FINAL example 4

SI-MKS

Speed of light in free space	$c = 2.99792458 \times 10^{8}\ \text{m s}^{-1}$
Planck's constant	$\hbar = 6.58211889 \times 10^{-16}\ \text{eV s}$
	$\hbar = 1.054571596 \times 10^{-34}\ \text{J s}$
Electron charge	$e = 1.602176462 \times 10^{-19}\ \text{C}$
Electron mass	$m_0 = 9.10938188 \times 10^{-31}\ \text{kg}$
Neutron mass	$m_n = 1.67492716 \times 10^{-27}\ \text{kg}$
Proton mass	$m_p = 1.67262158 \times 10^{-27}\ \text{kg}$
Boltzmann	$k_B = 1.3806503 \times 10^{-23}\ \text{J K}^{-1}$
constant	$k_B = 8.617342 \times 10^{-5}\ \text{eV K}^{-1}$
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}\ \text{F m}^{-1}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \text{H m}^{-1}$
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0 \mu_0}$
Avagadro's number	$N_A = 6.02214199 \times 10^{23}\ \text{mol}^{-1}$
Bohr radius	$a_B = 0.52917721 \times 10^{-10}\text{m}$

$$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$$

Inverse fine-structure constant $\alpha^{-1} = 137.0359976$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

PROBLEM 1

(a) In a uniform dielectric the dielectric function is a constant over space but depends on wave vector so that $\varepsilon = \varepsilon(\mathbf{q})$. Given an impurity potential at position $\mathbf{r}$ due to a charge e at position $\mathbf{R}_i$ is $v_q(\mathbf{r}-\mathbf{R}_i) = \frac{-e^2}{4\pi\varepsilon_q|\mathbf{r}-\mathbf{R}_i|}$, derive an expression for $v(\mathbf{q})$. (30%)

(b) Use the expression for $v(\mathbf{q})$ and Fermi's Golden rule to evaluate the total elastic scattering rate for an electron of initial energy $E(\mathbf{k})$ due to a single impurity in a dielectric with dielectric function $\varepsilon = \varepsilon(\mathbf{q})$. Describe any assumptions you have made. (30%)

(c) Extend your calculation to include elastic scattering of an electron energy E moving in the x-direction in the conduction band of a semiconductor with effective electron mass $m^* = 0.07 \times m_0$. The electron is incident on two identical ionized impurities, one at position $x = 0$ nm and the other at $x = 20$ nm. The semiconductor has low frequency relative permittivity $\varepsilon_{r0} = 13.2$. Explain you results. (40%)

PROBLEM 2

In first-order time-dependent perturbation theory a particle initially in eigenstate $|n\rangle$ of the unperturbed Hamiltonian scatters into state $|m\rangle$ with probability $|a_m(t)|^2$ after the perturbation $\hat{W}(x, t)$ is applied at time $t = 0$.

(a) Derive the expression for the time-dependent coefficient

$$a_m(t) = \frac{1}{i\hbar}\int_{t'=0}^{t'=t} W_{mn}e^{i\omega_{mn}t'}dt'$$

where the matrix element $W_{mn} = \langle m|\hat{W}(x, t)|n\rangle$ and $\hbar\omega_{mn} = E_m - E_n$ is the difference in eigenenergies of the states $|m\rangle$ and $|n\rangle$. (30%)

(b) An electron is in the ground state of a one-dimensional rectangular potential well for which $V(x) = 0$ in the range $0 < x < L$ and $V(x) = \infty$ elsewhere. It is decided to control the state of the electron by applying a pulse of electric field $\mathbf{E}(t) = \mathbf{E}_0 e^{-t^2/\tau^2}$ in the x-direction starting at time $t = 0$, where τ is a constant and $|\mathbf{E}_0|$ is the maximum strength of the applied electric-field. Calculate the probability P_{12} that the particle will be found in the first excited state in the long time limit, $t \to \infty$. (30%)

(c) If the electron is in a semiconductor and has an effective mass $m^* = 0.07 \times m_0$, where m_0 is the bare electron mass, and the potential well is of width $L = 10$ nm, calculate the minimum value of $|\mathbf{E}_0|$ for which $P_{12} = 1$. Comment on your result. (40%)

You may wish to make use of the standard integral $\int_{t'=0}^{t'=\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

PROBLEM 3

A two-level system described by Hamiltonian $\hat{H}_0$ has eigenstates $|1\rangle$ and $|2\rangle$ with energy separation $\hbar\omega_{21} = E_2 - E_1$. The system is initially in its ground state $|1\rangle$ and at time $t \geq 0$ it is illuminated by a small electric field $\mathbf{E} = \mathbf{E}_0(e^{i\omega t} + e^{-i\omega t})$ in the x direction. The electric field oscillates at frequency ω and has magnitude $|\mathbf{E}_0|$.

(a) Write down the Hamiltonian for time $t \geq 0$ in terms of $\hat{H}_0$ and a perturbation $\hat{W}$. (10%)

(b) The solution at time $t \geq 0$ is of the form $|x, t\rangle = a_1(t)e^{-i\omega_1 t}|1\rangle + a_2(t)e^{-i\omega_2 t}|2\rangle$ where $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$. Substitute this into the time-dependent Schrödinger equation and show that

$$i\hbar\left(\frac{d}{dt}a_1(t)\right)e^{-i\omega_1 t}|1\rangle + i\hbar\left(\frac{d}{dt}a_2(t)\right)e^{-i\omega_2 t}|2\rangle = a_1(t)e^{-i\omega_1 t}\hat{W}|1\rangle + a_2(t)e^{-i\omega_2 t}\hat{W}|2\rangle$$

then multiply both sides by $\langle 1|$ or $\langle 2|$ and obtain two equations

$$i\hbar\frac{d}{dt}a_1(t) = a_1(t)\langle 1|\hat{W}|1\rangle + a_2(t)e^{-i(\omega_2 - \omega_1)t}\langle 1|\hat{W}|2\rangle$$

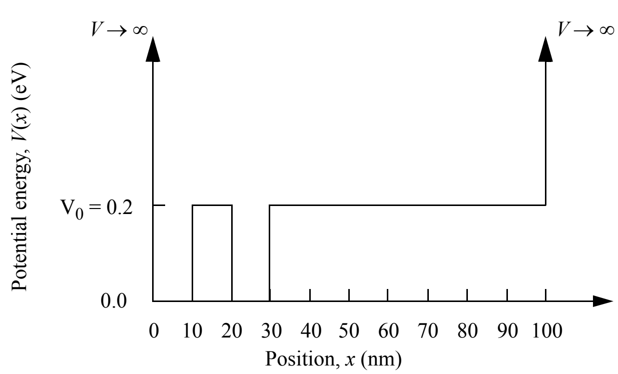
$$i\hbar\frac{d}{dt}a_2(t) = a_1(t)e^{i(\omega_2 - \omega_1)t}\langle 2|\hat{W}|1\rangle + a_2(t)\langle 2|\hat{W}|2\rangle$$

where $\langle j|\hat{W}|k\rangle = -e|\mathbf{E}_0|\langle j|\hat{x}|k\rangle(e^{i\omega t} + e^{-i\omega t}) = W_{jk}(e^{i\omega t} + e^{-i\omega t})$. (40%)

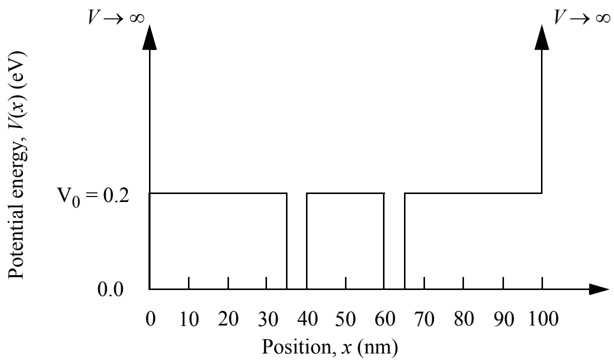
(c) If $\omega = \omega_{21}$, find the probability that the system will be in the excited state $|2\rangle$ at time $t > 0$. At what values of $t > 0$ will the system be in a pure ground state? (50%)

PROBLEM 4

(a) A particle mass m_1 moves in the one-dimensional double barrier potential of energy $V_0 = 0.2$ eV sketched in the following figure and is bounded by barriers of infinite energy for $x \leq 0$ nm and $x \geq 100$ nm. The ground state, first, and second excited state eigenenergies of the particle are $E_1 = 0.063$ eV, $E_2 = 0.098$ eV, and $E_3 = V_0 = 0.200$ eV respectively. Sketch and explain the shapes of the corresponding eigenfunctions. (40%)



(b) A particle mass m_2 moves in the symmetric one-dimensional double barrier potential of energy $V_0 = 0.2$ eV sketched in the following figure and is bounded by barriers of infinite energy for $x \leq 0$ nm and $x \geq 100$ nm. The ground state, first, and second excited state eigenenergies of the particle are $E_1 = 0.069$ eV, $E_2 = 0.070$ eV, and $E_3 = V_0 = 0.200$ eV respectively. Sketch and explain the shapes of the corresponding eigenfunctions. (40%)



(c) Explain the differences between your results in part (a) and (b). (20%)