FINAL example 2

SI-MKS

Speed of light in free space	$c = 2.99792458 \times 10^{8}\ \text{m s}^{-1}$
Planck's constant	$\hbar = 6.58211889 \times 10^{-16}\ \text{eV s}$
	$\hbar = 1.054571596 \times 10^{-34}\ \text{J s}$
Electron charge	$e = 1.602176462 \times 10^{-19}\ \text{C}$
Electron mass	$m_0 = 9.10938188 \times 10^{-31}\ \text{kg}$
Neutron mass	$m_\text{n} = 1.67492716 \times 10^{-27}\ \text{kg}$
Proton mass	$m_\text{p} = 1.67262158 \times 10^{-27}\ \text{kg}$
Boltzmann constant	$k_\text{B} = 1.3806503 \times 10^{-23}\ \text{J K}^{-1}$
	$k_\text{B} = 8.617342 \times 10^{-5}\ \text{eV K}^{-1}$
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}\ \text{F m}^{-1}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \text{H m}^{-1}$
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0\mu_0}$
Avagadro's number	$N_\text{A} = 6.02214199 \times 10^{23}\ \text{mol}^{-1}$
Bohr radius	$a_\text{B} = 0.52917721 \times 10^{-10}\text{m}$

$$a_\text{B} = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$$

Inverse fine-structure constant $\alpha^{-1} = 137.0359976$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

PROBLEM 1

The first four lowest energy states of a one-dimensional harmonic oscillator with characteristic frequency ω_0 are subject to the perturbation

$$\mathbf{W} = \begin{bmatrix} W_{00} & W_{01} & W_{02} & W_{03} \\ W_{10} & W_{11} & W_{12} & W_{13} \\ W_{20} & W_{21} & W_{22} & W_{23} \\ W_{30} & W_{31} & W_{32} & W_{33} \end{bmatrix} = \Delta\hbar\omega_0 \begin{bmatrix} 1 & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\Delta \ll 1$.

(a) Find the new eigenenergies to first-order in time-independent perturbation theory. (50%)

(b) Find the new eigenenergies to second-order in time-independent perturbation theory. (50%)

PROBLEM 2

In first-order time-dependent perturbation theory a particle initially in eigenstate $|n\rangle$ of the unperturbed Hamiltonian scatters into state $|m\rangle$ with probability $|a_m(t)|^2$ after the perturbation $\hat{W}(x, t)$ is applied at time $t = 0$.

(a) Derive the expression for the time-dependent coefficient

$$a_m(t) = \frac{1}{i\hbar}\int_{t'=0}^{t'=t} W_{mn} e^{i\omega_{mn}t'} dt'$$

where the matrix element $W_{mn} = \langle m|\hat{W}(x, t)|n\rangle$ and $\hbar\omega_{mn} = E_m - E_n$ is the difference in eigenenergies of the states $|m\rangle$ and $|n\rangle$. (40%)

(b) A particle in a continuum system described by Hamiltonian $\hat{H}_0$ is prepared in eigenstate $|n\rangle$ with eigenenergy $E_n = \hbar\omega_n$. Consider the effect of a perturbation turned on at time $t = 0$ that is harmonic in time such that $\hat{W}(x, t) = V(x)\cos(\omega t)$, where $V(x)$ is the spatial part of the potential and ω is the frequency of oscillation. Show that the scattering rate in the static limit ($\omega \to 0$) is given by Fermi's golden rule $\frac{1}{\tau_n} = \frac{2\pi}{\hbar}|W_{mn}|^2 D(E)\delta(E_m - E_n)$, where the matrix element $W_{mn} = \langle m|\hat{W}(x, t)|n\rangle$ couples state $|n\rangle$ to state $|m\rangle$ via the static potential $V(x)$, the density of final continuum states is $D(E)$, and $\delta(E_m - E_n)$ ensures energy conservation. (50%)

(c) Justify the use of time-dependent perturbation theory to describe an electron scattering from a static potential that has no explicit time dependence. (10%)

PROBLEM 3

A potential $V(x, y)$ is infinite except in a region $0 < x < L$, and $0 < y < L$ where $V(x, y) = 0$.

(a) Write down the time-independent Schrödinger equation for an electron confined to motion in the potential and solve for the eigenfunctions and eigenenergies. (20%)

(b) What is the degeneracy of the ground state and what is the degeneracy of the first excited state? (10%)

(c) The system is perturbed by introducing a constant potential $\hat{W} = V_0 = 0.1 \text{ eV}$ in a region for which $0 < x < \frac{L}{2}$, $0 < y < \frac{L}{2}$, and $L = 3 \text{ nm}$. The perturbation $\hat{W} = 0$ elsewhere. Use first-order perturbation theory to find the numerical value of the new ground state energy. (30%)

(d) What are the numerical values of the new eigenenergies of the first excited state? What are the new eigenfunctions of the first excited state? (40%)

You may wish to use the relation $2\sin(\theta)\sin(\phi) = \cos(\theta - \phi) - \cos(\theta + \phi)$

PROBLEM 4

(a) The time-dependence of the expectation value of an operator $\hat{\mathbf{A}}$ is found from $\frac{d}{dt}\langle\hat{\mathbf{A}}\rangle$. What general conclusions may be drawn concerning time-dependence of expectation value an operator $\hat{\mathbf{A}}$ that commutes with a Hamiltonian used to describe a physical system? (40%)

(b) Suppose a Hamiltonian with eigenfunctions ϕ_1 and ϕ_2 and corresponding eigenvalues E_1 and E_2 does *not* commute with an operator $\hat{\mathbf{A}}$. The operator $\hat{\mathbf{A}}$ has eigenfunctions $u_1 = (\phi_1 + \phi_2)/\sqrt{2}$ and $u_2 = (\phi_1 - \phi_2)/\sqrt{2}$ and corresponding eigenvalues a_1 and a_2. At time $t = 0$ the system is in state $\psi = u_1$. Show that at time *t* the state of the system is $\psi(t) = (\phi_1 e^{-iE_1 t/\hbar} + \phi_2 e^{-iE_2 t/\hbar})/\sqrt{2}$ and find how the expectation value of the operator $\hat{\mathbf{A}}$ varies with time. (60%)