FINAL example 3

SI-MKS

Speed of light in free space	$c = 2.99792458 \times 10^{8}\ \mathrm{m\ s^{-1}}$
Planck's constant	$\hbar = 6.58211889 \times 10^{-16}\ \mathrm{eV\ s}$
	$\hbar = 1.054571596 \times 10^{-34}\ \mathrm{J\ s}$
Electron charge	$e = 1.602176462 \times 10^{-19}\ \mathrm{C}$
Electron mass	$m_0 = 9.10938188 \times 10^{-31}\ \mathrm{kg}$
Neutron mass	$m_\mathrm{n} = 1.67492716 \times 10^{-27}\ \mathrm{kg}$
Proton mass	$m_\mathrm{p} = 1.67262158 \times 10^{-27}\ \mathrm{kg}$
Boltzmann constant	$k_\mathrm{B} = 1.3806503 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
	$k_\mathrm{B} = 8.617342 \times 10^{-5}\ \mathrm{eV\ K^{-1}}$
Permittivity of free space	$\varepsilon_0 = 8.8541878 \times 10^{-12}\ \mathrm{F\ m^{-1}}$
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{H\ m^{-1}}$
Speed of light in free space	$c = 1/\sqrt{\varepsilon_0\mu_0}$
Avagadro's number	$N_\mathrm{A} = 6.02214199 \times 10^{23}\ \mathrm{mol^{-1}}$
Bohr radius	$a_\mathrm{B} = 0.52917721 \times 10^{-10}\mathrm{m}$

$$a_\mathrm{B} = \frac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$$

Inverse fine-structure constant	$\alpha^{-1} = 137.0359976$

$$\alpha^{-1} = \frac{4\pi\varepsilon_0\hbar c}{e^2}$$

PROBLEM 1

In first-order time-dependent perturbation theory a particle initially in eigenstate $|n\rangle$ of the unperturbed Hamiltonian scatters into state $|m\rangle$ with probability $|a_m(t)|^2$ after the perturbation $\hat{W}(x, t)$ is applied at time $t = 0$.

(a) Derive the expression for the time-dependent coefficient

$$a_m(t) = \frac{1}{i\hbar}\int_{t'=0}^{t'=t} W_{mn} e^{i\omega_{mn}t'} dt'$$

where the matrix element $W_{mn} = \langle m|\hat{W}(x, t)|n\rangle$ and $\hbar\omega_{mn} = E_m - E_n$ is the difference in eigenenergies of the states $|m\rangle$ and $|n\rangle$. (30%)

(b) An electron is in the ground state of a one-dimensional rectangular potential well for which $V(x) = 0$ in the range $0 < x < L$ and $V(x) = \infty$ elsewhere. It is decided to control the state of the electron by applying a pulse of electric field $\mathbf{E}(t) = \mathbf{E}_0 e^{-t/\tau}$ in the *x*-direction starting at time $t = 0$, where τ is a constant and $|\mathbf{E}_0|$ is the maximum strength of the applied electric-field. Calculate the probability P_{12} that the particle will be found in the first excited state at time limit, $t \geq 0$. (30%)

(c) If the electron is in a semiconductor and has an effective mass $m^* = 0.07 \times m_0$, where m_0 is the bare electron mass, the potential well is of width $L = 20$ nm, and $\tau = 16.4$ fs, calculate the value of $|\mathbf{E}_0|$ for which $P_{12} = 0.5$ in the long time limit $t \to \infty$. Comment on your result. (40%)

PROBLEM 2

A particle of mass m_0 moves in a three-dimensional harmonic potential

$$V = \frac{\kappa}{2}(\hat{x}^2 + \hat{y}^2 + \hat{z}^2)$$

with natural frequency ω_0, where $\kappa = m_0\omega_0^2$ is a constant, and $\hat{x} = \left(\frac{\hbar}{2m_0\omega_0}\right)^{1/2}(\hat{b}_x + \hat{b}_x^\dagger)$, etc.

The potential is subject to perturbation $\hat{W} = \frac{\lambda\kappa}{2}\hat{x}\hat{y}$ with $\lambda \ll 1$.

(a) What is the degeneracy and eigenenergy of the ground-state and first excited-state of the unperturbed system? (10%)

(b) Write down the potential and the Hamiltonian for the complete system, including the perturbation. (10%)

(c) Find the ground-state energy up to second-order using time-independent perturbation theory. (40%)

(d) Find the first excited-state energy levels using time-independent perturbation theory. (40%)

PROBLEM 3

The ground-state wave function of a hydrogenic atom with nuclear charge Ze is $\psi_1(r) = Ae^{-r/r_1} = Ae^{-\beta r}$, where r is the distance between the electron of mass m_0 and the nucleus and $r_1 = 1/\beta$ is a characteristic length scale. The electron is subject to a radially-symmetric coulomb potential given by $V(r) = -Ze^2/4\pi\varepsilon_0 r$.

(a) Find the value of the normalization constant A. (10%)

(b) Show how to find the value of r_1 that minimizes the ground-state energy expectation value $\langle E_1\rangle$ and show that if the reduced mass $m_r = m_0$, then $\beta = \dfrac{Z}{a_B} = \dfrac{1}{r_1}$ where the Bohr radius is $a_B = \dfrac{4\pi\varepsilon_0\hbar^2}{m_0 e^2}$. (20%)

(c) Use the value of r_1 in (b) to find an expression for the ground state energy, E_1. (20%)

(d) Find an expression for the expectation value of electron radial position, $\langle r\rangle$. (20%)

(e) Sketch the radial probability distribution of the electron and find an expression for the value of r at which it peaks. (30%)

You may wish to use the standard integral $\int_{x=0}^{x=\infty} x^n e^{-\mu x} dx = n!\mu^{-n-1}$ for $\mathrm{Re}(\mu > 0)$ when solving this problem.

PROBLEM 4

(a) For a system described by wave function $\psi(x, t)$ which is *not* an eigenstate the spread in values of observable A associated with time independent operator $\hat{A}$ in the time interval Δt is $\Delta A = \left|\dfrac{d}{dt}\langle A\rangle\right|\Delta t$. Use the fact that $\dfrac{d}{dt}\langle A\rangle = \dfrac{i}{\hbar}\langle[H,\hat{A}]\rangle$ and the generalized uncertainty relation $\Delta A\Delta B \geq \left|\dfrac{1}{2}\ \langle[\hat{A}, \hat{B}]\rangle\right|$ for time independent operators $\hat{A}$ and $\hat{B}$ to show that $\Delta E\Delta t \geq \dfrac{\hbar}{2}$. (40%)

(b) Show that the spread in photon number Δn and phase $\Delta\phi$ for light of frequency ω is

$$\Delta n\Delta\phi \geq \frac{1}{2}$$

and that for a normal distribution of such photons

$$\Delta\phi \geq \frac{1}{2\sqrt{\langle n\rangle}}. \text{ (30\%)}$$

(c) What is the minimum average optical power in a 100 ps pulse of normally distributed λ = 1500 nm wavelength light that is required to measure optical phase to an accuracy of 3°? What is the average number of photons in the pulse? (30%)