Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014) CHAPTER 9

PROBLEM 9.1. Linear Theory of Magnetoconvection.
Derive the linear dispersion relation for magnetoconvection

$$(\omega+\kappa k^2)(\omega+\eta k^2)(\omega+\nu k^2)k^2$$
$$= -\frac{(\mathbf{B}_0\cdot\mathbf{k})^2}{\mu\rho_0}(\omega+\kappa k^2)k^2+\frac{g\alpha_T\Delta T}{d}(\omega+\eta k^2)(k_x^2+k_y^2) \qquad (1)$$

in the Boussinesq approximation.

SOLUTION.

The incompressible MHD equations, including uniform viscous, magnetic and (radiative) thermal diffusion and gravity, are (Sec.2.4.3) $\boldsymbol{\nabla}\cdot\mathbf{v}=\boldsymbol{\nabla}\cdot\mathbf{B}=0$ with

$$\rho\frac{d\mathbf{v}}{dt}=-\boldsymbol{\nabla}p+\mathbf{j}\times\mathbf{B}+\rho\nu\nabla^2\mathbf{v}-\rho g\hat{\mathbf{z}},$$
$$\frac{\partial\mathbf{B}}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})+\eta\nabla^2\mathbf{B}, \qquad \frac{dT}{dt}=\kappa\nabla^2T.$$

Departures from an equilibrium plasma, with a linear temperature profile $[T_0(z)]$, temperature difference ΔT and uniform magnetic field $(\mathbf{B}_0)$, are written $\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1, \mathbf{v}=\mathbf{v}_1, T=T_0(z)+T_1$. The equations are linearised and for simplicity the *Boussinesq approximation* is adopted, which holds when convection is highly subsonic and the depth (d) is much smaller than a scale-height. This filters out sound waves and incorporates density variations only in the buoyancy force, where they are written $\rho_1=-\rho_0\alpha_T T_1$.

The linearised equations are then $\boldsymbol{\nabla}\cdot\mathbf{v}_1=\boldsymbol{\nabla}\cdot\mathbf{B}_1=0$ with

$$\rho_0\frac{\partial\mathbf{v}_1}{\partial t}=-\boldsymbol{\nabla}p_1+\mathbf{j}_1\times\mathbf{B}_0+\rho_0\nu\nabla^2\mathbf{v}_1-\rho_1 g\hat{\mathbf{z}},$$

$$\frac{\partial\mathbf{B}_1}{\partial t}=\boldsymbol{\nabla}\times(\mathbf{v}_1\times\mathbf{B}_0)+\eta\nabla^2\mathbf{B}_1, \qquad \frac{\partial T_1}{\partial t}+(\mathbf{v}_1\cdot\boldsymbol{\nabla})T_0=\kappa\nabla^2T_1.$$

In other words, $\boldsymbol{\nabla} \cdot \mathbf{v}_1 = \boldsymbol{\nabla} \cdot \mathbf{B}_1 = 0$ with

$$\rho_0 \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \mathbf{v}_1 = -\boldsymbol{\nabla} p_1 + \mathbf{j}_1 \times \mathbf{B}_0 + \rho_0 g \alpha_T T_1 \hat{\mathbf{z}},$$

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \mathbf{B}_1 = (\mathbf{B}_0 \cdot \boldsymbol{\nabla}) \mathbf{v}_1, \qquad \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) T_1 = v_{1z} \frac{dT_0}{dz}.$$

Taking the curl of the equation of motion and dividing by ρ_0 gives

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \boldsymbol{\nabla} \times \mathbf{v}_1 = \frac{(\mathbf{B}_0 \cdot \boldsymbol{\nabla})}{\mu \rho_0} \boldsymbol{\nabla} \times \mathbf{B}_1 + g \alpha_T \boldsymbol{\nabla} \times (T_1 \hat{\mathbf{z}}).$$

Operating on this equation with $(\partial/\partial t - \eta \nabla^2)$ and $(\partial/\partial t - \kappa \nabla^2)$, and substituting for $\mathbf{B}_1$ and T_1 then gives an equation for $\mathbf{v}_1$ alone, namely,

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \boldsymbol{\nabla} \times \mathbf{v}_1$$

$$= \frac{(\mathbf{B}_0 \cdot \boldsymbol{\nabla})^2}{\mu \rho_0} \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \boldsymbol{\nabla} \times \mathbf{v}_1 + \frac{g \alpha_T \Delta T}{d} \left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \boldsymbol{\nabla} \times (v_{1z} \hat{\mathbf{z}}).$$

Now $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{v}_1) = -\nabla^2 \mathbf{v}_1$ and $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times v_{1z} \hat{\mathbf{z}}) = -\boldsymbol{\nabla} \partial v_{1z}/\partial z - \nabla^2 v_{1z} \hat{\mathbf{z}}$, whose z-component is just $-(\partial^2/\partial x^2 + \partial^2/\partial y^2) v_{1z}$. Thus, after taking the curl of the above equation, its z-component reduces to an equation for v_{1z}, namely

$$\left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \nabla^2 v_{1z}$$

$$= \frac{(\mathbf{B}_0 \cdot \boldsymbol{\nabla})^2}{\mu \rho_0} \left(\frac{\partial}{\partial t} - \kappa \nabla^2 \right) \nabla^2 v_{1z} + \frac{g \alpha_T \Delta T}{d} \left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) \left(\frac{\partial^2 v_{1z}}{\partial x^2} + \frac{\partial^2 v_{1z}}{\partial y^2} \right).$$

A solution of the form

$$v_{1z} \sim e^{\omega t} e^{i(k_x + k_y y)} \sin k_z z$$

vanishes at the boundaries $(z = 0, d)$ if $k_z = \pi/d$ implies that $\partial/\partial t$ is replaced by ω and $\boldsymbol{\nabla}$ by $i\mathbf{k}$. The above equation therefore reduces to

$$(\omega + \kappa k^2)(\omega + \eta k^2)(\omega + \nu k^2) k^2$$

$$= -\frac{(\mathbf{B}_0 \cdot \mathbf{k})^2}{\mu \rho_0} (\omega + \kappa k^2) k^2 + \frac{g \alpha_T \Delta T}{d} (\omega + \eta k^2)(k_x^2 + k_y^2), \tag{2}$$

as required.

PROBLEM 9.2. Convection in the Absence of a Magnetic Field.

Use Eq.(2) for the case of no magnetic field with rolls ($k_y = 0$) aligned with the y-direction, to show that convection sets in as overturning motion with the disturbance growing monotonically with wavenumber k_x when $Ra > (\pi^2 + k_x^2 d^2)^3 / k_x^2 d^2$. Hence show that the minimum Rayleigh number is $27\pi^4/4$ and that it occurs when $k_x = \pi d/\sqrt{2}$, a classic result due to Rayleigh (1916) (Sec.7.5.6), which gives the precise value for the order of magnitude criterion.

SOLUTION.
With $B_0 = 0$, $k_y = 0$ and dimensionless values defined as $\bar{k} = kd$, $\bar{k}_x = k_x d$, $\bar{\omega} = \omega d^2/\kappa$ and $Ra = g\alpha_T \Delta T d^3/(\kappa\nu$, Eq.(2) reduces to

$$\bar{\omega}^2 + \bar{k}^2(1+\kappa/\nu)\bar{\omega} + \bar{k}^4 - Ra\ \bar{k}_x^2/\bar{k}^2 = 0,$$

where $\bar{k}_x^2 = \bar{k}^2 - \pi^2$.

Instability onset occurs when $\bar{\omega}$ is real and increases through zero to positive values. The condition for $\bar{\omega}$ (given by the above quadratic equation) to be real is simply

$$\bar{k}^4(1+\kappa/\nu)^2 > 4(\bar{k}^4 - Ra\ \bar{k}_x^2/\bar{k}^2).$$

The condition for $\bar{\omega}$ to be positive is simply

$$\bar{k}^4 - Ra\ \bar{k}_x^2/\bar{k}^2 < 0,$$

or, after substituting for $\bar{k}_x$,

$$Ra > \frac{\bar{k}^6}{\bar{k}^2 - \pi^2} \qquad \text{or} \qquad Ra > \frac{(\bar{k}_x^2 + \pi^2)^3}{\bar{k}_x^2},$$

where $\bar{k}_x^2 = k_x\ d$, as required. The equation $Ra = \bar{k}^6/(\bar{k}^2 - \pi^2)$ when plotted for Ra as a function of $\bar{k}^2$ in the range $\bar{k}^2 > \pi^2$ possesses a minimum value of $Ra = 27\pi^4/4$ when $\bar{k}^2 = 3/2\pi^2$ and so $\bar{k}_x^2 = 1/2\pi^2$.

Thus, we have proved that the minimum Rayleigh number for the onset of convection is $27\pi^4/4$ at $k_x = \pi d/\sqrt{2}$, as required.

PROBLEM 9.3. Convection in a Horizontal Magnetic Field.
Use Eq.(2) for the case of a horizontal magnetic field ($\mathbf{B}_0 = B_0\hat{\mathbf{x}}$), with no dissipative effects ($\kappa = \eta = \nu = 0$).

(a) Show that rolls parallel to the field (with $k_x = 0$) are unstable ($\omega > 0$) for all wavenumbers $k_y \neq 0$;

(b) For a given $k_x \neq 0$, prove that convection is inhibited for all k_y if the magnetic field is so strong that $B_0^2/(\mu\rho_0) > g\alpha_T\Delta T/(dk_x^2)$.

SOLUTION.
With no dissipation and a horizontal magnetic field ($\mathbf{B}_0 = B_0\hat{\mathbf{x}}$), the dispersion relation Eq.(2) becomes

$$\omega^2 = -\frac{B_0^2}{\mu\rho_0}k_x^2 + \frac{g\alpha_T\Delta T}{d}\frac{k_x^2 + k_y^2}{k^2},$$

where $k^2 = k_x^2 + k_y^2 + \pi^2/d^2$.

(a) For rolls parallel to the field (with $k_x = 0$), this becomes

$$\omega^2 = \frac{g\alpha_T\Delta T}{d}\frac{k_y^2}{k_y^2 + \pi^2/d^2},$$

and so such rolls are unstable ($\omega > 0$) for all wavenumbers $k_y \neq 0$, as required. When $k_y = 0$, then $\omega = 0$.

(b) Also, for a given $k_x \neq 0$, the plasma is stable ($\omega^2 < 0$) provided

$$\frac{B_0^2}{\mu\rho_0} > \frac{g\alpha_T\Delta T}{dk_x^2}\frac{k_x^2 + k_y^2}{k_x^2 + k_y^2 + \pi^2/d^2}.$$

However, the maximum value of $(k_x^2 + k_y^2)/(k_x^2 + k_y^2 + \pi^2/d^2)$ as k_y varies is unity in the limit as $k_y \to \infty$. Thus, for a given non-zero k_x, convection is inhibited for all k_y if the magnetic field is so strong that

$$\frac{B_0^2}{\mu\rho_0} > \frac{g\alpha_T\Delta T}{dk_x^2},$$

as required.

PROBLEM 9.4. Concentration of a Flux Tube by a Stagnation-Point Flow.

(a) By seeking steady-state solutions of the induction equation with no electric field, show that a sheet (B $\hat{\mathbf{y}}$) of flux $2B_0a$ is concentrated by a 2D stagnation-point flow ($v_x = -V_0x/a, v_y = V_0y/a$) to the form

$$B(x) = 2B_0 \left(\frac{R_m}{\pi}\right)^{1/2} \exp\left(-\frac{R_m x^2}{a^2}\right),$$

where $R_m = aV_0/(2\eta)$. Deduce that its thickness is $2a/\sqrt{R_m}$ and its peak field strength is $2B_0\sqrt{(R_m/\pi)}$.

(b) Prove that the corresponding effect of a 3D flow ($v_R = -V_0R/a, v_z = 2V_0z/a$) in cylindrical polars on a field $B(R)$ $\hat{\mathbf{z}}$ is to make

$$B(R) = B_0R_m \exp(-R_mR^2/a^2).$$

SOLUTION.

(a) The integral of the induction equation is Ohm's law, namely,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \boldsymbol{\nabla} \times \mathbf{B},$$

which, for the above flow and magnetic field, becomes

$$E - \frac{V_0x}{a}B = \eta\frac{dB}{dx},$$

where E is constant. If $E = 0$, the equation separates and the solution is

$$\log_e B - \log_e c = -\frac{V_0x^2}{2\eta a},$$

where $\log_e c$ is a constant of integration. Taking exponentials gives

$$B(x) = c\exp(-R_mx^2/a^2),$$

where $R_m = aV_0/(2\eta)$.

Now, since the flux in this sheet is assumed to be $2B_0a$, we have

$$2B_0a = \int_{-\infty}^{\infty} B(x)dx = \int_{-\infty}^{\infty} c\ \exp(-R_mx^2/a^2)\ dx.$$

Change the variable from x to $X = x\sqrt{R_m}/a$. Then the integral becomes

$$2B_0a = \frac{ac}{\sqrt{R_m}}\int_{-\infty}^{\infty} \exp(-X^2)dX.$$

But $\int_{-\infty}^{\infty} \exp(-X^2)dX = \sqrt{\pi}$, and so

$$2B_0 a = ac\sqrt{\frac{\pi}{R_m}},$$

or in other words c is given by

$$c = 2B_0 \left(\frac{R_m}{\pi}\right)^{1/2},$$

as required. Thus we see that the sheet thickness is $2a/\sqrt{R_m}$ and the peak field strength is $2B_0\sqrt{(R_m/\pi)}$.

(b) The corresponding case in cylindrical geometry is very similar with Ohm's law becoming

$$E - \frac{V_0 R}{a}B = \eta\frac{dB}{dR}$$

and the solution when E=0 being

$$B(R) = c\exp(-R_m R^2/a^2).$$

Now, if the flux is $\pi B_0 a^2$, we have

$$\begin{aligned} \pi B_0 a^2 &= \int_0^{\infty} 2\pi B(R)RdR = \int_0^{\infty} 2\pi c\exp(-R_m R^2/a^2)RdR \\ &= \frac{\pi a^2}{R_m}[-\exp(-R_m R^2/a^2)]_0^{\infty} = \frac{\pi a^2 c}{R_m}, \end{aligned}$$

which implies that $c = R_m B_0$. Thus, the radius of the tube is a/R_m and the peak field is $R_m B_0$.

PROBLEM 9.5. The Shape of a Buoyant Flux Tube.

Show that the equation of a vertical slender flux tube in equilibrium in the yz-plane between magnetic buoyancy and magnetic tension in an isothermal medium of scale-height H is given by

$$\cos\frac{y}{2H} = e^{(z-z_s)/(2H)},$$

where $(y = 0, z = z_s)$ is the summit of the tube. Hence deduce that the maximum footpoint separation for such a tube is $2\pi H$.

SOLUTION.

If the footpoints of a flux tube are separated by less than a few scale-heights (H), a buoyant tube in the convection zone can form an arch if it is in equilibrium under a balance between the forces of magnetic buoyancy and magnetic tension. The shape $[z = z(y)]$ of such an arch in a vertical yz-plane may be estimated as follows.

With z directed vertically upwards, the gradient of the arch is given in terms of its inclination $[\theta(z)]$ to the horizontal by

$$\frac{dz}{dy} = \tan\theta. \qquad (*)$$

Furthermore, the tension per unit area along the tube is B^2/μ, and so the assumption that the tube is in horizontal equilibrium implies that the horizontal component of the total tension summed across the cross-sectional area A is constant, namely,

$$\frac{AB^2}{\mu}\cos\theta = \frac{A_s B_s^2}{\mu},$$

where B^2/μ and B_s^2/μ are the tensions per unit area in the tube at height z and at the summit $(y = 0, z = z_s)$, while A and A_s are the corresponding cross-sectional areas. But the flux AB is constant along the tube, and so the above equation becomes

$$B\cos\theta = B_s.$$

Now, the field strength $[B(z)]$ for a tube in a hydrostatic isothermal medium is given (Sec.9.2.1) by

$$B(z)^2 = B_s^2 e^{(z_s - z)/H}.$$

Thus, eliminating B between the above two equations gives

$$\cos\theta = e^{(z - z_s)/(2H)}. \qquad (**)$$

Then, d/dz of Equation $(**)$ implies

$$\frac{d\theta}{dz} = -\frac{\cot\theta}{2H},$$

which, when combined with Equation $(*)$, gives

$$\frac{d\theta}{dy} = -\frac{1}{2H}.$$

This may be integrated to give

$$\theta = -\frac{y}{2H},$$

and so Equation (**) implies that the equation of the arch is simply

$$\cos\frac{y}{2H} = e^{(z-z_s)/(2H)},$$

As $z \to -\infty$, so $y \to \pm\pi H$, and the maximum footpoint separation is therefore $2\pi H$.

PROBLEM 9.6. Magnetic Buoyancy Instability.
Consider a 1D equilibrium $[p_0(z), \rho_0(z), T_0(z), B_0(z)\hat{\mathbf{x}}]$ satisfying the perfect gas law and magnetostatic balance with uniform sound and Alfvén speeds. Find the dispersion relation for perturbations of the form $f(z)e^{i(kx+ny-\omega t)}$ when $n^{-1} \ll k^{-1}, H$. Deduce that, when $0 < k^2 H < -(1/B_0)dB_0/dz$, the configuration is unstable.

SOLUTION.
In the basic stability analysis (Schatzman, 1963; Gilman, 1970) the equilibrium variables $[p_0(z), \rho_0(z), T_0(z), B_0(z)]$ satisfy the perfect gas law

$$p_0 = (k_B/m)\rho_0 T_0$$

and magnetostatic balance

$$d/dz[p_0 + B_0^2/(2\mu)] + \rho_0 g = 0.$$

If the (isothermal) sound and Alfvén speeds, $c_{s0} = (p_0/\rho_0)^{1/2}, v_{A0} = B_0/(\mu\rho_0^{1/2})$, are constant, independent of height, the resulting dispersion relation has constant coefficients, and the equilibrium profiles are

$$p_0(z) = p^* e^{-z/H_B}, \quad \rho_0(z) = \rho^* e^{-z/H_B}, \quad B_0(z) = B^* e^{-z/(2H_B)},$$

where p^*, ρ^*, B^* are the values at $z = 0$ and $H_B = [p^* + B^{*2}/(2\mu)]/(\rho^* g)$ is the scale-height $[H = p^*/(\rho^* g)]$ increased by the presence of the magnetic field.

Perturbations of the form $f(z)e^{i(kx+ny-\omega t)}$ in the linearised MHD equations yield

$$(c_{s0}^2+v_{A0}^2)\omega^4 - v_{A0}^2[(2c_{s0}^2+v_{A0}^2)k^2 + c_{s0}^2/(2HH_B)]\omega^2 + k^2 v_{A0}^4 c_{s0}^2(k^2 - (2HH_B)^{-1}) = 0$$

as the dispersion relation when $n^{-1} \ll k^{-1}, H$. The perturbed field in a horizontal plane possess a long-wavelength structure along its length and a short-wavelength structure in the y-direction, reminiscent of neighbouring flux loops rising and sinking from a unidirectional field. As a loop rises, its field strength tends to increase due to horizontal stretching but also to decrease due to vertical expansion. When there are no variations along the field ($k = 0$), the above dispersion relation gives stability.

When $dB_0/dz < 0$ and the wavelength ($2\pi/k$) along the field is so large that

$$0 < k^2 H < \frac{1}{2H_B} \equiv -\frac{1}{B_0}\frac{dB_0}{dz},$$

the last term in the dispersion relation is negative, and so the configuration is unstable ($\omega^2 < 0$). This is an updated version of Parker's condition, with magnetic tension being overcome by magnetic buoyancy for small enough wavenumbers k.

The influence of normal buoyancy is to modify the above instability criterion to

$$\frac{1}{HB_0}\frac{dB_0}{dz} < -k^2 - \frac{\gamma N^2}{v_A^2},$$

while magnetic and thermal diffusion reduce the stabilising effect of stratification and modify it to

$$\frac{1}{HB_0}\frac{dB_0}{dz} < -k^2 - \frac{\eta}{\kappa}\frac{\gamma N^2}{v_A^2}$$

PROBLEM 9.7. Self-Similar Model for a Sunspot.

The Schlüter-Temesvary model for a sunspot has self-similar magnetic field

$$B_R = -\tfrac{1}{2} R f(\zeta) dB_i/dz, \ \ B_z = f(\zeta) B_i(z),$$

in terms of the similarity variable $\zeta = RB_i^{1/2}(z)$ where $f(0) = 1$. It satisfies $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and the equations for magnetohydrostatic equilibrium. Calculate the flux (F_m) through the spot and show how $B_i(z)$ and $p_e(z)$ may be determined if the temperature structure, flux (F_m) and shape factor $f(\zeta)$ are prescribed, together with B_i and dB_i/dz at $z = 0$.

SOLUTION.

A simple self-similar model for magnetostatic equilibrium of a sunspot is based on the force balance

$$0 = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} - \rho g \hat{\mathbf{z}}, \tag{3}$$

with gravity acting along the negative z-axis. For a cylindrically symmetric ($\partial/\partial\phi = 0$), untwisted ($B_\phi = 0$) magnetic field $[B_R(R,z), 0, B_z(R,z)]$, the force balance has components

$$0 = -\frac{\partial p}{\partial R} + \frac{B_z}{\mu}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right),$$

$$0 = -\frac{\partial p}{\partial z} - \frac{B_R}{\mu}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) - \rho g.$$

A similarity solution automatically satisfying $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ has the form

$$B_R = -\frac{R}{2} f(\zeta)\frac{dB_i}{dz}, \qquad B_z = f(\zeta) B_i(z),$$

where $\zeta = R B_i^{1/2}(z)$ and $f(0) = 1$, so that $B_i(z)$ represents the field on the axis of symmetry ($R = 0$).

The flux (F_m) through the spot is

$$F_m = 2\pi \int_0^\infty \zeta f(\zeta) d\zeta.$$

The above radial component of force balance may then be integrated with respect to R from 0 to ∞ at constant z, to give

$$0 = 2[p_e(z) - p_i(z)] + 4a_2 B_i^{1/2}\frac{d^2 B_i^{1/2}}{dz^2} - B_i^2, \tag{$*$}$$

where the subscripts e and i denote values at $R = \infty$ and $R = 0$, respectively, and $a_2 = \int_0^\infty \frac{1}{2}\zeta f^2(\zeta) d\zeta$.

On the other hand, the vertical component of the force balance gives $dp_i/dz = -\rho_i g$, at $R = 0$ (where B_R vanishes), and

$$\frac{dp_e}{dz} = -\rho_e g, \tag{$**$}$$

provided $f \to 0$ fast enough at infinity. Thus, if the temperature structure, flux (F_m) and shape factor $f(\zeta)(= e^{-\zeta^2}$, for instance) are prescribed, plus B_i and dB_i/dz at $z = 0$, Eqs.($*$) and($**$) determine $B_i(z)$ and $p_e(z)$.