Chapter 4: Differentiation

Part B: Applications of Differentiation

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Absolute Maximum and Minimum

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Absolute maxima and minima are collectively known as **absolute extremes**.

Local Maximum and Minimum

We say that $f: D \to \mathbb{R}$ has a **local** or **relative maximum** at a point c if there is an open interval I containing c such that $f(c) \geq f(x)$ for every $x \in I \cap D$.

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Local maxima and minima are collectively known as **local extremes**.

An absolute maximum will also be a local maximum, and an absolute minimum will be a local minimum. But local extremes need not be absolute extremes, and a function could well have local extremes without having any absolute extreme.

Fermat's Theorem

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Theorem 1

Let $f(x)$ have a local extreme at an interior point c of an interval in its domain. Then either $f'(c)$ does not exist or $f'(c) = 0$.

Proof. Suppose $f'(c)$ exists. We have to show that $f'(c) = 0$. Suppose $f'(c) > 0$, that is,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} > 0.$$

Since the limit is positive, the values $\dfrac{f(x) - f(c)}{x - c}$ must themselves be positive once we are close to c. That is, there must be a $\delta > 0$ such that $0 < |x - c| < \delta \implies \dfrac{f(x) - f(c)}{x - c} > 0$. Then,

$$c - \delta < x < c \implies f(x) < f(c) \implies c \text{ is not a point of local minimum,}$$
$$c < x < c + \delta \implies f(x) > f(c) \implies c \text{ is not a point of local maximum.}$$

This rules out $f'(c) > 0$. We similarly rule out $f'(c) < 0$. $\square$

Two Examples

An example of a local extreme which occurs at a point where f' does not exist:

Example 2

Consider $f(x) = |x|$. It has a local minimum at $x = 0$ but $f'(0)$ is not defined.

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An example of a point where f' is zero but it is not a local extreme:

Example 3

Consider $f(x) = x^3$. Then $f'(0) = 0$ but there isn't a local extreme at $x = 0$.

Critical Points

We call c a **critical point** or **critical number** of $f(x)$ if it is an interior point c of an interval in the domain of f and either $f'(c)$ does not exist or $f'(c) = 0$.

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Let $f : [a, b] \to \mathbb{R}$. By Fermat's Theorem, the local extremes of f occur either at critical points or at the end-points of $[a, b]$.

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Example 4

Consider $f(x) = x^3 - 3x + 1$ with domain $[0,3]$.

1. Function values at endpoints: $f(0) = 1$ and $f(3) = 19$.
2. Critical points: Since f is differentiable we look for $f'(c) = 0$. This gives $3c^2 - 3 = 0$ or $c = \pm 1$. Thus $c = 1$ is the only critical point (in the given domain).
3. Function values at the critical points: $f(1) = -1$.

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Thus the candidates for absolute extremes are only $f(0) = 1$, $f(1) = -1$ and $f(3) = 19$. So the absolute maximum is at $x = 3$ and the absolute minimum is at $x = 1$.

Monotonicity Theorem

Theorem 5

Suppose I is an interval and $f : I \to \mathbb{R}$ is differentiable on I.

1. *If $f'(x) > 0$ for every $x \in I$ then f is strictly increasing.*
2. *If $f'(x) \geq 0$ for every $x \in I$ then f is increasing.*

We also have the corresponding statements regarding negative derivatives and decreasing functions.

Proof. Suppose $f'(x) > 0$ for every $x \in I$. Let $p, q \in I$ with $p < q$. We have to show that $f(p) < f(q)$.
By continuity, f achieves its maximum and minimum over $[p, q]$.
By Fermat's Theorem the points of maximum and minimum can only be p or q.
If the maximum and minimum values are equal, then f is a constant function, and $f' = 0$. So $f(p) \neq f(q)$.
(continued)

Monotonicity Theorem

(Proof continued)
Suppose $f(q)$ is the minimum value over $[p, q]$. Then

$$f'(q) = \lim_{x \to q-} \frac{f(x) - f(q)}{x - q} \leq 0.$$

This contradicts the positivity of f'. It follows that $f(q)$ is the maximum value over $[p, q]$ and hence $f(p) < f(q)$.
Now, suppose we only have $f'(x) \geq 0$ for every $x \in I$. Let $p, q \in I$ with $p < q$. Take any $\epsilon > 0$ and consider the function $g(x) = f(x) + \epsilon x$. Then $g'(x) = f'(x) + \epsilon > 0$ and g is strictly increasing. Now,

$$g(p) < g(q) \implies f(q) - f(p) > \epsilon(p - q).$$

Thus $f(q) - f(p)$ is greater than every negative number and hence must be non-negative. $\square$

Example

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We have $f(-1) = -3 < 0$ and $f(0) = 1 > 0$. By Intermediate Value Theorem we have a $c \in (-1, 0)$ such that $f(c) = 0$, i.e. $c^3 + 3c + 1 = 0$.

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We have $f(-1) = -3 < 0$ and $f(0) = 1 > 0$. By Intermediate Value Theorem we have a $c \in (-1, 0)$ such that $f(c) = 0$, i.e. $c^3 + 3c + 1 = 0$.

Now $f'(x) = 3x^2 + 3 \geq 3 > 0$.

Hence f is strictly increasing, therefore one-one. So there can only be one c with $f(c) = 0$.

Zero derivative implies constancy

Theorem 6

Let f, g be differentiable functions from an interval I to $\mathbb{R}$.

1. *If $f'(x) = 0$ for each $x \in I$ then $f(x)$ is constant.*
2. *If $f'(x) = g'(x)$ for each $x \in I$ then $f(x) = g(x) + constant$.*

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Proof. Assume $f'(x) = 0$ for each $x \in I$.
$f' \geq 0$ implies f is increasing.
$f' \leq 0$ implies f is decreasing.
Since f is both increasing and decreasing, it is constant.

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$f' \geq 0$ implies f is increasing.
$f' \leq 0$ implies f is decreasing.
Since f is both increasing and decreasing, it is constant.

Assume $f'(x) = g'(x)$ for each $x \in I$. Apply the first part of the theorem to $f(x) - g(x)$. $\square$

Characterizing the Exponential Function

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Theorem 7

If $f'(x) = k\,f(x)$ on an interval I then $f(x) = Ae^{kx}$.

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Proof. Consider $g(x) = f(x)e^{-kx}$. Then

$$g'(x) = f'(x)e^{-kx} - kf(x)e^{-kx} = kf(x)e^{-kx} - kf(x)e^{-kx} = 0.$$

Hence $g(x) = A$, a constant, and $f(x) = Ae^{kx}$. $\square$

Task 1

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f' = f$ and $f(0) = 1$. Show that $f(x) = e^x$.

Characterizing Sine and Cosine

The sine and cosine functions satisfy the relation $f'' = -f$. More generally, every combination $a\cos x + b\sin x$ satisfies this relation. Are they the only ones?

Task 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $f'' = -f$ and $f(0) = f'(0) = 0$. Show that $f(x) = 0$. (Hint: Differentiate the function $f^2 + (f')^2$).

Task 3

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f'' = -f$. Show that if $f(0) = a$ and $f'(0) = b$ then $f(x) = a\cos x + b\sin x$.

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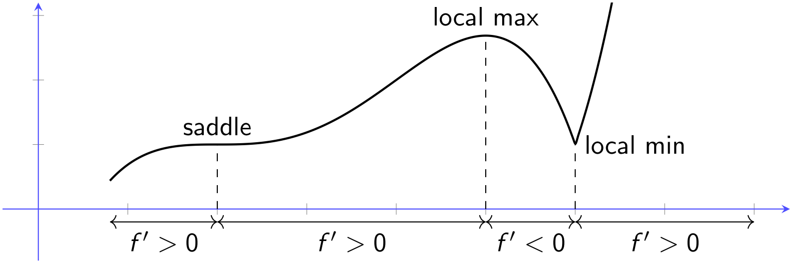
Saddle Points

A critical point that is not a local extreme is called a **saddle point**.



Observe the changes in the sign of f' as we pass through different types of critical points.

First Derivative Test

Theorem 8

Let f be continuous on (a, b) and let $c \in (a, b)$ be a critical point of f. Suppose f is differentiable on (a, b) except perhaps at c. Then,

1. *If $f'(x) \geq 0$ for $x \in (a, c)$ and $f'(x) \leq 0$ for $x \in (c, b)$ then f has a local maximum at c.*
2. *If $f'(x) \leq 0$ for $x \in (a, c)$ and $f'(x) \geq 0$ for $x \in (c, b)$ then f has a local minimum at c.*
3. *If f' has the same sign on either side of c then f has a saddle point at c.*

First Derivative Test – Proof

Suppose $f'(x) \geq 0$ for $x \in (a, c)$ and $f'(x) \leq 0$ for $x \in (c, b)$. By the Monotonicity Theorem, f is increasing on (a, c) and decreasing on (c, b).

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The continuity of f then gives us that f is increasing on $(a, c]$ and decreasing on $[c, b)$. For, suppose there is $x_1 < c$ with $f(x_1) > f(c)$. By the Intermediate Value Theorem, there is $x_2 \in (x_1, c)$ with $f(x_2) = \frac{1}{2}(f(x_1) + f(c)) < f(x_1)$, violating the fact that f is increasing on (a, c). This shows that f is increasing on $(a, c]$. Similarly, f is decreasing on $[c, b)$.

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Similarly, if $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, there is a local minimum at c.

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Similarly, if $f'(x) < 0$ for $x \in (a, c)$ and $f'(x) > 0$ for $x \in (c, b)$, there is a local minimum at c.

But if $f'(x)$ has the same sign on both sides of c then values on one side are higher and on the other are lower. Hence there is neither a local maximum nor a local minimum at c. $\square$

Example

Consider $f(x) = x^2 e^x$. We have $f'(x) = 2xe^x + x^2 e^x = x(x+2)e^x$.

$$f'(c) = 0 \iff c(c+2) = 0 \iff c = 0, -2$$

Identify the the sign of the derivative on either side of each critical point:

	$x < -2$	$-2 < x < 0$	$x > 0$
$f'(x)$	$+$	$-$	$+$

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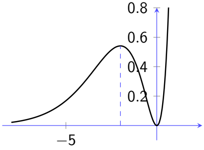
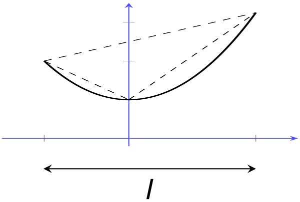
$$\lim_{x\to\infty} x^2e^x = \infty \text{ and } \lim_{x\to-\infty} x^2e^x = \lim_{x\to\infty} x^2/e^x = 0$$



Convexity

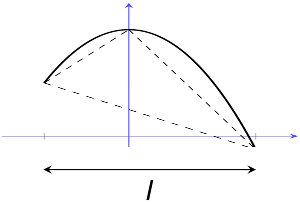
A function $f : I \to \mathbb{R}$ is said to be **convex** on I if its graph over *every* interval $[a, b]$ in I lies *below* the secant line through the endpoints of the graph over that interval.



The graph of a convex function turns upwards as we move from left to right.

Concavity

$f : I \to \mathbb{R}$ is said to be **concave** on I if its graph over *every* interval $[a, b]$ in I lies *above* the secant line through the endpoints of the graph over that interval.



The graph of a concave function turns downwards as we move from left to right.

Formal Definitions of Convex/Concave

1. f is called **convex** on I if for every $a, x, b \in I$ with $a < x < b$, we have

$$f(x) \leq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{1}$$

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Task 4

Can a function be both convex and concave?

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2. f is called **concave** on I if for every $a, x, b \in I$ with $a < x < b$, we have

$$f(x) \geq f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

Task 4

Can a function be both convex and concave?

If the inequalities (1) and (2) are strict, we call f **strictly convex** and **strictly concave** respectively.

Inflection and Convexity Test

A point where the function is continuous and switches from strictly convex on one side to strictly concave on the other, is called an **inflection point** of the function.

Theorem 9

Let f be twice differentiable on an interval I. Then

1. $f'' \geq 0$ *on* I *implies* f *is convex on* I.
2. $f'' \leq 0$ *on* I *implies* f *is concave on* I.
3. *If* f'' *is continuous at an inflection point* c *then* $f''(c) = 0$.

If the inequalities are strict, so is the convexity.

Convexity Test – Proof

First, suppose $f'' \geq 0$ on I. Let $c, d \in I$ with $c < d$. Consider

$$g(x) = f(c) + \frac{f(d) - f(c)}{d - c}(x - c) - f(x), \text{ for } x \in (c, d).$$

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We wish to show that for each $x \in (c, d)$, $g(x) \geq 0$. Suppose that $g(x) < 0$ at some point $x \in (c, d)$. By the Monotonicity Theorem, we obtain α, β as follows:

- $\alpha \in (c, x)$ and $g'(\alpha) < 0$,
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If $f'' \leq 0$ on I, apply the first part to $-f$.

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If $f'' \leq 0$ on I, apply the first part to $-f$.

For the third part, suppose $f''(c) > 0$. Then, by continuity, $f'' > 0$ in an interval I centered at c. So f is convex on I and c is not an inflection point.

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If $f'' \leq 0$ on I, apply the first part to $-f$.
For the third part, suppose $f''(c) > 0$. Then, by continuity, $f'' > 0$ in an interval I centered at c. So f is convex on I and c is not an inflection point.This rules out $f''(c) > 0$. We can similarly rule out $f''(c) < 0$.

Second Derivative Test

Theorem 10

Let f have a critical point at c and f'' be continuous in an open interval containing c. Then

1. $f''(c) > 0$ *implies there is a local minimum at* c.
2. $f''(c) < 0$ *implies there is a local maximum at* c.

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Let f have a critical point at c and f'' be continuous in an open interval containing c. Then

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Then f' is strictly increasing in that interval.

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Proof. Let $f''(c) > 0$. By continuity, $f'' > 0$ in an open interval containing c.
Then f' is strictly increasing in that interval.
Hence f' changes from negative to positive at c, and there is a local minimum at c (by the First Derivative Test).

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Proof. Let $f''(c) > 0$. By continuity, $f'' > 0$ in an open interval containing c.
Then f' is strictly increasing in that interval.
Hence f' changes from negative to positive at c, and there is a local minimum at c (by the First Derivative Test).
For the second part, apply the first part to $-f$. □

Example

Let $f(x) = x^2 e^x$. We saw earlier that this has a local maximum at -2 and a local (as well as absolute) minimum at 0.

$$f'(x) = (x^2 + 2x)e^x \implies f''(x) = (x^2 + 4x + 2)e^x.$$

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We identify the possible inflection points.

$$f''(c) = 0 \iff c^2 + 4c + 2 = 0 \iff c = -2 \pm \sqrt{2} \approx -3.4, -0.6.$$

	$x < -2 - \sqrt{2}$	$-2 - \sqrt{2} < x < -2 + \sqrt{2}$	$x > -2 + \sqrt{2}$
$f''(x)$	$+$	$-$	$+$
Convexity	Convex	Concave	Convex

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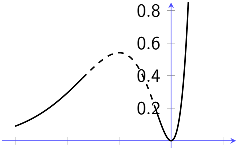
Note that $f''(-2) = -2e^{-2} < 0$ confirms the local maximum at -2 and $f''(0) = 2 > 0$ confirms the local minimum at 0.



Curve Sketching

We have seen how first and second derivative calculations can give us key features of a graph.

We can capture all the essential aspects of a function's behaviour by supplementing these with the following: domain, axis-intercepts, points of discontinuity, symmetry (even, odd, periodic), asymptotes (vertical, horizontal, slant).

Let's demonstrate this with an example.

Example

Consider $f(x) = \arctan\left(\dfrac{x-1}{x+1}\right)$.

Domain: The domain is $\mathbb{R} \setminus \{-1\}$. Note also that $f(x) \in (-\pi/2, \pi/2)$.

Intercepts: The function is zero at $x = 1$. It cuts the y-axis at $y = f(0) = \arctan(-1) = -\pi/4$.

Symmetry: We have $f(2) = \arctan(1/3)$ and $f(-2) = \arctan(3)$. They are positive and unequal (arctan is 1-1) so $f(x)$ is neither even nor odd.

Vertical Asymptotes: As $f(x)$ is continuous on its domain, the only possibility of vertical asymptotes is at $x = -1$. So we calculate the one-sided limits there:

$$\lim_{x\to-1+} \frac{x-1}{x+1} = \lim_{t\to 0+} \frac{t-2}{t} = -\infty \implies \lim_{x\to-1+} \arctan\left(\frac{x-1}{x+1}\right) = -\frac{\pi}{2},$$

$$\lim_{x\to-1-} \frac{x-1}{x+1} = \lim_{t\to 0+} \frac{t+2}{t} = \infty \implies \lim_{x\to-1-} \arctan\left(\frac{x-1}{x+1}\right) = \frac{\pi}{2}.$$

Since the limits are finite there isn't a vertical asymptote at $x = -1$.

Example - continued

Horizontal Asymptotes:

$$\lim_{x\to\infty}\left(\frac{x-1}{x+1}\right)=1 \quad\Longrightarrow\quad \lim_{x\to\infty}\arctan\left(\frac{x-1}{x+1}\right)=\arctan(1)=\frac{\pi}{4},$$
$$\lim_{x\to-\infty}\left(\frac{x-1}{x+1}\right)=1 \quad\Longrightarrow\quad \lim_{x\to-\infty}\arctan\left(\frac{x-1}{x+1}\right)=\arctan(1)=\frac{\pi}{4}.$$

Therefore $y=\pi/4$ is a horizontal asymptote on both sides.
Critical Points:

$$\begin{aligned}\frac{d}{dx}\arctan\left(\frac{x-1}{x+1}\right) &= \frac{(x+1)^2}{(x+1)^2+(x-1)^2}\times\frac{(x+1)-(x-1)}{(x+1)^2}\\ &= \frac{1}{x^2+1}.\end{aligned}$$

There are no critical points. In fact $f'(x)>0$ and so f is strictly increasing on $(-\infty,-1)$ and also on $(-1,\infty)$.

Example - continued

Convexity: $f''(x) = \dfrac{d}{dx}\dfrac{1}{x^2+1} = \dfrac{-2x}{(x^2+1)^2}$.

We have $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$.

So f is convex on $(-\infty, -1)$ and on $(-1, 0)$.

It is concave on $(0, \infty)$. The only inflection point is $x = 0$.

Here is the graph of f.