Mathematical appendix 1

The Equilibrium Tide

The following arguments develop the ideas outlined in Section 2.3 (p. 37) and Appendix 1.

Substitute for $\cos\phi$ in Equation (A1.1) from the expansion on p. 235. After some rearrangement this leads to:

$$-\Omega_{\mathrm{P}} = \frac{3}{2} ag \frac{m_1}{m_{\mathrm{e}}} \left(\frac{a}{r_2}\right)^3 \left[\frac{3}{2}\left(\sin^2 d_2 - \frac{1}{3}\right)\left(\sin^2\phi_{\mathrm{P}} - \frac{1}{3}\right) + \frac{1}{2}\sin 2d_2 \sin 2\phi_{\mathrm{P}} \cos C_{\mathrm{P}} + \frac{1}{2}\cos^2 d_2 \cos^2\phi_{\mathrm{P}} \cos 2C_{\mathrm{P}}\right] \tag{1}$$

In the equilibrium theory of tides the free surface is assumed to be a level surface under the combined influence of the Earth's gravity and the tidal disturbing force. The horizontal tide generating forces may be regarded as causing a deflection of the vertical, as shown in Figure 1.

The force is $-(\partial\Omega/\partial x)$ where x is a direction at right angles to the direction of undisturbed gravity (see Appendix 1, p. 233)

Consider the magnitude of the forces:

$$\tan\alpha = -\left(\frac{\partial\Omega_{\mathrm{P}}}{\partial x}\right) \Big/ g$$

and also:

$$\tan\alpha = \left(\frac{\partial\bar{\zeta}}{\partial x}\right)$$

so that:

$$g\frac{\partial\bar{\zeta}}{\partial x} + \frac{\partial\Omega_{\mathrm{P}}}{\partial x} = 0$$

or:

$$\frac{\partial}{\partial x}\left(g\bar{\zeta} + \Omega_{\mathrm{P}}\right) = 0$$

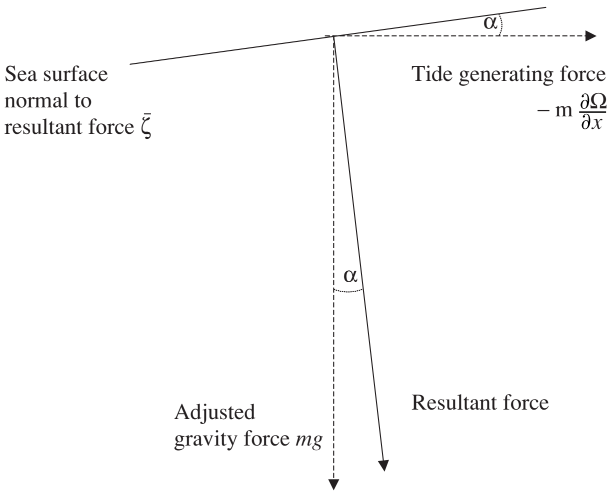

Figure 1

and similarly:

$$\frac{\partial}{\partial y}\left(g\bar{\zeta} + \Omega_{\mathrm{P}}\right) = 0$$

Integrating over a finite area gives:

$$g\bar{\zeta} + \Omega_{\mathrm{P}} = \text{constant}$$

But if the integral is taken over the whole area of the ocean so that the total volume of water is conserved, then the constant is zero. Applying this condition to Equation 1, the Equilibrium Tide becomes:

$$\bar{\zeta} = a\left(\frac{m_2}{m_\mathrm{e}}\right)\left[C_0(t)\left(\frac{3}{2}\sin^2\phi_\mathrm{P} - \frac{1}{2}\right) + C_1(t)\sin 2\phi_\mathrm{P} + C_2(t)\cos^2\phi_\mathrm{P}\right]$$

where the time-dependent coefficients are:

$$C_0(t) = \left(\frac{a}{r_2}\right)^3\left(\frac{3}{2}\sin^2 d_2 - \frac{1}{2}\right)$$
$$C_1(t) = \left(\frac{a}{r_2}\right)^3\left(\frac{3}{4}\sin 2d_2\cos C_\mathrm{P}\right)$$
$$C_2(t) = \left(\frac{a}{r_2}\right)^3\left(\frac{3}{4}\cos^2 d_2\cos 2C_\mathrm{P}\right)$$

The three coefficients characterise the three main species of tides: the long-period species, the diurnal species at a frequency of one cycle per day ($\cos C_\mathrm{P}$), and the semidiurnal species at two cycles per day ($\cos 2C_\mathrm{P}$). The magnitudes of all three species are modulated by a common term that varies inversely as the cube of the lunar distance r_1.

The long-period tidal species is also produced because of the monthly variations in lunar declination d_2. It has maximum amplitude at the poles and zero amplitude at latitudes $35^\circ\ 16'$, north and south of the equator.

The diurnal species is modulated at twice the frequency with which the lunar declination varies and it has its maximum amplitude at 45° latitude, and zero amplitude at the equator and the poles. The variations north and south of the equator are in opposite phase.

The semidiurnal species is also modulated at twice the frequency of the lunar declination, but is a maximum when the declination is zero. It has a maximum amplitude at the equator, and zero amplitude at the poles.