Solutions to exercises in chapter 7

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1. **Strain guages**

   a) The resistance of the strain guage is $R = \rho l/A$. Under strain, there is a change in length, given by $\Delta l = l_2 - l_1 = \epsilon \cdot l_1$ or: $l_2 = l_1(1+\epsilon)$ as well as a change in cross-section (contraction due to Poisson's ratio) to $A_2 = A_1(1-m\epsilon)^2$ - both width and height decrease by a factor of $(1-m\epsilon)$. Therefore, the resistance is changed to: $R = R(0)\frac{1+\epsilon}{(1-m\epsilon)^2}$ after application of a strain ϵ.

   b) For this we have to invert the relation between R and ϵ, $R(\epsilon)$. To do this, we use the fact that strains are typically small, such that we can approximate the relation via a first order Taylor expansion using: $\frac{\partial R}{\partial \epsilon} = R(0)\frac{(1-m\epsilon)+2m(1+\epsilon)}{(1-m\epsilon)^3}$. We are expanding around $\epsilon = 0$, hence we need to evaluate this at zero and obtain $R(0)(1+2m)$, therefore the resistance is approximately: $R = R(0)(1+2m)\epsilon$ or inverted $\epsilon = \frac{R}{1.68 \cdot R(0)}$. Hence the uncertainty of the determination of ϵ is also given by one percent.

2. **Freely hanging rope**

   a) The stress in the rope at height z is due to the weight of the rope underneath this height, i.e. $\sigma(z) = \rho g(l-z)A/A = \rho g(l-z)$. The maximum stress therefore occurs at the top at $z = 0$ and is $\sigma_{max} = \rho g l$. If the rope breaks, this has to be equal to the yield stress, i.e. $\sigma_{max} = \rho g l = \sigma_Y$ and hence the maximum length is given by $l = \frac{\sigma_Y}{\rho g} = 5 \cdot 10^8 kg/(s^2 m) 8 \cdot 10^3 kg/m^3 10 m/s^2 = 6 \cdot 10^3 m = 6km$

   b) This maximum length is a ratio of σ_Y and ρ, such that $r_l^2 = r_\rho^2 + r_\sigma^2$, with $r_\rho = 0.01$ and $r_\sigma = 0.02$. This gives $r_l = \sqrt{5}\%$.

   c) The rope's extension Δl increases with the length of the rope and the strain ϵ is just the change in extension with this length, i.e. $\epsilon = \frac{d\Delta l}{dz}$. According to Hooke ($\sigma = E\epsilon$), we have with the stress from a): $E\frac{d\Delta l}{dz} = \rho g(l-z)$ or $\int d\Delta l = \frac{\rho g}{E}\int(l-z)dz$. Integrating gives: $\Delta l = \frac{\rho g}{E}(l^2 - l^2/2) = \frac{\rho g l^2}{2E}$. Finally, we insert the maximum length and obtain: $\Delta l = \frac{\sigma_Y^2}{2E\rho g} = \frac{25 \cdot 10^{16} Pa N/m^2}{4 \cdot 10^{11} Pa 8 \cdot 10^4 N/m^3} = 250/32 m \simeq 8m$

   d) The extension is a ration of σ_Y^2, E and ρ, such that $r_{\Delta l}^2 = 4r_\rho^2 + r_\sigma^2 + r_E^2$, where $r_\rho = r_E = 0.01$ and $r_\sigma = 0.02$. This gives $r_{\Delta l} = 3\%$.

3. **Beam bending**

   a) The maximum tension (on top at the fixed end) for a bending force G is given by: $\sigma_{max} = \frac{GLs}{2I}$, where L is the length of the beam, s its width and I the area moment of inertia. For a beam with a square cross section, this is $I = s^4/12$, and hence $\sigma_{max} = \frac{6GL}{s^3}$. For the beam to break, this needs to exceed the yield stress, i.e. $\sigma_{max} = \sigma_Y$, which gives the maximum force as: $G = \frac{\sigma_Y s^3}{6L} = \frac{5 \cdot 10^8 Pa 8^3 \cdot 10^{-27} m^3}{10^{-4} m} = 5 \cdot 8^3 \cdot 10^{-15}$ N $\simeq$ 2.5pN.

   b) The width enters with the third power, whereas the other parameters (σ_Y und L) each enter linearly. Since only σ_Y has an uncertainty, the relative uncertainty of the force is the same as that of the yield stress, i.e. $r_G = 10\%$.

   c) Here we have to change the area moment of inertia for a hollow beam. With d the outer width and w the wall thickness, the area moment of inertia is $I = \frac{d^4}{12} - \frac{(d-2w)^4}{12} = \frac{d^4}{12}(1-(1-\frac{2w}{d})^4) \simeq \frac{d^4}{12}(1-(1-4\frac{2w}{d})) = \frac{d^4}{12}4\frac{2w}{d} = \frac{2d^3w}{3}$. Inserting this in the maximum stress gives $\sigma_{max} = \frac{GLd}{2I} = \frac{3GL}{4d^2w}$ and hence $G = \frac{4\sigma_Y d^2 w}{3L} = \frac{4 \cdot 5 \cdot 10^8 Pa 25^2 cdot 4 \cdot 10^{-27} m^3}{10^{-4} m} = 80 \cdot 625 \cdot 10^{-15}$ N $= 50$ pN.

4. **Elastic properties of materials**

   a) Hooke says: $\sigma = E\epsilon$ with the values given therefore $\sigma_Y = 2 \cdot 10^9$ Pa $\cdot 0.5 = 10^9$ Pa. This is twice the yield stress of steel ($\sigma_Y = 5 \cdot 10^8$ Pa)!

   b) Der relative Fehler in σ_Y ist $r_\sigma^2 = r_E^2 + r_\epsilon^2$. Nach Angaben sind die Fehler $r_E = 0.05$ und $r_\epsilon = 0.1$, das heisst: $r_\sigma^2 = \cdot(0.05)^2 + 0.1^2 = 1.25 \cdot 0.1^2$. Damit wird der relative Fehler der Zerreiss-Spannung: $r_\sigma \simeq 11\%$.

c) Again according to Hooke: $\epsilon = \sigma/E$. Using the given values: $\epsilon = \frac{5\cdot10^8 Pa}{2\cdot10^11 Pa} = 2.5\cdot10^{-3}$. So it breaks after extension by only one quarter percent!

5. **Interaction between molecules**

   a) $F = -\frac{\partial E_{pot}}{\partial r}$. For the first term we obtain $\frac{-6M}{r^7}$, and for the second term $\frac{12N}{r^{13}}$. The first force (negative sign) acts against the separation and therefore is attractive. The second one in contrast is repulsive.

   (b) The equilibrium position is at the place where the two forces from a) are equal and opposite, i.e. $\frac{6M}{r_{min}^7} = \frac{12N}{r_{min}^{13}}$. We could get the same criterion from determining the minimum of the total energy. Solving this equation gives: $r_{min}^6 = 2N/M$. At this position, the potential energy is $E_{pot}(r_{min}) = -\frac{M}{r_{min}^6} + \frac{N}{r_{min}^{12}} = -\frac{M}{2N/M} + \frac{N}{4N^2/M^2} = -\frac{M^2}{2N} + \frac{NM^2}{4N^2} = -\frac{M^2}{2N} + \frac{M^2}{4N} = -\frac{M^2}{4N}$.

   (c) Drawing

   (d) From (b) we have: $r_{min} = (2N/M)^{1/6}$ and $E_{pot}(r_{min}) = -\frac{M^2}{4N}$. Thus we have for the relative errors: $r_r^2 = 1/36(r_M^2 + r_N^2)$ and $r_E^2 = 4r_M^2 + r_N^2$. With $r_M = r_N = 0.01$, we therefore obtain: $r_r = 0.01\cdot\sqrt{2}/6 = \sqrt{2}/6\% \simeq 0.25\%$. and $r_E = \sqrt{5}\cdot 0.01 = \sqrt{5}\% \simeq 2.2\%$.

6. **Laplace pressure**

   a) Laplace pressure for a sphere is: $p = 2\sigma/R$. With $\sigma = 0.075(3)$ N/m and $R = 50(2)\mu$m, we obtain a pressure of $2\cdot 7.5\cdot 10^{-2}/5\cdot 10^{-5}$ Pa = 3000 Pa. The relative error of this is given by the relative errors of σ and R: $r_p^2 = r_\sigma^2 + r_R^2 = (2/50)^2 + (3/75)^2 = 2/25^2$. Hence $r_p = \sqrt{2}\cdot 4\% \simeq 6\%$ or $\sigma_p = 180$ Pa.

   b) Rearranging Laplace gives $R = 2\sigma/p$. With a pressure difference of 1000 Pa and a surface tension of 0.03 N/m, we obtain $R = \frac{0.06N/m}{1000N/m^2} = \frac{6\cdot10^{-2}}{10^3}$m $= 6\cdot10^{-5}$ m = 60 μ m.

7. **Laplace pressure 2**

   a) For a cylinder, Laplace pressure is: $p = \sigma/R$. For a blood pressure of 4 kPa and a radius of 6 μm, there is a tension of $\sigma = pR = 4\cdot10^3 N/m^2 6\cdot10^{-6}m = 24\cdot10^{-3}N/m$ or $0.024N/m$. This is less than the surface tension of water.

   b) Maximum pressure: $\frac{500N/m}{0.5cm} = \frac{5\cdot10^2}{5\cdot10^{-3}}Pa = 10^5 Pa$. Aneurisms therefore should normally not be a problem.

8. **Equation of continuity**

   The equation of continuity demands $vA = const$. A tube with half the diameter has one quarter of the cross-sectional area. Two tubes with half the diameter each thus have half the total cross-sectional area. Such that vA remains constant, the speed needs to double.

9. **Equation of continuity and Bernoulli**

   a) According to Bernoulli: $p + \rho v^2/2 = const$. When pressing on the syringe, there is an additional pressure of F/A. Therefore the speed from the syringe is $\sqrt{2F/(A\rho)}$. Numerically: $v = \sqrt{2N/(10^3kg/m^3 8010^{-6}m^2)} = \sqrt{\frac{2kgm/s^2}{810^{-2}kg/m}} = \sqrt{1/410^2m^2/s^2} = 1/210m/s = 5m/s$

   b) From $v = \sqrt{2F/(A\rho)}$ we see, that the uncertainties come from powers of F and A. Therefore, the relative error of v is given by $r_v = 1/2\sqrt{r_F^2 + r_A^2}$. The problem states that the errors are 5% each, thus $r_v = 5/\sqrt{2}\% \simeq 3.7\%$. or for the absolute error: $\sigma_v \simeq 0.2m/s$.

10. **Resistance to flow**

    For tubes in parallel, we have: $1/R = 1/R_1 + 1/R_2$, with $R \propto L/r^4$. Therefore: $\rho^4/L = r^4/L + r^4/L = 2r^4/L$. Multiplying by L and taking the fourth root gives: $\rho = \sqrt[4]{2}r$.

11. **Resistance to flow 2**

    For two tubes in series, we have: $R = R_1 + R_2$ and again $R \propto L/r^4$. Therefore: $2L/\rho^4 = L/r_1^4 + L/r_2^4 = L/(2r_2)^4 + L/r_2^4 = L/r_2^4(1 + 1/16)$. Dividing by L and solving for ρ we get: $\rho = \left(\frac{2}{1+1/16}\right)^{1/4} r_2 \simeq \sqrt[4]{2}r_2$.