FINAL example 1

SI-MKS

Speed of light in free space	$c = 2.99792458 \times 10^{8}\ \text{m s}^{-1}$
Planck's constant	$\hbar = 6.58211889 \times 10^{-16}\ \text{eV s}$
	$\hbar = 1.054571596 \times 10^{-34}\ \text{J s}$
Electron charge	$e = 1.602176462 \times 10^{-19}\ \text{C}$
Electron mass	$m_0 = 9.10938188 \times 10^{-31}\ \text{kg}$
Neutron mass	$m_n = 1.67492716 \times 10^{-27}\ \text{kg}$
Proton mass	$m_p = 1.67262158 \times 10^{-27}\ \text{kg}$
Boltzmann constant	$k_B = 1.3806503 \times 10^{-23}\ \text{J K}^{-1}$
	$k_B = 8.617342 \times 10^{-5}\ \text{eV K}^{-1}$
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}\ \text{F m}^{-1}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \text{H m}^{-1}$
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0\mu_0}$
Avagadro's number	$N_A = 6.02214199 \times 10^{23}\ \text{mol}^{-1}$
Bohr radius	$a_B = 0.52917721 \times 10^{-10}\text{m}$

$$a_B = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$$

Inverse fine-structure constant $\alpha^{-1} = 137.0359976$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

PROBLEM 1

The first four lowest energy states of a one-dimensional harmonic oscillator with characteristic frequency ω_0 are subject to the perturbation

$$\mathbf{W} = \begin{bmatrix} W_{00} & W_{01} & W_{02} & W_{03} \\ W_{10} & W_{11} & W_{12} & W_{13} \\ W_{20} & W_{21} & W_{22} & W_{23} \\ W_{30} & W_{31} & W_{32} & W_{33} \end{bmatrix} = \Delta\hbar\omega_0 \begin{bmatrix} 1 & \frac{-1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \end{bmatrix}$$

where $\Delta \ll 1$.

(a) Find the new eigenenergies to first-order in time-independent perturbation theory. (50%)

(b) Find the new eigenenergies to second-order in time-independent perturbation theory. (50%)

PROBLEM 2

In first-order time-dependent perturbation theory a particle initially in eigenstate $|n\rangle$ of the unperturbed Hamiltonian scatters into state $|m\rangle$ with probability $|a_m(t)|^2$ after the perturbation $\hat{W}(x, t)$ is applied at time $t = 0$.

(a) Derive the expression for the time dependent coefficient

$$a_m(t) = \frac{1}{i\hbar} \int_{t'=0}^{t'=t} W_{mn} e^{i\omega_{mn}t'} dt'$$

where the matrix element $W_{mn} = \langle m|\hat{W}(x, t)|n\rangle$ and $\hbar\omega_{mn} = E_m - E_n$ is the difference in eigenenergies of the states $|m\rangle$ and $|n\rangle$. (40%)

(b) An electron is initially in the ground state of a one-dimensional harmonic oscillator with Hamiltonian $\hat{H} = \hbar\omega(\hat{b}^\dagger\hat{b} + 1/2)$ where ω is the oscillator's characteristic frequency and the operator $\hat{b} = \left(\frac{m_0\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i\hat{p}_x}{m_0\omega}\right)$. At time $t = 0$ a perturbation $\hat{W}(x, t) = V_0 x^3 e^{-t/\tau}$ is applied where V_0 and τ are constants. What are the allowed transitions? Calculate the probability of transition to each excited state of the system in the long time limit, $t \to \infty$. (50%)

(c) What value of τ maximizes the transition probability? Explain your result. (10%)

PROBLEM 3

A one-dimensional potential $V(x)$ is infinite except in a region $0 < x < L$ where $V(x) = 0$.

(a) Write down the time-independent Schrödinger equation for an electron confined to motion in the potential and solve for the eigenfunctions, $|n\rangle$, and eigenenergies, E_n. (20%)

(b) The electron is initially in state $|n\rangle$. Suddenly, in a time $t \ll 2\pi\hbar/E_n$, the region where $V(x) = 0$ is increased in length to $2L$ so that $V(0 < x < 2L) = 0$. Afterwards, what is the probability that the particle will be found in an energy eigenstate with energy E_n? (40%)

(c) If the electron in (a) is initially in the ground-state and the potential is suddenly removed so that $V(-\infty < x < \infty) = 0$, use the Fourier transform of the initial real-space wave function to find the probability distribution for *momentum* of the freed particle. (40%)

You may wish to use the standard indefinite integral

$$\int e^{ax}\sin(bx)dx = e^{ax}\frac{(a\ \sin(bx) - b\ \cos(bx))}{a^2 + b^2}$$

when solving this problem.

PROBLEM 4

(a) Use first-order perturbation theory to find the correction to the $1s$ ground-state energy of a hydrogen atom subject to a uniform electric field **E** in the z direction. (10%)

(b) If spin effects are neglected, the four states of the hydrogen atom with quantum number $n = 2$ have the same energy, $E_2^{(0)}$. A uniform z-directed electric field $\mathbf{E}$ is applied to hydrogen atoms in these states. To first-order, find the new eigenenergies and eigenfunctions for these states.

Treat the z-directed electric field as a perturbation on the separable, orthonormal, unperturbed electron wave functions $\psi_{nlm}(r, \theta, \phi) = R_n(r)\Theta_l^m(\theta)\Phi_m(\phi)$, where r, θ, and ϕ are the standard spherical coordinates. You may use the unperturbed wave functions

$$\psi_{200} = \frac{2}{(2a_{\mathrm{B}})^{3/2}}\left(1 - \frac{r}{2a_{\mathrm{B}}}\right) e^{-r/2a_{\mathrm{B}}}\left(\frac{1}{4\pi}\right)^{1/2}$$

$$\psi_{210} = \frac{1}{\sqrt{3}\,(2a_{\mathrm{B}})^{3/2}}\,\frac{r}{a_{\mathrm{B}}}\,e^{-r/2a_{\mathrm{B}}}\,\frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2}\cos(\theta)$$

and you may wish to use the relation $\int_0^\infty r^n e^{-r/a_{\mathrm{B}}}dr = n!a_{\mathrm{B}}^{n+1}$. (70%)

(c) If the magnitude of the electric field in (b) is $10^7\ \mathrm{V\ cm^{-1}}$, what is the maximum percentage change in eigenenergy? (20%)