Chapter 3

Question 1

(i) $y_{t+2} = y_{t+1} - .5\,y_t + 1$ second-order, linear autonomous, non-homogeneous

(ii) $y_{t+2} = 2\,y_t + 3$ second-order, linear autonomous, non-homogeneous

(iii) $\dfrac{y_{t+1} - y_t}{y_t} = 4$ first-order, linear, autonomous, homogeneous

(iv) $y_{t+2} - 2\,y_{t+1} + 3\,y_t = t$ second-order, linear, non-autonomous, non-homogeneous

Question 2

```
> rsolve( {P(t)=(1+r)*P(t-1)-R,P(0)=P0},P(t));
```

$$P0\,(1+r)^t - \frac{R\,(1+r)^t}{r} + \frac{R}{r}$$

```
> simplify(P0*(1+r)^t-R*(1+r)^t/r+R/r);
```

$$-\frac{-P0\,(1+r)^t\,r + R\,(1+r)^t - R}{r}$$

Question 3

For periods 0 to n

0 P_0

1 $(1+r)\,P_0 - R_1$

2 $(1+r)\,[(1+r)\,P_0 - R_1] - R_2 = (1+r)^2\,P_0 - (1+r)\,R_1 - R_2$

3 $(1+r)\,[(1+r)^2\,P_0 - (1+r)\,R_1 - R_2] - R_3 = (1+r)^3\,P_0 - (1+r)^2\,R_1 - (1+r)\,R_2 - R_3$

4 $(1+r)\,[(1+r)^3\,P_0 - (1+r)^2\,R_1 - (1+r)\,R_2 - R_3] - R_4 =$

$(1+r)^4\,P_0 - (1+r)^3\,R_1 - (1+r)^2\,R_2 - (1+r)\,R_3 - R4$

: :

n $(1+r)^n\,P_0 - (1+r)^{(n-1)}\,R_1 - (1+r)^{(n-2)}\,R_2 \ldots -(1+r)\,R_{n-1} - R_n$

Hence

$$P_n = (1+r)^n\,P_0 - (1+r)^{(n-1)}\,R_1 - (1+r)^{(n-2)}\,R_2 \ldots -(1+r)\,R_{n-1} - R_n$$

It is possible to check the consistency of this answer with the previous one. Let $R_k = R$ for all k. Then

$$P_n = (1+r)^n\,P_0 - (1+r)^{(n-1)}\,R - (1+r)^{(n-2)}\,R \ldots -(1+r)\,R - R$$

Consider the terms involving R. Let the sum of these be denoted S. Then

```
> S=sum((1+r)^k*R, k=0..n-1);
```

$$S = \frac{R\,(1+r)^n}{r} - \frac{R}{r}$$

Substituting this into P_n we obtain

$$P_n = (1+r)^n P_0 - \frac{R[(1+r)^n - 1]}{n} = (1+r)^n \left(P_0 - \frac{R}{r}\right) + \frac{R}{r}$$

Question 4

(i)

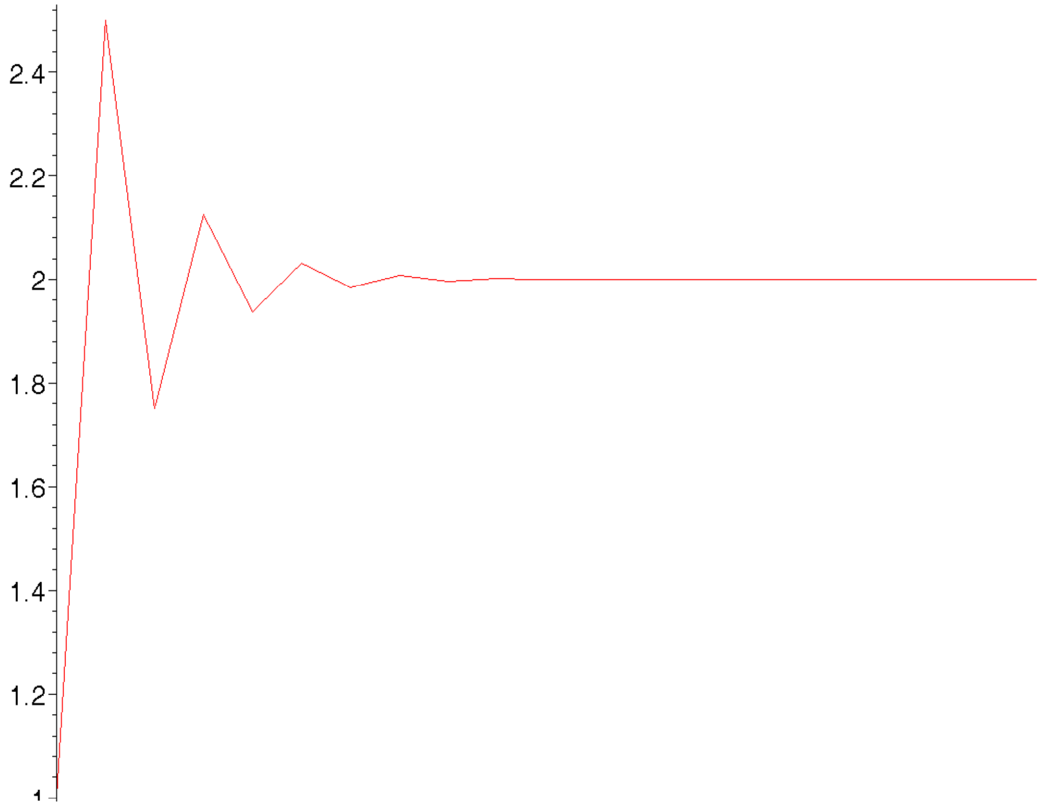
```
> rsolve({y(t+1)=-(1/2)*y(t)+3,y(0)=y0},y(t));
```

$$y0\left(\frac{-1}{2}\right)^t - 2\left(\frac{-1}{2}\right)^t + 2$$

```
> y1:=subs(y0=1,y0*(-1/2)^t-2*(-1/2)^t+2);
```

$$y1 := -\left(\frac{-1}{2}\right)^t + 2$$

```
> points1:=seq( [t,y1],t=0..20):
> plot([points1]);
```



(ii)

```
> rsolve({2*y(t+1)=-3*y(t)+4,y(0)=y0},y(t));
```

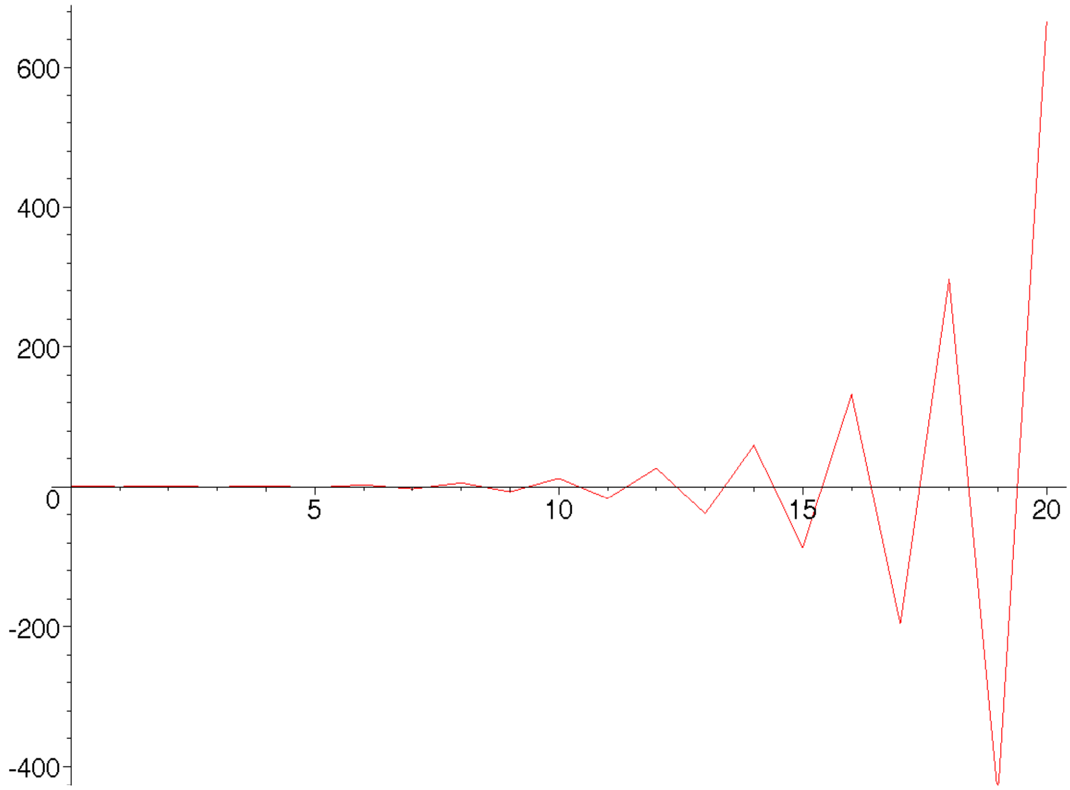
$$y0\left(\frac{-3}{2}\right)^t - \frac{4}{5}\left(\frac{-3}{2}\right)^t + \frac{4}{5}$$

```
> y2:=subs(y0=1,y0*(-3/2)^t-4/5*(-3/2)^t+4/5);
```

$$y2 := \frac{1}{5}\left(\frac{-3}{2}\right)^t + \frac{4}{5}$$

```
> points2:=seq( [t,y2],t=0..20):
```

```
> plot([points2]);
```



(iii)

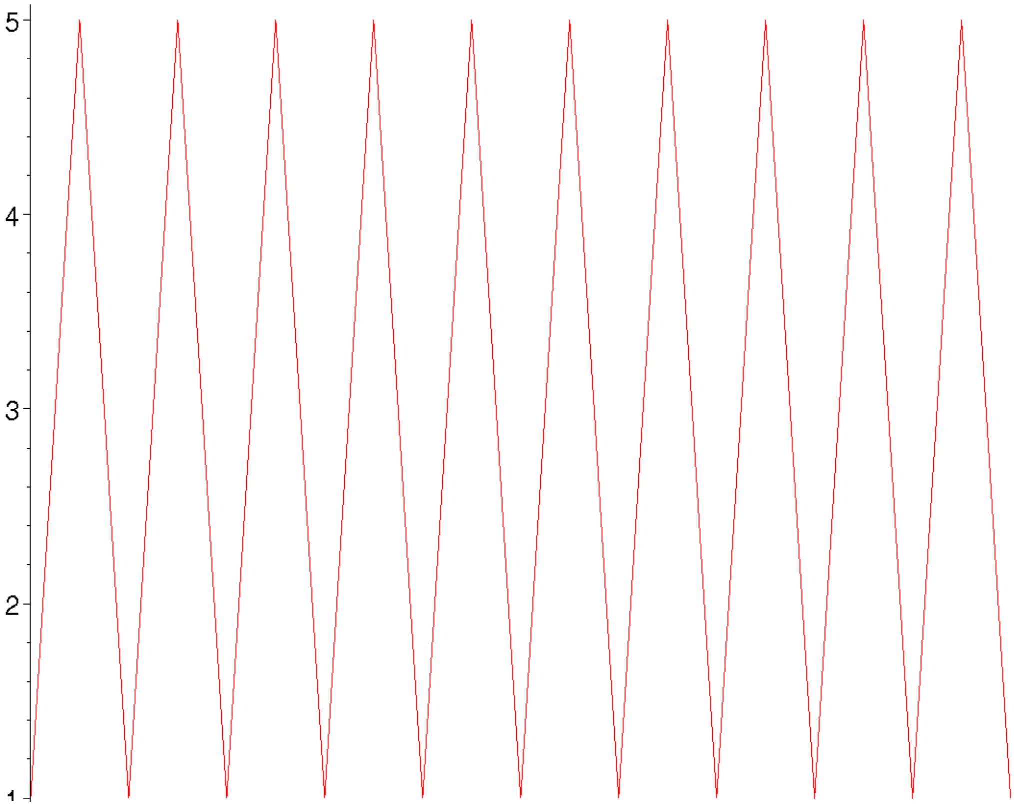
```
> rsolve({y(t+1)=-y(t)+6,y(0)=y0},y(t));
```

$$y0\,(-1)^t - 3\,(-1)^t + 3$$

```
> y3:=subs(y0=1,y0*(-1)^t-3*(-1)^t+3);
```

$$y3 := -2\,(-1)^t + 3$$

```
> points3:=seq( [t,y3],t=0..20):
> plot([points3]);
```



(iv)

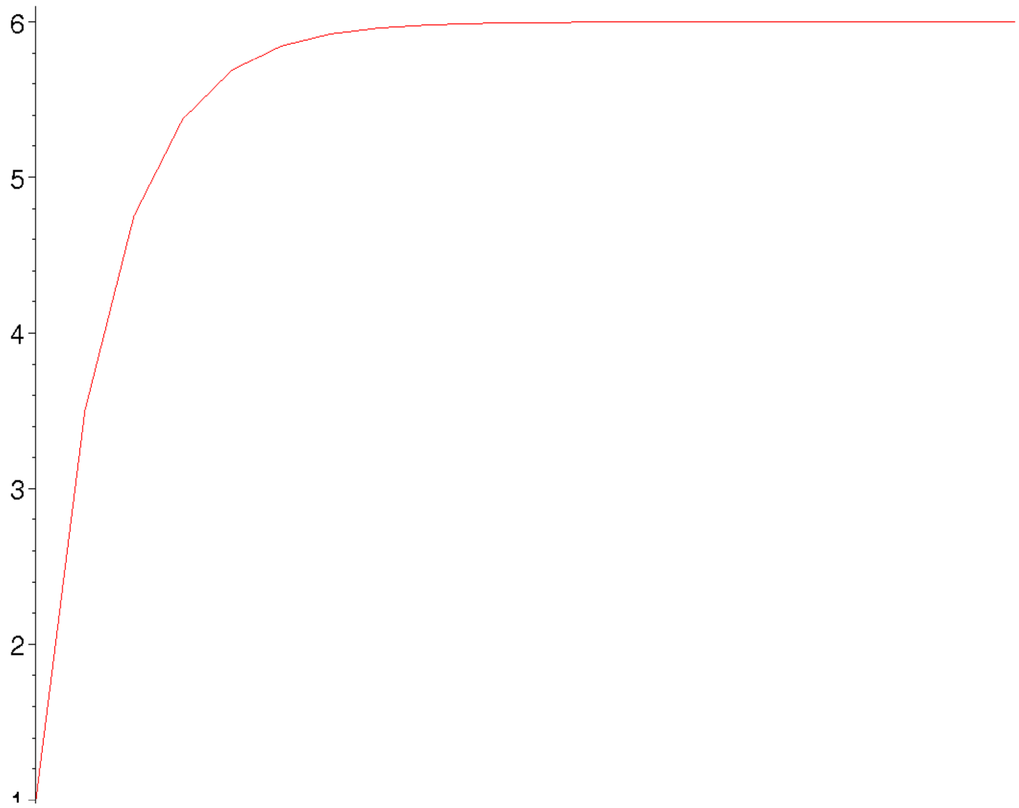
```
> rsolve({y(t+1)=(1/2)*y(t)+3,y(0)=y0},y(t));
```

$$y0\left(\frac{1}{2}\right)^t - 6\left(\frac{1}{2}\right)^t + 6$$

```
> y4:=subs(y0=1,y0*(1/2)^t-6*(1/2)^t+6);
```

$$y4 := -5\left(\frac{1}{2}\right)^t + 6$$

```
> points4:=seq( [t,y4],t=0..20):
> plot([points4]);
```



(v)

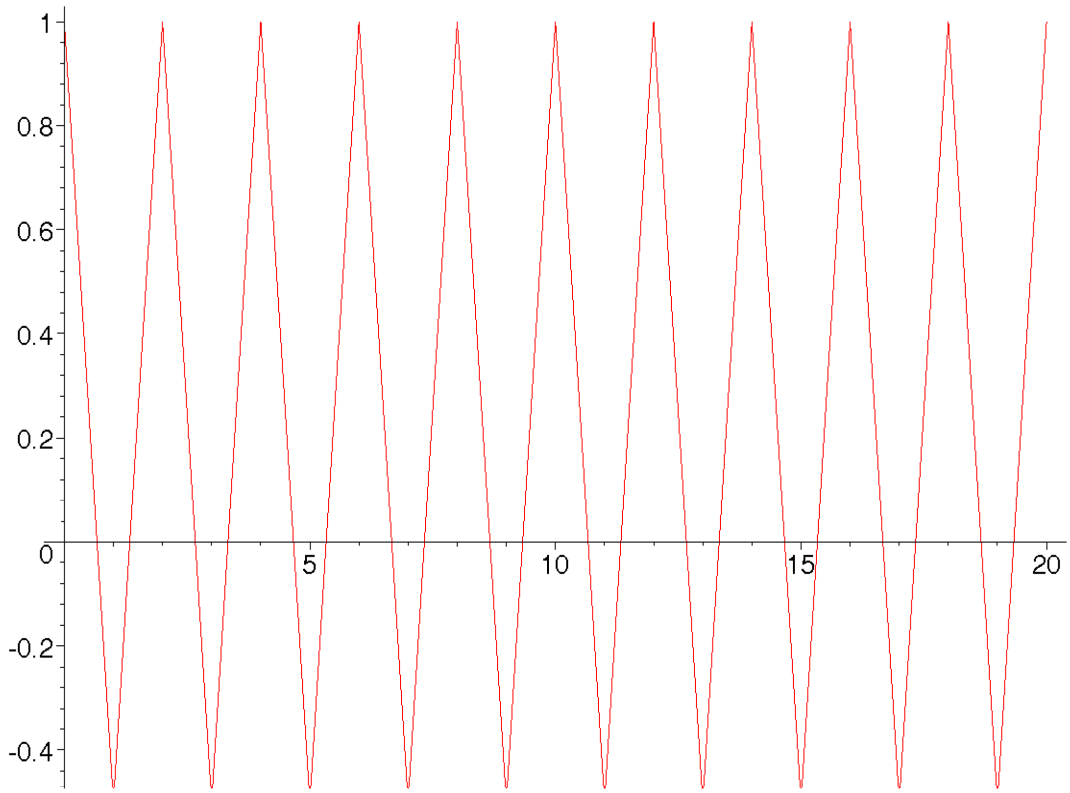
```
> rsolve({4*y(t+2)+4*y(t+1)-2=0,y(0)=y0},y(t));
```

$$y0\,(-1)^t-\frac{1}{4}(-1)^t+\frac{1}{4}$$

```
> y5:=subs(y0=1,y0*(-1)^t-1/4*(-1)^t+1/4);
```

$$y5:=\frac{3}{4}(-1)^t+\frac{1}{4}$$

```
> points5:=seq([t,y5],t=0..20):
> plot([points5]);
```



Question 5

(i)

```
> solve(a=a^3-a^2+1,a);
```

$$-1, 1, 1$$

```
> factor(a^3-a^2-a+1);
```

$$(a+1)(a-1)^2$$

(ii)

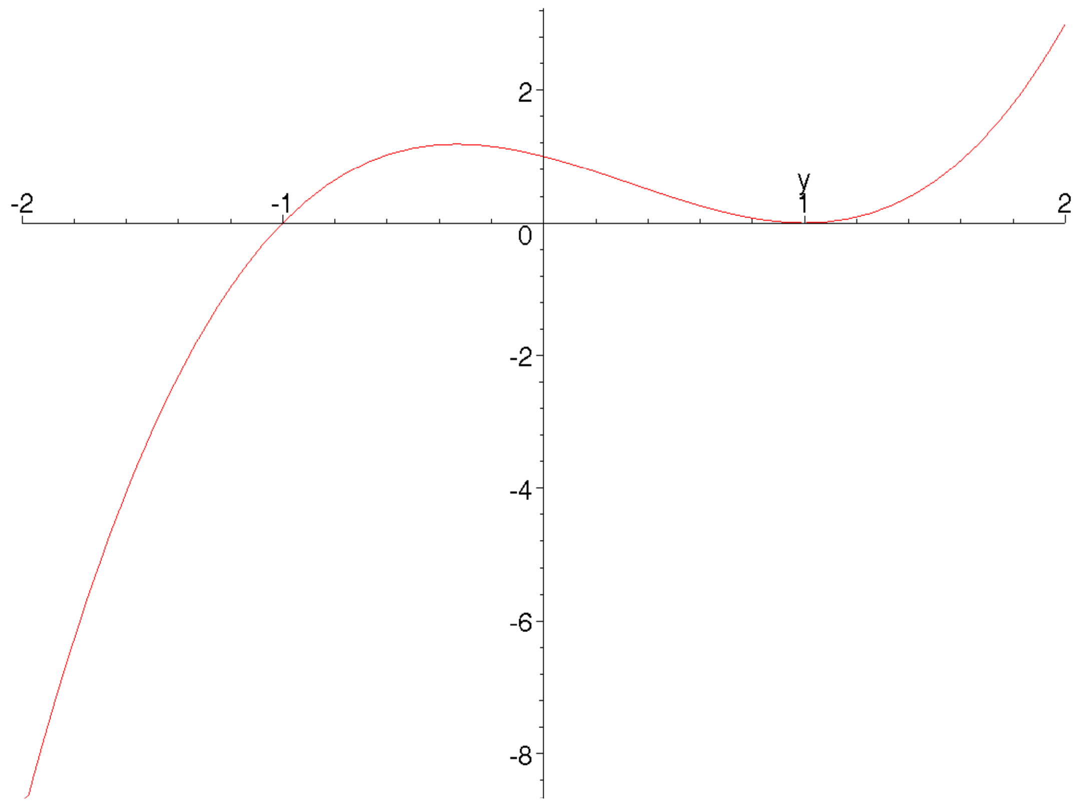
```
> f:=y->y^3-y^2-y+1;
```

$$f := y \to y^3 - y^2 - y + 1$$

```
> f(1);
```

$$0$$

```
> plot(f(y),y=-2..2);
```



Question 6

Although this requests a spreadsheet, we can use *Maple* equally well.

(i)

```
> f:=y->(1+a)*y-b*y^2;
```

$$f := y \to (a+1)\,y - b\,y^2$$

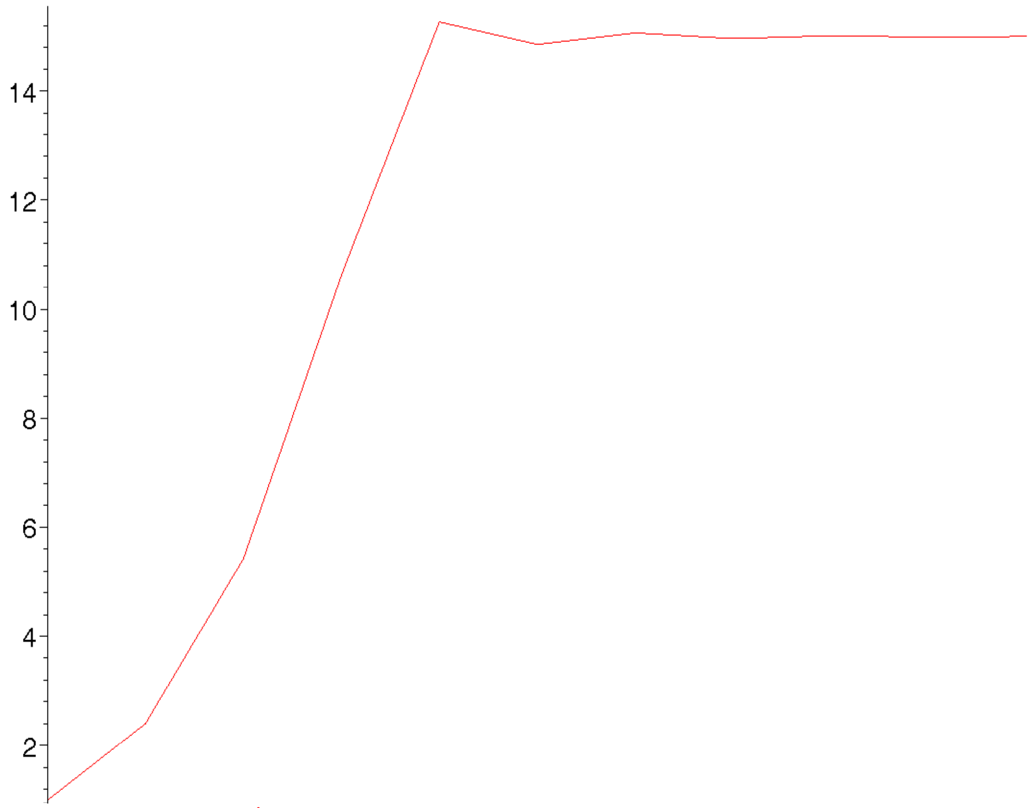
```
> f1:=y->(1+1.5)*y-0.1*y^2;
```

$$f1 := y \to 2.5\,y - .1\,y^2$$

```
> fn:=(f1,n)->(f1@@(n))(1);
```

$$fn := (f1, n) \to (f1^{(n)})(1)$$

```
> seq(fn(f1,n),n=0..10):
> data1A:=seq([n,fn(f1,n)],n=0..10):
> plot([data1A]);
```




```
> g:=y->((1+a)*y)/(1+b*y);
```

$$g := y \rightarrow \frac{(a+1)\,y}{1+b\,y}$$

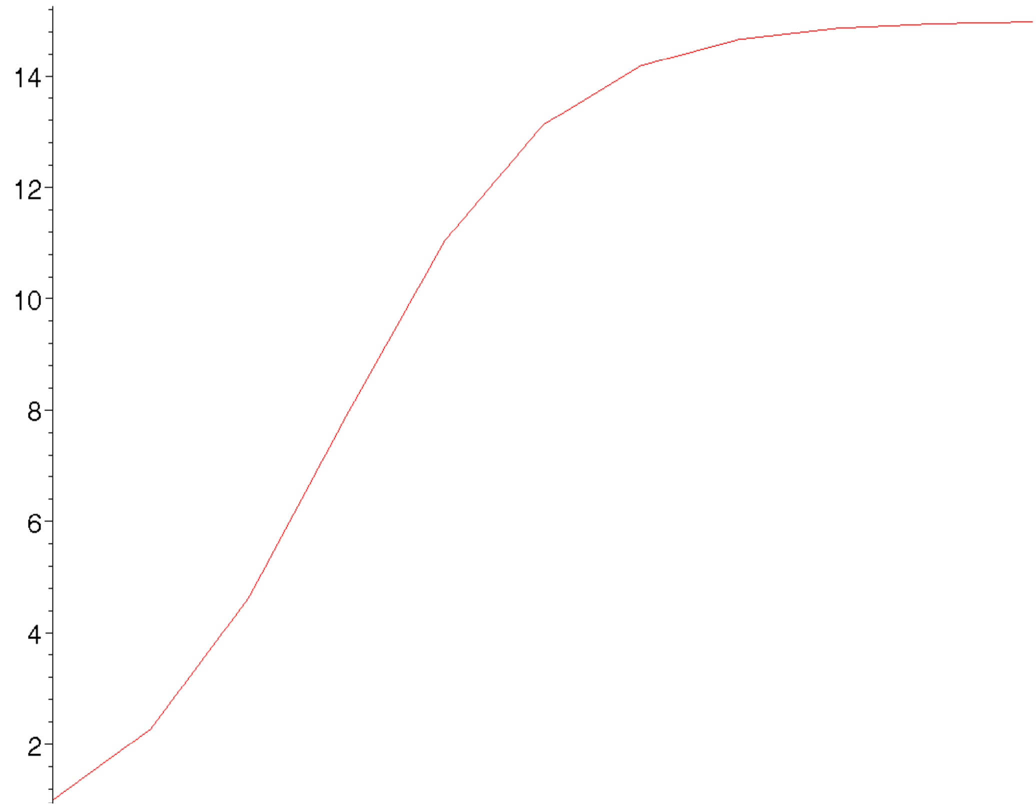
```
> g1:=y->((1+1.5)*y)/(1+0.1*y);
```

$$g1 := y \rightarrow 2.5\,\frac{y}{1+.1\,y}$$

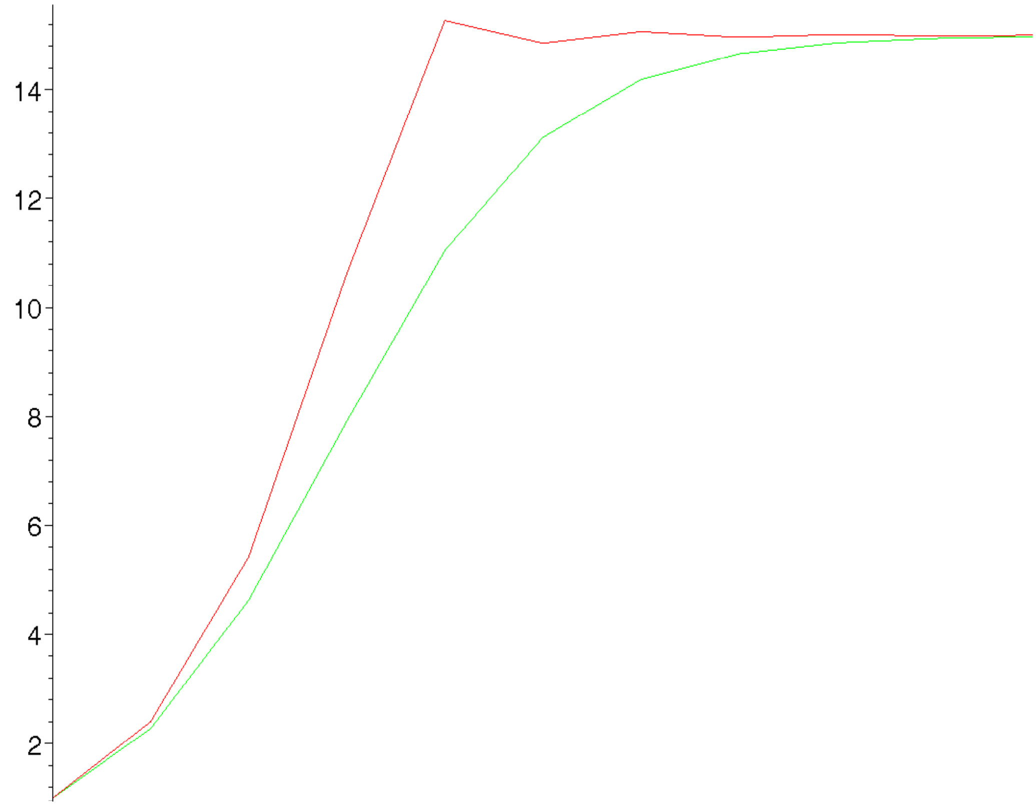
```
> gn:=(g1,n)->(g1@@(n))(1);
```

$$gn := (g1, n) \rightarrow (g1^{(n)})(1)$$

```
> seq(gn(g1,n),n=0..10):
> data1B:=seq([n,gn(g1,n)],n=0..10):
> plot([data1B]);
```




```
> plot({[data1A],[data1B]});
```



(ii)

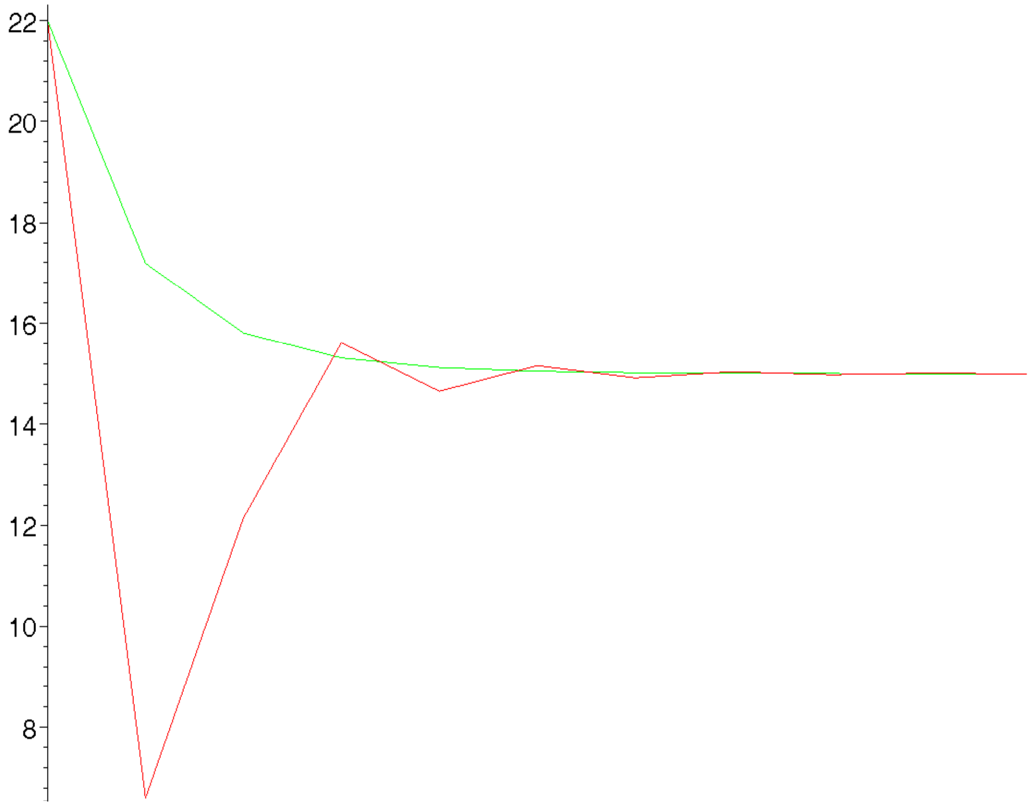
```
> fn2:=(f1,n)->(f1@@(n))(22);
```

$$fn2 := (f1, n) \rightarrow (f1^{(n)})(22)$$

```
> seq(fn2(f1,n),n=0..10):
> data2A:=seq([n,fn2(f1,n)],n=0..10):
> gn2:=(g1,n)->(g1@@(n))(22);
```

$$gn2 := (g1, n) \to (g1^{(n)})(22)$$

```
> seq(gn2(g1,n),n=0..10):
> data2B:=seq([n,gn2(g1,n)],n=0..10):
> plot({[data2A],[data2B]});
```



(iii)

```
> f3:=y->(1+2.2)*y-0.1*y^2;
```

$$f3 := y \to 3.2\, y - .1\, y^2$$

```
> fn3:=(f3,n)->(f3@@(n))(1);
```

$$fn3 := (f3, n) \to (f3^{(n)})(1)$$

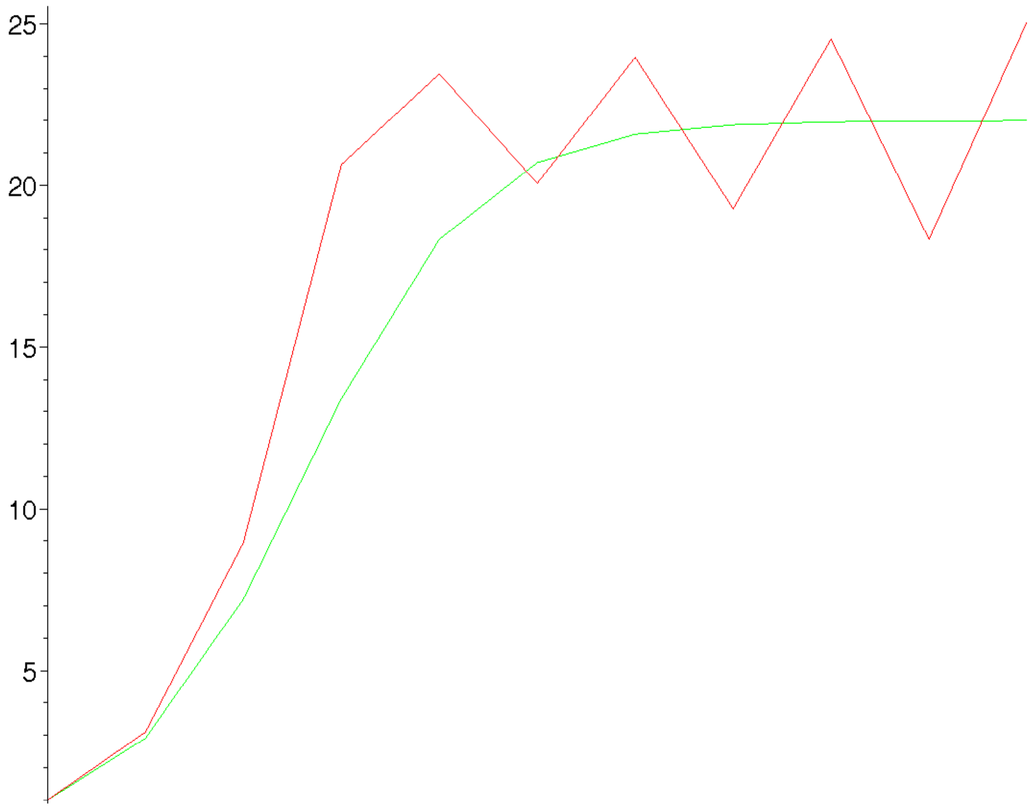
```
> seq(fn3(f3,n),n=0..10):
> data3A:=seq([n,fn3(f3,n)],n=0..10):
> g3:=y->((1+2.2)*y)/(1+0.1*y);
```

$$g3 := y \to 3.2 \frac{y}{1 + .1\, y}$$

```
> gn3:=(g3,n)->(g3@@(n))(1);
```

$$gn3 := (g3, n) \to (g3^{(n)})(1)$$

```
> seq(gn3(g3,n),n=0..10):
> data3B:=seq([n,gn3(g3,n)],n=0..10):
> plot({[data3A],[data3B]});
```



(iv)

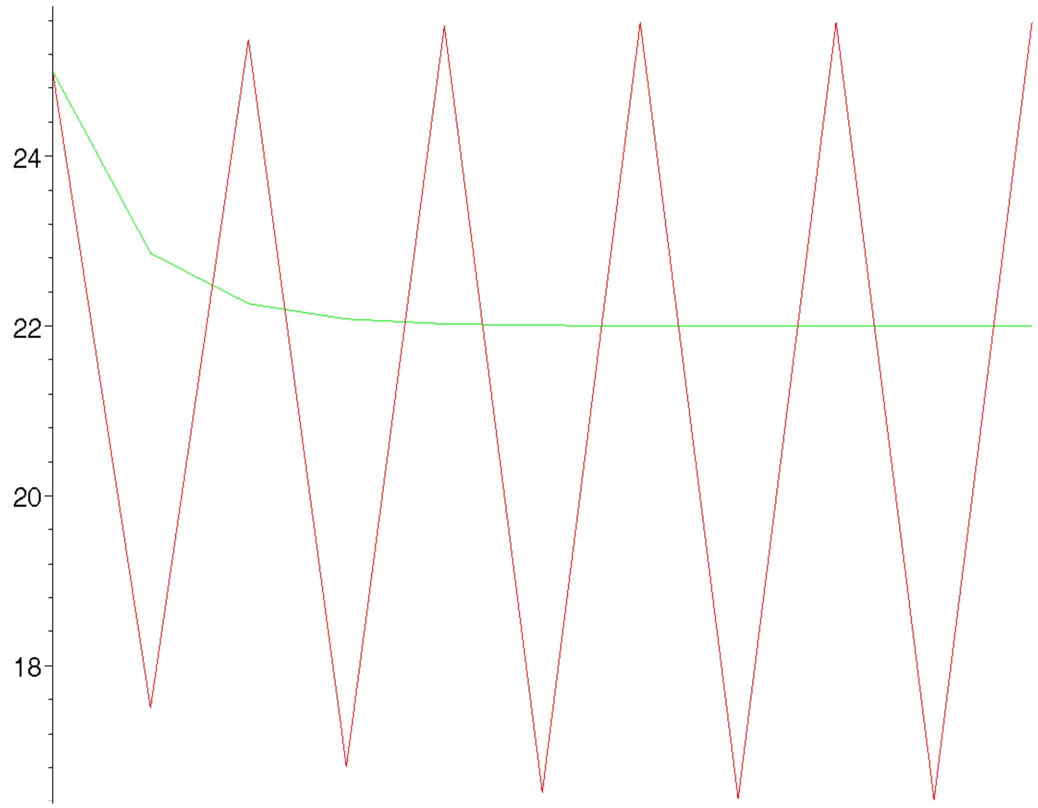
```
> fn4:=(f3,n)->(f3@@(n))(25);
```

$$fn4 := (f3, n) \to (f3^{(n)})(25)$$

```
> seq(fn4(f3,n),n=0..10):
> data4A:=seq([n,fn4(f3,n)],n=0..10):
> gn4:=(g3,n)->(g3@@(n))(25);
```

$$gn4 := (g3, n) \to (g3^{(n)})(25)$$

```
> seq(gn4(g3,n),n=0..10):
> data4B:=seq([n,gn4(g3,n)],n=0..10):
> plot({[data4A],[data4B]});
```



(v)

```
> f5:=y->(1+1.8)*y-0.15*y^2;
```

$$f5 := y \to 2.8\, y - .15\, y^2$$

```
> fn5:=(f5,n)->(f5@@(n))(11.5);
```

$$fn5 := (f5, n) \to (f5^{(n)})(11.5)$$

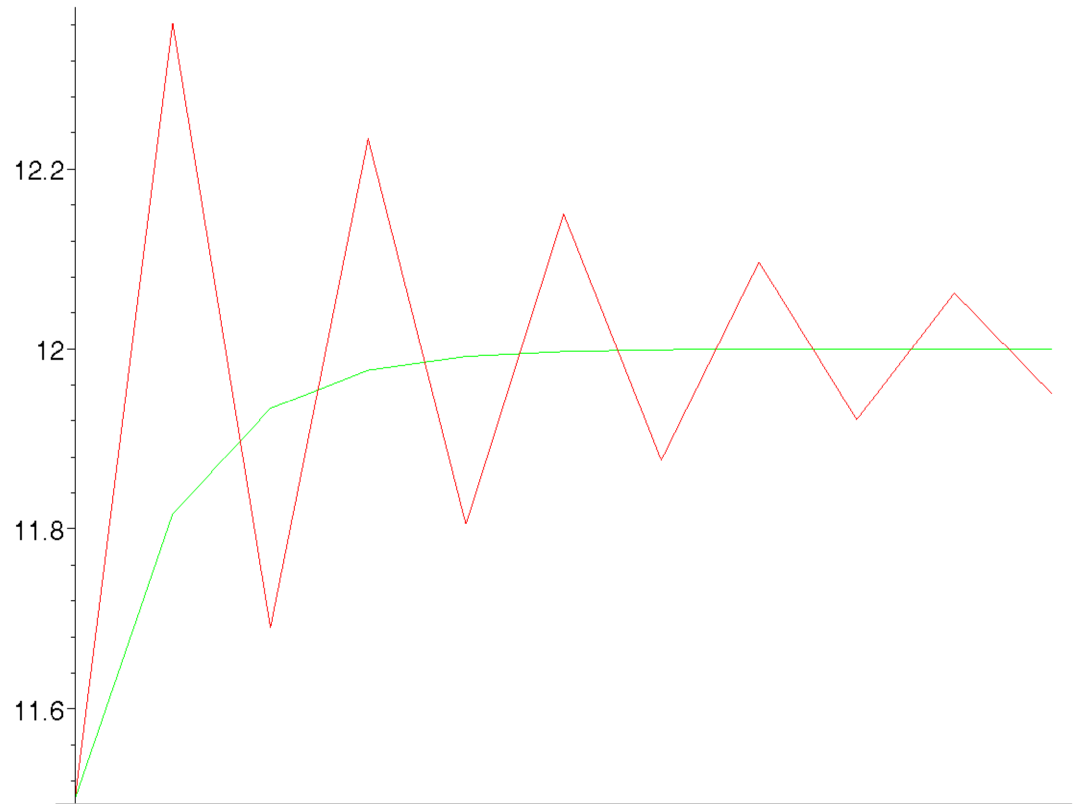
```
> seq(fn5(f5,n),n=0..10):
> data5A:=seq([n,fn5(f5,n)],n=0..10):
> g5:=y->((1+1.8)*y)/(1+0.15*y);
```

$$g5 := y \to 2.8\, \frac{y}{1 + .15\, y}$$

```
> gn5:=(g5,n)->(g5@@(n))(11.5);
```

$$gn5 := (g5, n) \to (g5^{(n)})(11.5)$$

```
> seq(gn5(g5,n),n=0..10):
> data5B:=seq([n,gn5(g5,n)],n=0..10):
> plot({[data5A],[data5B]});
```



Question 7

```
> a:='a':b:='b':c:='c':d:='d':p:='p':
> sol:=rsolve({p(t)=((a-c)/b)-(d/b)*p(t-1),p(0)=p0},p(t));
```

$$sol := p0\left(-\frac{d}{b}\right)^t - \frac{(a-c)\left(-\frac{d}{b}\right)^t}{b+d} + \frac{a-c}{b+d}$$

```
> sol1:=subs({a=10,b=3,c=2,d=1,p0=1},sol);
```

$$sol1 := -\left(\frac{-1}{3}\right)^t + 2$$

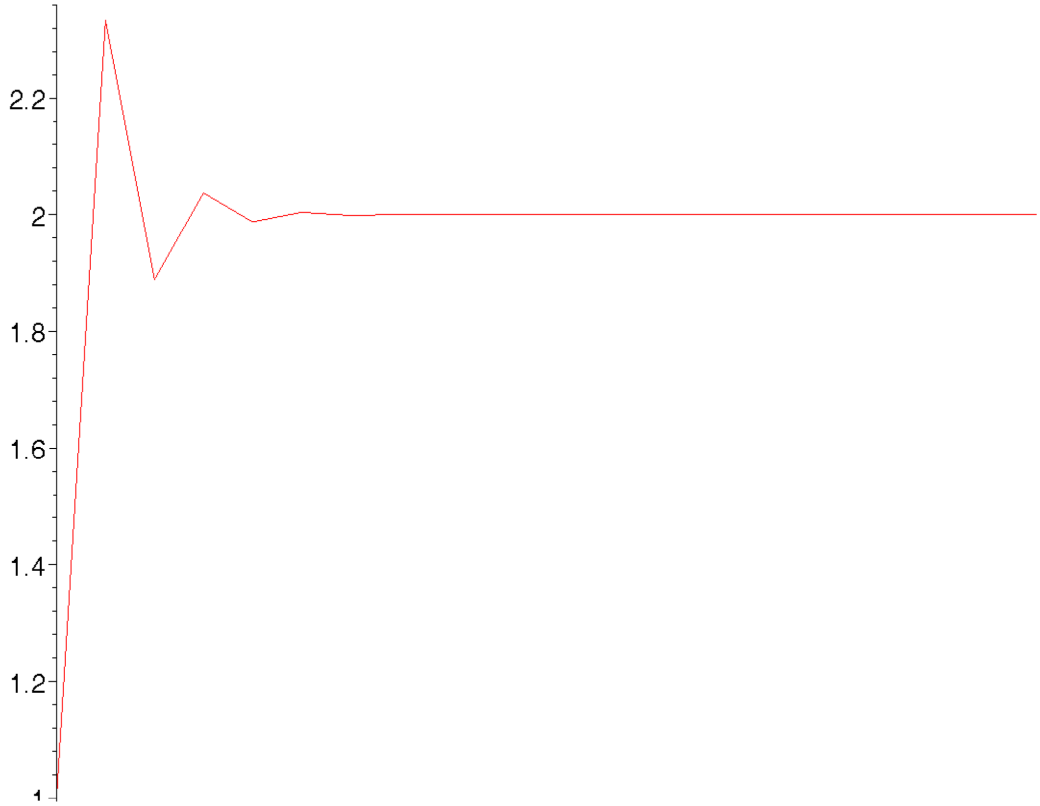
```
> sol2:=subs({a=25,b=4,c=3,d=4,p0=1},sol);
```

$$sol2 := -\frac{7}{4}(-1)^t + \frac{11}{4}$$

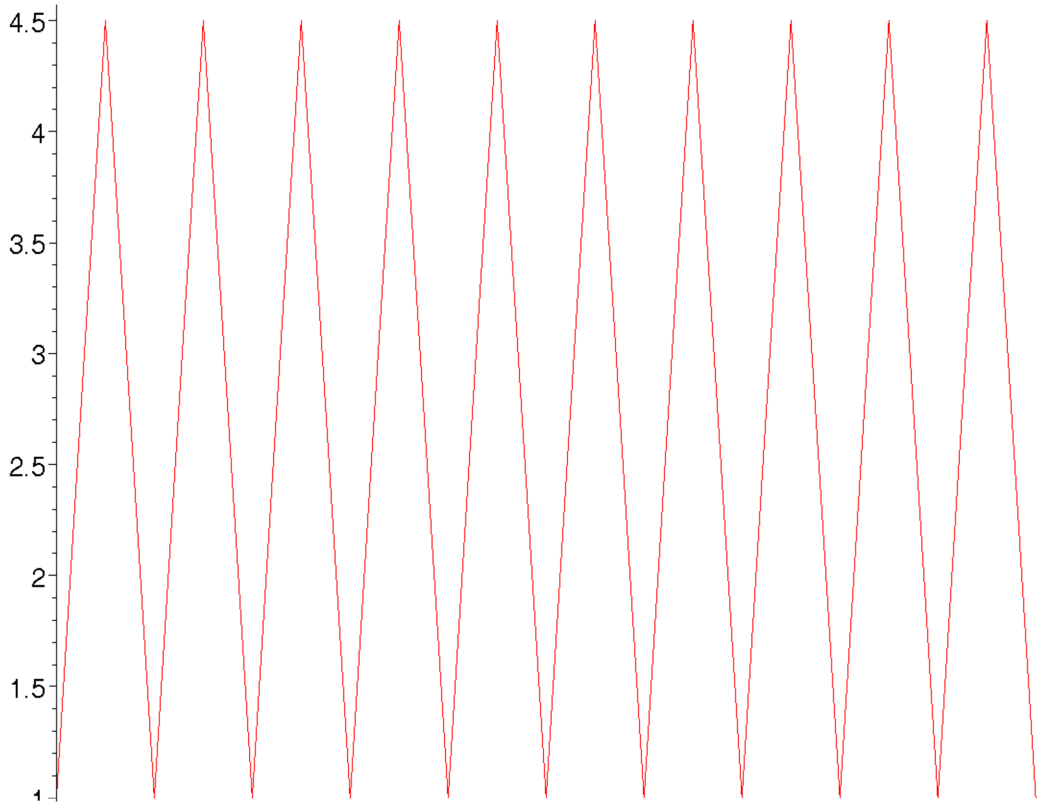
```
> sol3:=subs({a=45,b=5/2,c=5,d=15/2,p0=1},sol);
```

$$sol3 := -3\,(-3)^t + 4$$

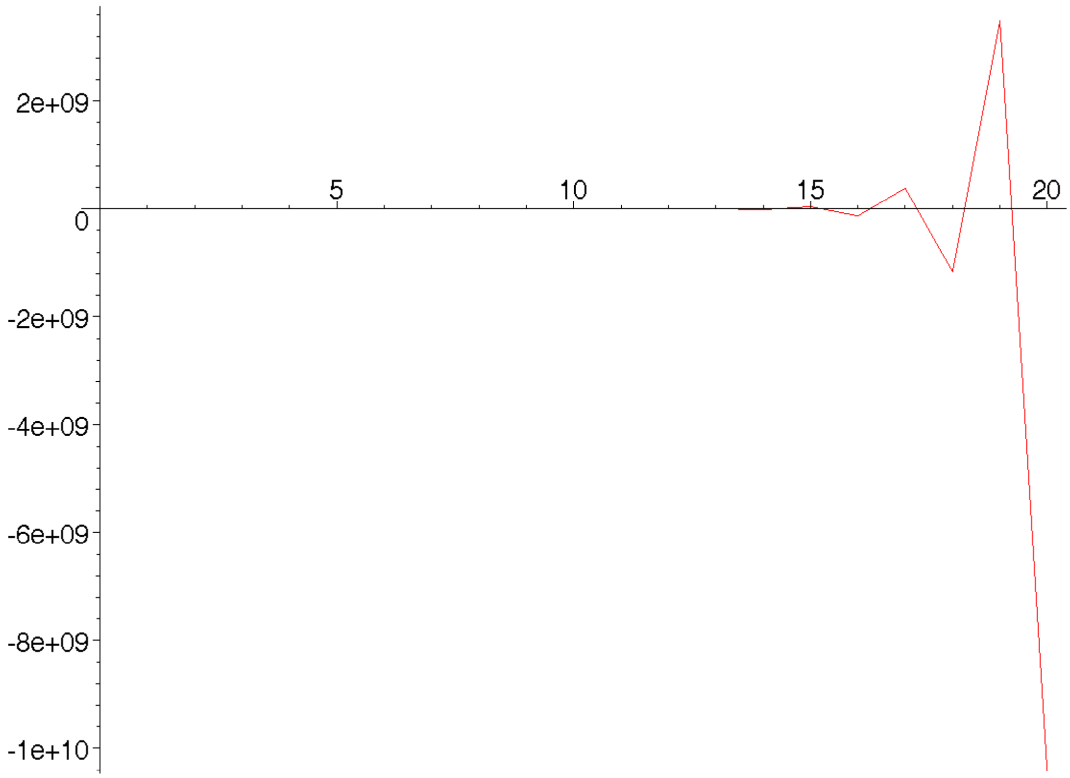
```
> p1:=seq([t,sol1],t=0..20):
> plot([p1]);
```




```
> p2:=seq([t,sol2],t=0..20):
> plot([p2]);
```




```
> p3:=seq([t,sol3],t=0..20):
> plot([p3]);
```



Question 8

The system of equations is

$$C_t = a + b\,Y_{t-1}$$
$$E_t = C_t + I_t + G_t$$
$$Y_t = E_t$$

Substituting, where I and G are exogenous, we obtain

$$Y_t = a + I + G + b\,Y_{t-1}$$

```
> solve(Y=(a+I+G)+b*Y,Y);
```

$$-\frac{a+I+G}{-1+b}$$

```
> rsolve({Y(t)=(a+I+G)+b*Y(t-1),Y(0)=Y0},Y(t));
```

$$Y0\,b^t + \frac{(a+I+G)\,b^t}{-1+b} - \frac{a+I+G}{-1+b}$$

Hence, the general solution is

$$Y_n = \frac{a+I+G}{1-b} + b^n\left(Y_0 - \frac{a+I+G}{1-b}\right)$$

which is stable so long as $0 < b < 1$.

Question 9

Readily obtain the difference equation $\mathrm{Y}(t) = a + I + G + b\,\mathrm{Y}(t-1)$

```
> Y:='Y':
> solY:=rsolve({Y(t)=(50+10+20)+0.8*Y(t-1),Y(0)=20},Y(t));
```

$$solY := -380\left(\frac{4}{5}\right)^t + 400$$

```
> Ystar:=(50+10+20)/(1-0.8);
```

$$Ystar := 400.0000000$$

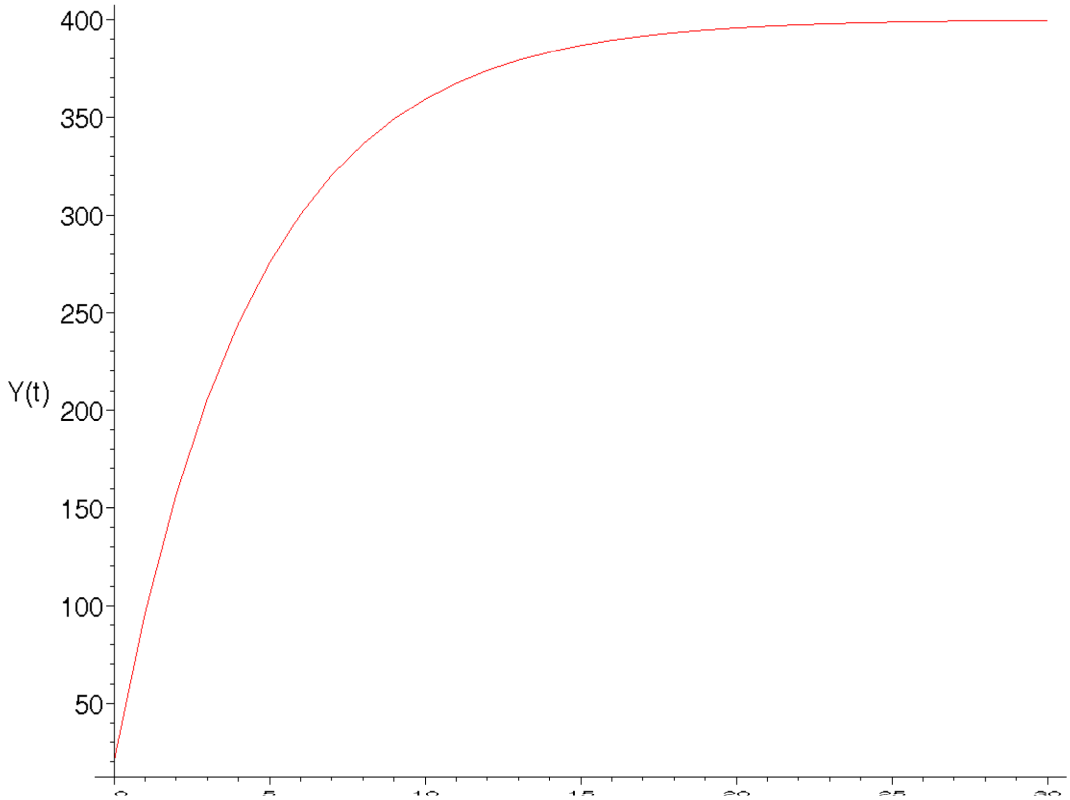
```
> YU:=(1+0.01)*Ystar;
```

$$YU := 404.0000000$$

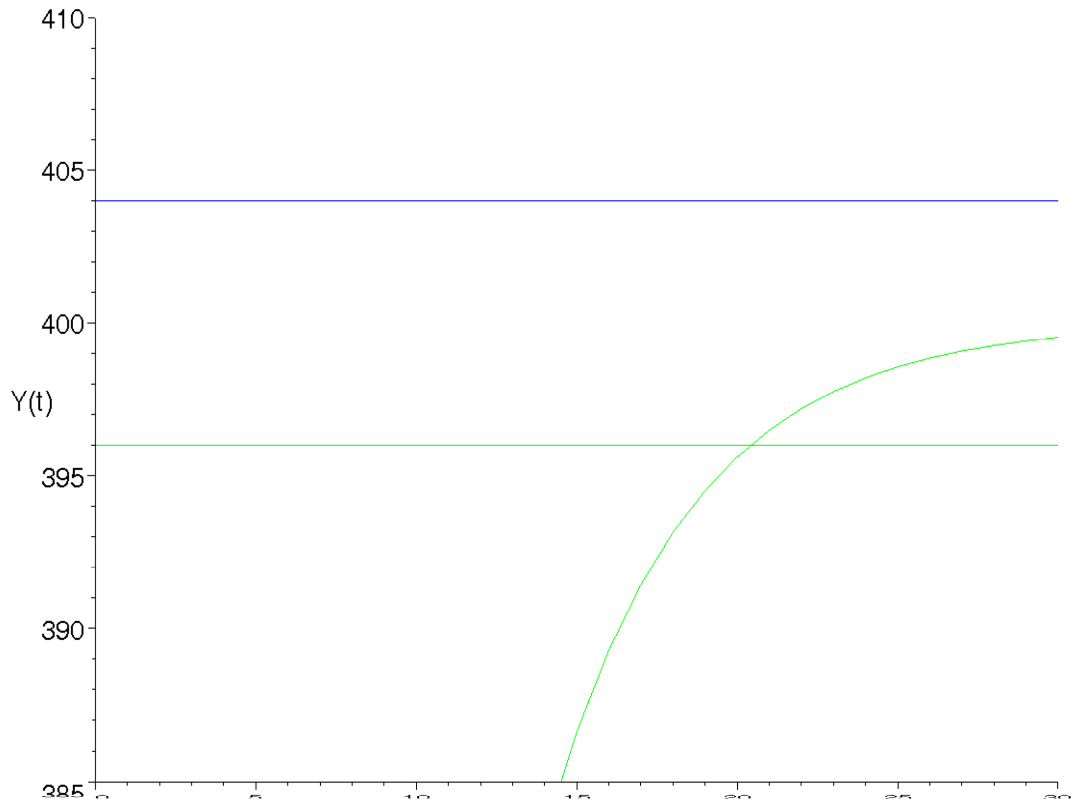
```
> YL:=(1-0.01)*Ystar;
```

$$YL := 396.0000000$$

```
> dataY:=evalf(seq([t,solY],t=0..30)):
> plot([dataY],labels=["t","Y(t)"]);
```



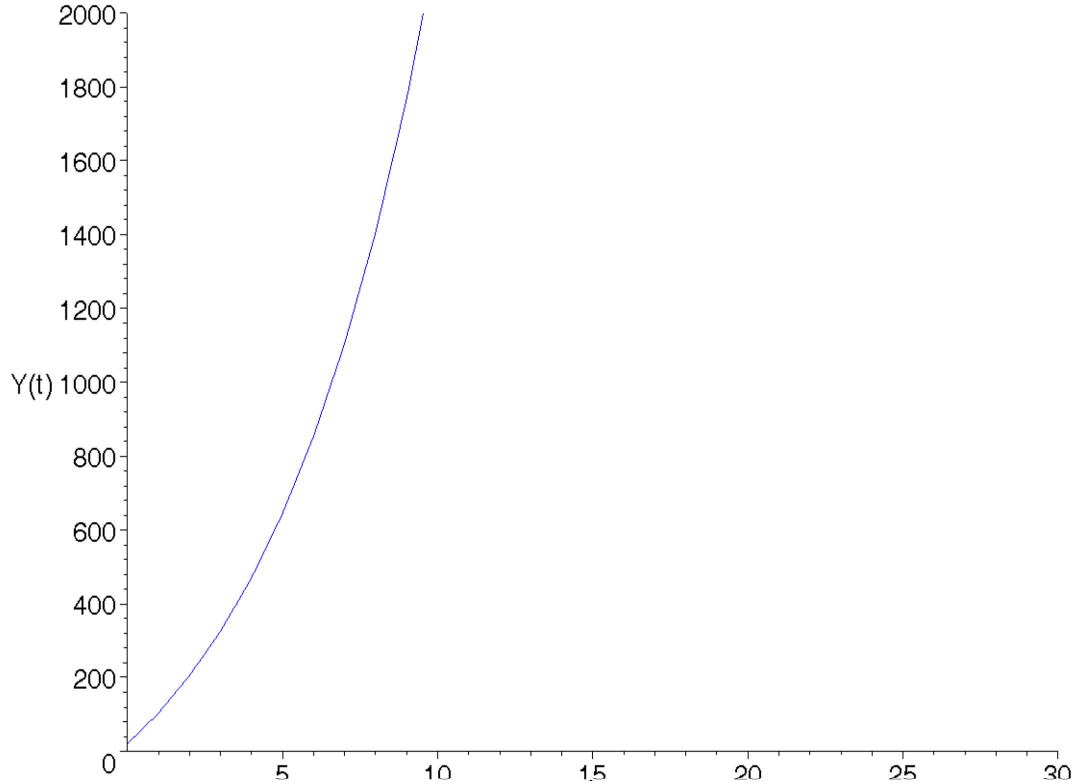

```
> lowerline:=seq([t,YL],t=0..30):
> upperline:=seq([t,YU],t=0..30):
> plot({[dataY],[lowerline],[upperline]},0..30,385..410,color=[
  blue,green,green],labels=["t","Y(t)"]);
```




```
> solY2:=rsolve({Y(t)=(50+10+20)+1.2*Y(t-1),Y(0)=20},Y(t));
```

$$solY2 := 420\left(\frac{6}{5}\right)^t - 400$$

```
> dataY2:=evalf(seq([t,solY2],t=0..30)):
> plot([dataY2],0..30,0..2000,color=blue,labels=["t","Y(t)"]);
```




```
> solY3:=rsolve({Y(t)=(50+10+20)+0.9*Y(t-1),Y(0)=20},Y(t));
```

$$solY3 := -780\left(\frac{9}{10}\right)^{t} + 800$$

```
> dataY3:=evalf(seq([t,solY3],t=0..30)):
> Ystar3:=(50+10+20)/(1-0.9);
```

$$Ystar3 := 800.0000000$$

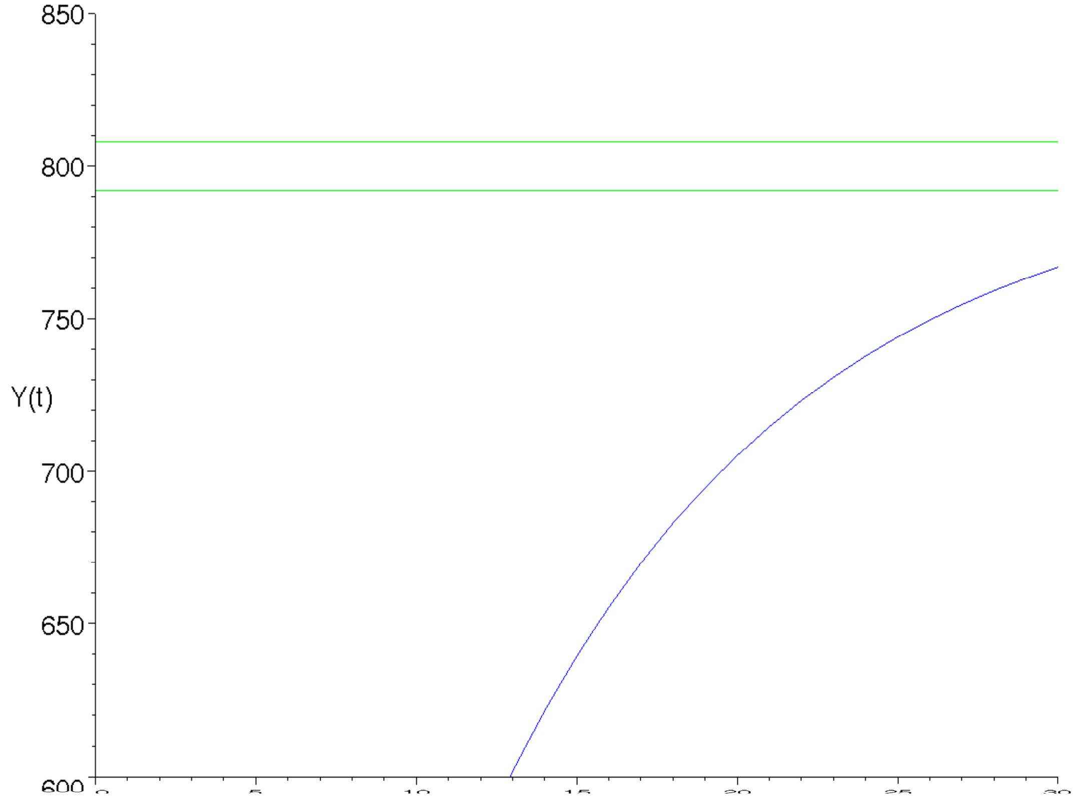
```
> YL3:=(1-0.01)*Ystar3;
```

$$YL3 := 792.0000000$$

```
> YU3:=(1+0.01)*Ystar3;
```

$$YU3 := 808.0000000$$

```
> lowerline3:=seq([t,YL3],t=0..30):
> upperline3:=seq([t,YU3],t=0..30):
> plot({[dataY3],[lowerline3],[upperline3]},0..30,600..850,colo
  r=[blue,green,green],labels=["t","Y(t)"]);
```



Question 10

(i)

Substituting we obtain the difference equation

$$p(t) = \frac{a-c}{b} - \frac{d\,(1-e)\,p(t-1)}{b} - \frac{d\,e\,p(t-2)}{b}$$

which is a second-order nonhomogeneous difference equation.

(ii)

```
> p:='p':p0:='p0':p1:='p1':
```

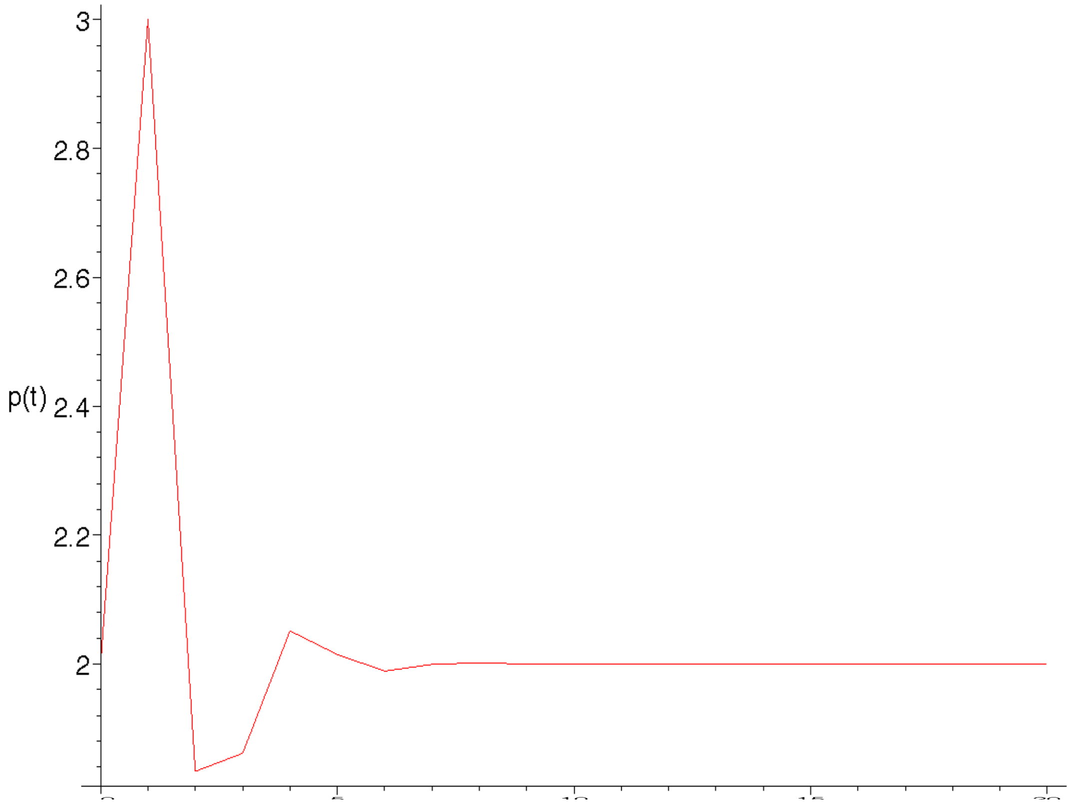
Substituting the values $a = 10$, $b = 3$, $c = 2$, $d = 1$ and $e = .5$ we can solve the following difference equation

```
> solp:=rsolve({p(t)=(8/3)-(1/6)*p(t-1)-(1/6)*p(t-2),p(0)=2,p(1
```

```
)=3},p(t));
```

$$solp := \frac{\left(-\frac{4}{23} I\sqrt{23}+20\right)\left(-2\frac{1}{1-I\sqrt{23}}\right)^t}{1-I\sqrt{23}} + \frac{\left(\frac{4}{23} I\sqrt{23}+20\right)\left(-2\frac{1}{1+I\sqrt{23}}\right)^t}{1+I\sqrt{23}} + 2 + \frac{\left(\frac{10}{23} I\sqrt{23}-14\right)\left(-2\frac{1}{1-I\sqrt{23}}\right)^t}{1-I\sqrt{23}} + \frac{\left(-\frac{10}{23} I\sqrt{23}-14\right)\left(-2\frac{1}{1+I\sqrt{23}}\right)^t}{1+I\sqrt{23}}$$

```
> seriesp:=seq([t,evalc(solp)],t=0..20):
> plot([seriesp],labels=["t","p(t)"]);
```



```
> eqp:=evalf(solp);
```

$$eqp := (0.+1.251086484\,I)\,(-.08333333334-.3996526270\,I)^t + (0.-1.251086484\,I)\,(-.08333333334+.3996526270\,I)^t+2.$$

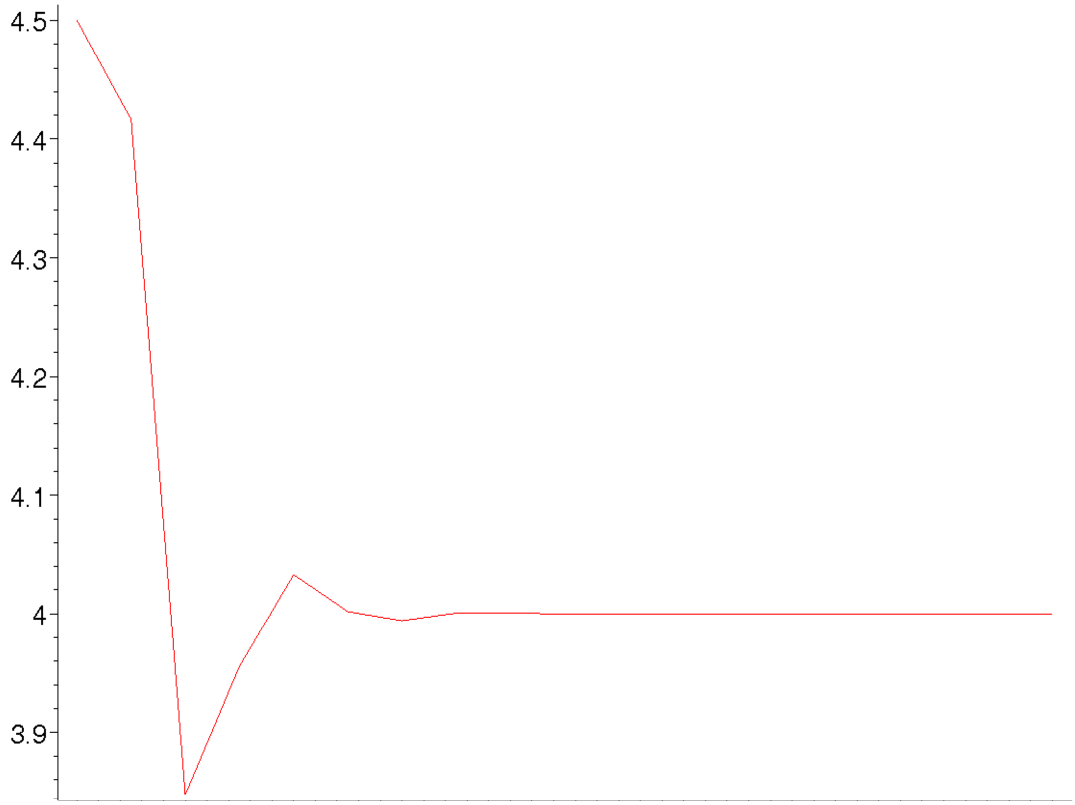
.Furthermore, $q(t) = c + d\,(p(t-1) - e\,(p(t-1) - p(t-2)))$. Substituting the same values we find $q(t) = 2 + .5\,p(t-1) + .5\,p(t-2)$.

```
> f:=t->(1.000000000+3.961773867*I)*(-.8333333334e-1-.399652627
0*I)^t+(1.000000000-3.961773867*I)*(-.8333333334e-1+.39965262
70*I)^t+2.+(-1.000000000-2.710687383*I)*(-.8333333334e-1-.399
6526270*I)^t+(-1.000000000+2.710687383*I)*(-.8333333334e-1+.3
996526270*I)^t;
```

$$f := t \to (0.+1.251086484\,I)\,(-.08333333334-.3996526270\,I)^t + (0.-1.251086484\,I)\,(-.08333333334+.3996526270\,I)^t+2.$$

```
> seriesq:=evalf(seq(2+0.5*f(t-1)+0.5*f(t-2),t=2..20)):
> dataq:=evalf(seq([t,2+0.5*f(t-1)+0.5*f(t-2)],t=2..20)):
```

```
> plot([dataq]);
```



Question 11

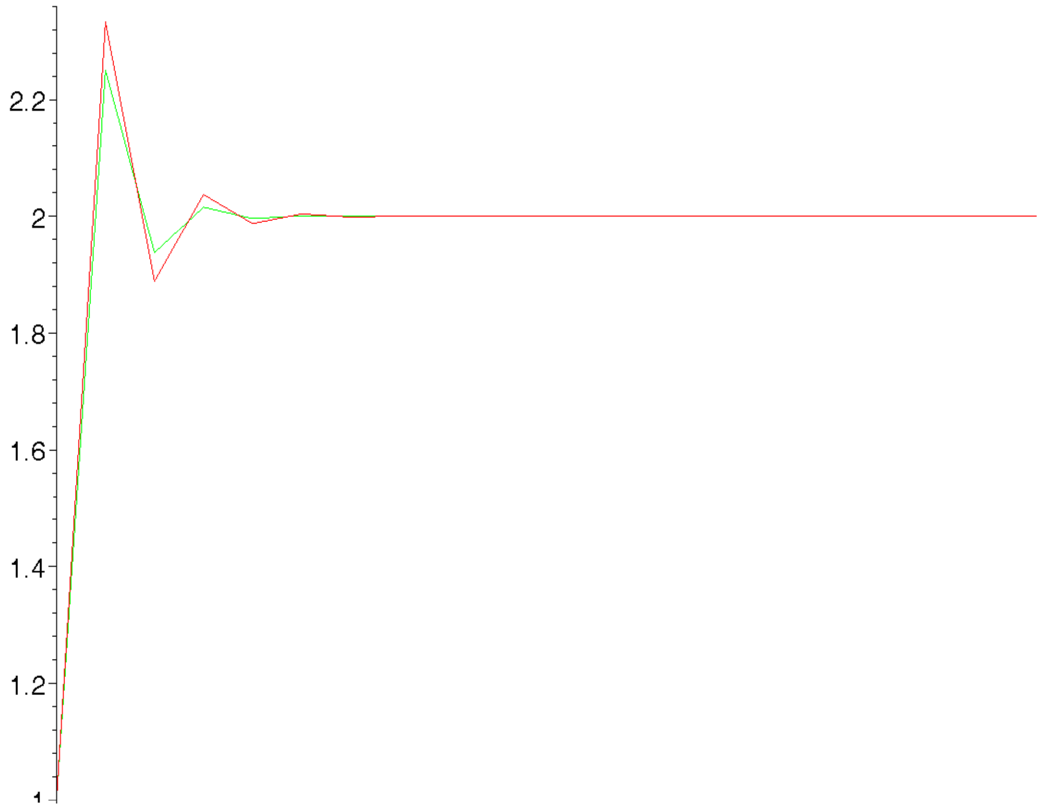
```
> p:='p':
> ds1:=rsolve({p(t)=(8/3)-(1/3)*p(t-1),p(0)=1},p(t));
```

$$ds1 := -\left(\frac{-1}{3}\right)^t + 2$$

```
> ds2:=rsolve({p(t)=(10/4)-(1/4)*p(t-1),p(0)=1},p(t));
```

$$ds2 := -\left(\frac{-1}{4}\right)^t + 2$$

```
> ds1list:=evalf(seq([t,-(-1/3)^t+2],t=0..20)):
> ds2list:=evalf(seq([t,-(-1/4)^t+2],t=0..20)):
> plot({[ds1list],[ds2list]});
```



2.2
2
1.8
1.6
1.4
1.2
1

t	$-\left(\frac{-1}{3}\right)^t+2$	$-\left(\frac{-1}{4}\right)^t+2$
1	2.3333	2.2500
2	1.8888	1.9375
3	2.0370	2.0156
4	1.9876	1.9960
5	2.0041	2.0009
6	1.9986	1.9997
7	2.0004	2.0000
8	1.9998	1.9999
9	2.0000	2.0000
10	1.9999	1.9999
11	2.0000	2.0000
12	1.9999	1.9999
13	2.0000	2.0000
14	1.9999	1.9999
15	2.0000	2.0000
16	1.9999	2.0000
17	2.0000	2.0000
18	1.9999	2.0000
19	2.0000	2.0000
20	2.0000	2.0000

```
> pstar1:=solve(p=(8/3)-(1/3)*p,p);
```

$$pstar1 := 2$$

```
> pstar2:=solve(p=(10/4)-(1/4)*p,p);
```

$$pstar2 := 2$$

```
> ds3:=rsolve({p(t)=(14/3)-(4/3)*p(t-1),p(0)=1},p(t));
```

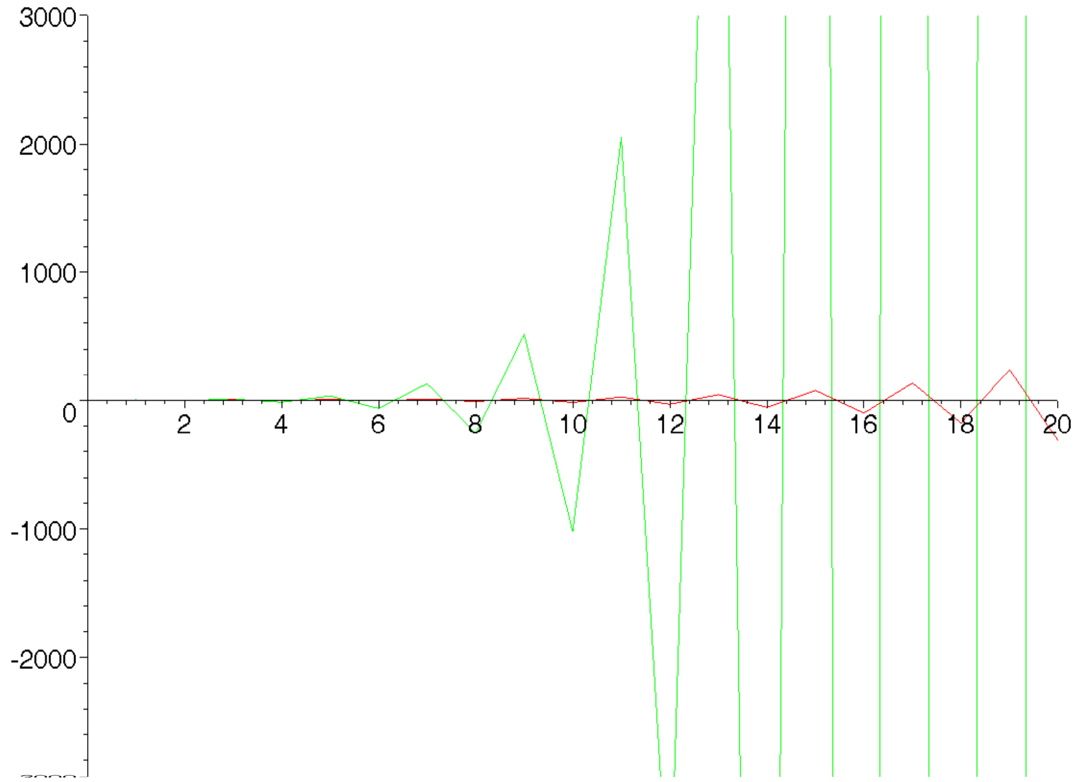
$$ds3 := -\left(\frac{-4}{3}\right)^t + 2$$

```
> ds4:=rsolve({p(t)=6-2*p(t-1),p(0)=1},p(t));
```

$$ds4 := -(-2)^t + 2$$

```
> d3list:=evalf(seq([t,-(-4/3)^t+2],t=0..20)):
```

```
> d4list:=evalf(seq([t,-(-2)^t+2],t=0..20)):
> plot({[d3list],[d4list]},0..20,-3000..3000);
```



Question 12

```
> p:='p':
> pbar:=solve(25-4*p=3+4*p,p);
```

$$pbar := \frac{11}{4}$$

```
> qbar:=25-4*pbar;
```

$$qbar := 14$$

```
> newp:=rsolve({25-4*p(t)=3+4*p(t-1),p(0)=p0},p(t));
```

$$newp := p0\,(-1)^t - \frac{11}{4}(-1)^t + \frac{11}{4}$$

```
> newp1:=subs(p0=1,newp);
```

$$newp1 := -\frac{7}{4}(-1)^t + \frac{11}{4}$$

```
> newlistp1:=seq(newp1,t=0..20);
```

$$newlistp1 := 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1, \frac{9}{2}, 1$$

```
> newp2:=subs(p0=3,newp);
```

$$newp2 := \frac{1}{4}(-1)^t + \frac{11}{4}$$

```
> newlistp2:=seq(newp2,t=0..20);
```

$$newlistp2 := 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3, \frac{5}{2}, 3$$

Question 13

Here we use *Maple* to solve the 3-cycle problem.

```
> f:='f':x:='x':eq1:='eq1':sol1:='sol1':
> f:=x->3.84*x*(1-x);
```

$$f := x \to 3.84\, x\, (1 - x)$$

```
> eq1:=x->(f@@3)(x);
```

$$eq1 := f^{(3)}$$

```
> sol1=solve(eq1(x)=x,x);
```

$$sol1 = (0., .7395833333, .1494068966, .1694338197, .4880043871, .5403878416, .9537362774, .9594474442)$$

Question 14

Assume $y_1 = Ystar$ and $y_2 = t\, Ystar$ are linearly dependent, then

$$b_1\, Ystar + b_2\, t\, Ystar = 0$$

which implies that $b_1 + b_2\, t = 0$ for all t only if $b_1 = b_2 = 0$. Hence $Ystar$ and $t\, Ystar$ are linearly dependent.

Question 15

(i)

Assume F is linearly homogeneous, then

$$\frac{Y_t}{A_t L_t} = \mathrm{F}\left(\frac{K_t}{A_t L_t}, 1\right) = \mathrm{f}(khat_t)$$

where we shall denote efficiency units throughout as $xhat$ meaning $xhat = \frac{x}{A\, L}$ for any arbitrary variable x.

Since $K_{t+1} - (1 - \delta)\, K_t = s\, Y_t$ then

$$\frac{K_{t+1}}{A_t L_t} - \frac{(1-\delta)\, K_t}{A_t L_t} = \frac{s\, Y_t}{A_t L_t}$$

$$\frac{K_{t+1} A_{t+1} L_{t+1}}{A_t L_t A_{t+1} L_{t+1}} - \frac{(1-\delta)\, K_t}{A_t L_t} = \frac{s\, Y_t}{A_t L_t}$$

$$khat_{t+1}\left(\frac{A_{t+1} L_{t+1}}{A_t L_t} - (1-\delta)\, khat_t = s\, \mathrm{f}(khat_t)\right)$$

But

$$\frac{A_{t+1} L_{t+1}}{A_t L_t} = \frac{\gamma^{(t+1)} A_0 L_{t+1}}{L_t \gamma^t A_0} = \gamma\, (n+1)$$

Therefore

$$\gamma\, (n+1)\, khat_t - (1-\delta)\, khat_t = s\, \mathrm{f}(khat_t)$$

or

$$khat_{t+1} = \frac{(1-\delta)\, khat_t + f\, \mathrm{f}(khat_t)}{\gamma\,(1+n)}$$

(ii)

Let *khat_star* denote the positive equilibrium, then

$$khat_{t+1} = \mathrm{f}(khat\_star) \left(\frac{\partial}{\partial khat} khat\_star \right) (khat_t - khat\_star)$$

But

$$\mathrm{f}(khat\_star) = khat\_star$$

and

$$\frac{\partial}{\partial khat} khat\_star = \frac{1-\delta}{\gamma\,(1+n)} + \frac{s\, \mathrm{f}(khat\_star)}{\gamma\,(1+n)}$$

hence

$$khat_{t+1} = khat\_star + \left(\frac{1-\delta}{\gamma\,(1+n)} + \frac{s\, \mathrm{f}(khat\_star)}{\gamma\,(1+n)} \right) (khat_t - khat\_star)$$

Question 16

Solving for $\mathrm{pe}(t-1)$ gives $\mathrm{pe}(t-1) = \frac{a-c}{d} - \frac{b\,\mathrm{p}(t-1)}{d}$, using this we obtain the difference equation

$$\mathrm{p}(t) = \left(1 - \frac{\lambda\,(b+d)}{b}\right) \mathrm{p}(t-1) + \frac{\lambda\,(a-c)}{b}$$

```
> pbar:='pbar':qbar:='qbar':p:='p':q:='q':a:='a':b:='b':c:='c':
  d:='d':
> pbar:=solve(p=(1-lambda*(b+d)/b)*p+(lambda*(a-c)/b),p);
```

$$pbar := \frac{a-c}{b+d}$$

```
> qbar:=a-b*pbar;
```

$$qbar := a - \frac{b\,(a-c)}{b+d}$$

```
> simplify(qbar);
```

$$\frac{a\,d + b\,c}{b+d}$$

```
> rsolve({u(t)=(1-lambda*(b+d)/b)*u(t-1),u(0)=u0},u(t));
```

$$u0 \left(-\frac{-b + \lambda\, b + \lambda\, d}{b} \right)^t$$

Which can be expressed,

$$u0 \left(1 - \frac{(b+d)\,\lambda}{b} \right)^t$$

```
> solve(1-(b+d)*lambda/b=-1,lambda);
```

$$2\frac{b}{b+d}$$

```
> solve(1-(b+d)*lambda/b=1,lambda);
```

$$0$$

Question 17

Our equation is $P(n) = (1+r)^n\left(P0 - \frac{R}{r}\right) + \frac{R}{r}$. In the present problem, taking note that we are dealing in months and the interest rate is per annum, we require zero payment after n = 3x12 = 36 monthly payments. Hence:

$P(n) = 0$, $P0 = 8000$, $r = \frac{.075}{12} = 0.00625$ and $R = m$ = fixed monthly payment.

```
> fsolve((1+0.00625)^36*(8000-(m/(0.075/12)))+(m/(0.075/12))=0,
  m);
```

$$248.8497447$$

Hence the monthly payment is £248.85.

Question 18

Since,

$n_t = 2\,n_{t-1}$ and $n_0 = 1$

```
> rsolve({n(t)=2*n(t-1),n(0)=1},n(t));
```

$$2^t$$

```
> fsolve(5*10^6=2^t,t);
```

$$22.25349666$$

Hence, the bacteria becomes contagious in just over twenty-two minutes.

Question 19

Given the recursive equation

$x_{t+1} = \frac{x_t}{1+x_t}$ with $x(0) = x_0$

```
> rsolve({x(t+1)=x(t)/(1+x(t)),x(0)=x0},x(t));
```

$$\mathrm{rsolve}\left(\left\{x(0) = x0,\ x(t+1) = \frac{x(t)}{1+x(t)}\right\},\ x(t)\right)$$

The fact that *Maple* returns the expression as entered means that it cannot solve it.

But now define

```
> f:=x->x/(1+x);
```

$$f := x \rightarrow \frac{x}{1+x}$$

```
> seq(simplify((f@@n)(x0)),n=0..10);
```

$$x0, \frac{x0}{1+x0}, \frac{x0}{1+2\,x0}, \frac{x0}{1+3\,x0}, \frac{x0}{1+4\,x0}, \frac{x0}{1+5\,x0}, \frac{x0}{1+6\,x0}, \frac{x0}{1+7\,x0}, \frac{x0}{1+8\,x0}, \frac{x0}{1+9\,x0},$$

$$\frac{x0}{1+10\,x0}$$

The solution is clearly, then,

$$x_n = \frac{x_0}{1+n\,x_0}$$

Question 20

(a) Here we shall use *Maple's* spreadsheet to do this.

	A	B
1	n	$x_{n-1}+x_{n-2}$
2	0	1
3	1	1
4	2	2
5	3	3
6	4	5
7	5	8
8	6	13
9	7	21
10	8	34
11	9	55
12	10	89

(b) Solving

```
> soln:=rsolve({x(n)=x(n-1)+x(n-2),x(0)=1,x(1)=1},x(n));
```

$$soln := \frac{2}{5}\,\frac{\sqrt{5}\left(2\,\frac{1}{-1+\sqrt{5}}\right)^n}{-1+\sqrt{5}} + \frac{\frac{2}{5}\sqrt{5}\left(-2\,\frac{1}{1+\sqrt{5}}\right)^n}{1+\sqrt{5}}$$

```
> evalf(seq(soln,n=0..10));
```

.9999999999, .9999999998, 1.999999998, 2.999999996, 4.999999991, 7.999999982, 12.99999997, 20.99999994, 33.99999989, 54.99999980, 88.99999959

Which to even four decimal places gives the same sequence of numbers as in the spreadsheet.