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Problems for Chapter 12 of *Advanced Mathematics for Applications*

The Bessel Equation

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1. By using the general power series expansion of $J_\nu(z)$ prove that

$$J_{1/2} = \left(\frac{2}{\pi z}\right)^{1/2} \sin z \,, \qquad J_{-1/2} = \left(\frac{2}{\pi z}\right)^{1/2} \cos z \,.$$

   From these results obtain the explicit expression of $J_{3/2}(z)$. Recall that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(1+z) = z\Gamma(z)$.

2. Prove that

$$J_\nu(z)J_{1-\nu}(z) + J_{-\nu}(z)J_{\nu-1}(z) = 2\frac{\sin \pi\nu}{\pi z} \,.$$

3. Prove that

$$K_n(z)I_n'(z) - K_n'(z)I_n(z) = \frac{1}{z} \,..$$

4. Show that

$$\int_{-\infty}^{\infty} J_{m+1/2}(x)J_{n+1/2}(x)\frac{\mathrm{d}x}{x} = \frac{2}{2n+1}\delta_{mn} \,.$$

5. By relying on the power series expansion of J_n prove that

$$\int_0^\infty e^{-b^2x^2} J_n(ax)\, x^{n+1}\, \mathrm{d}x = \frac{a^n}{(2b^2)^{n+1}} \exp\left(-\frac{a^2}{4b^2}\right)$$

   in which a and b are real constants and n a non-negative integer.

6. Obtain the first-order differential equation satisfied by the function

$$\mathcal{J}_n = \frac{J_{n+1}(z)}{zJ_n(z)} \,.$$

7. Show that, if $R^2 = r^2 + s^2 - 2rs\cos\theta$, then

$$J_0(R) = \sum_{m=-\infty}^{\infty} J_m(r)J_m(s)e^{im\theta} = J_0(r)J_0(s) + 2\sum_{n=1}^{\infty} J_m(r)J_m(s)\, \cos n\theta \,.$$

   For this purpose find the partial differential equation satisfied by the series and show that $J_0(R)$ is a solution of the same equation satisfying the same boundary conditions. This is a particular instance of an addition theorem for the Bessel functions.

8. The Schlömilch generating function (p. 320) for the Bessel functions is given by

$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z)\,.$$

Use this result to show that

$$e^{iz\sin\theta} = J_0(z) + 2\sum_{n=1}^{\infty} J_{2n}(z)\cos 2n\theta + 21\sum_{n=0}^{\infty} J_{2n+1}(z)\sin(2n+1)\theta$$

and deduce the Fourier series for $\cos(z\sin\theta)$ and $\sin(z\sin\theta)$.

9. By a general transformation of the independent variable and a linear transformation of the dependent variable (i.e., a transformation of the type $v = f(z)u$, where z and u are the original variables) derive the general form of a linear second-order equation reducible to Bessel's equation. Using this result write down the general solution of the following equations (a, b, c, ν arbitrary real constants, m an integer):

$$\frac{(zu')'}{z} + \left[\left(bcz^{c-1}\right)^2 - \frac{(c\nu)^2}{z^2}\right]u = 0\,,$$

$$u'' + \frac{1-2a}{z}u' + \left[\left(bcz^{c-1}\right)^2 + \frac{a^2 - c^2\nu^2}{z^2}\right]u = 0\,,$$

$$u'' + bz^m u = 0\,,$$

$$u'' + z^{-4}\left(e^{2/z} - \nu^2\right)u = 0\,.$$

10. By using a suitable transformation of the variables solve the equation

$$\sqrt{x}\,u'' + \lambda u = 0\,,$$

subject to $u(0) = 0$.

11. Determine the smallest value of λ for which the equation

$$u'' + a^2((\lambda - x)^2 u = 0 \qquad 0 < x < \lambda\,,$$

where a is a known constant, has a solution satisfying the conditions $u(0) = 0$, $u'(\lambda) = 0$.

12. Show that, if P_n is a Legendre polynomial of integer order n (chapter 13), then

$$\int_{-1}^{1} e^{izt} P_n(t)\,\mathrm{d}t = \left(\frac{2\pi}{z}\right)^{1/2} i^n J_{n+1/2}(z)\,.$$

You can (a) prove the statement by induction starting with $n = 0$, or (b) prove that both sides of the equality satisfy the same differential equation and the same boundary conditions, or (c) that they both satisfy the same recurrence equation and conditions sufficient to make them equal rather than merely proportional.

13. By making use of the power series for $J_n(x)$ and term-by-term integration show that

$$\int_0^{\infty} e^{-b^2x^2} J_n(ax)x^{n+1}\,\mathrm{d}x = \frac{a^n}{(2b^2)^{n+1}}\exp\left(-\frac{a^2}{4b^2}\right)\,.$$