation (16.43) Replace the trace with evaluating the maximum eigenvalue. Numerically compare the receiver operating curves of the form in Equation (16.43) and the modified form.

16.7 For a SISO channel, under the assumptions discussed in Section 16.4.1, evaluate an approximate optimal spectral efficiency to minimize interference with a legacy or hidden-node network under the assumption that the frame is small compared to the legacy waveform in time and bandwidth.

17 Multiple-Antenna Acquisition and Synchronization

Problems

17.1 Reformulate the Cramer-Rao bound in Section 17.2 for frequency estimation.

17.2 Reformulate the Cramer-Rao bound in Section 17.2 under the assumptions of an uninformed MIMO transmitter, n_I strong interferers, and a random flat-fading i.i.d. complex circular Gaussian channel matrix.

17.3 Reformulate the correlation, MMSE, and GLRT test statistics in terms of frequency synchronization.

17.4 Evaluate the GLRT under the assumption that the interference-plus-covariance matrix is known and given by $\mathbf{I}$.

17.5 For a channel with a number of receivers greater than or equal to transmitters ($n_r \geq n_t$), develop a test statistic that replaces the beamformers in $\mathbf{W}$ from Equation (17.21) with zero-forcing beamformers using the estimated channel. Numerically compare performance of the statistic to the performances in Figure 17.1.

17.6 For the correlation, MMSE, and GLRT test statistics determine numerically the SNR required for a 4×4 link to achieve a probability of false alarm of less than 10^{-6} and a probability of detection of at least 0.9 for 32 observations under an INR per receive antenna of

(a) $-\infty$ dB

(b) 20 dB

18 Practical Issues

no problems.