Solutions to the Tutorial Problems in the book "Magnetohydrodynamics of the Sun" by ER Priest (2014) CHAPTER 3

PROBLEM 3.1. Vector Potential.
In general, Faraday's law implies that $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Phi$, where Φ is arbitrary. Show that $\mathbf{A}$ may be redefined to include Φ. In the case of infinite magnetic Reynolds number, deduce the equation that determines $\mathbf{A}$ when $\mathbf{v}$ is given, and also the change $\delta\mathbf{A}$ in $\mathbf{A}$ produced by a displacement $\boldsymbol{\xi}$.

SOLUTION. We follow here the discussion in Roberts, P.H. (An Introduction to Magnetohydrodynamics, Longmans, 1967). Suppose

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A},$$

where there are many different choices for $\mathbf{A}$ since it is arbitrary to within the gradient of a gauge (Ψ). In other words, adding $\boldsymbol{\nabla}\Psi$ to $\mathbf{A}$, where Ψ is any single-valued function, will not change $\mathbf{B}$.

Now, Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

implies that

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0},$$

so that

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} - \boldsymbol{\nabla}\Phi,$$

where Φ is any single-valued scalar. In a steady state, Φ is the usual electrostatic potential, but more generally it depends on the choice of gauge of $\mathbf{A}$.

The arbitrariness in $\mathbf{A}$ may be removed by requiring it to satisfy some extra condition that simplifies the formalism of whatever problem is being tackled. In MHD a common condition is to impose $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, whereas for electromagnetic radiation one often assumes $\boldsymbol{\nabla} \cdot \mathbf{A} = -\partial\Phi/\partial t$. Here instead let us define $\mathbf{A}$ to include the Φ term – i.e., to introduce a new function such that

$$\tilde{\mathbf{A}} = \mathbf{A} + \int \boldsymbol{\nabla}\Phi dt,$$

so that, after dropping the tilde, the equation for $\partial \mathbf{A}/\partial t$ reduces to

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}.$$

Note that, since free charges are usually present, in this case $\boldsymbol{\nabla} \cdot \mathbf{E} \neq \mathbf{0}$ and so $\boldsymbol{\nabla} \cdot \mathbf{A} \neq \mathbf{0}$.

In particular, for an ideal plasma (when the magnetic Reynolds number is infinite), Ohm's law ($\mathbf{E} = -\mathbf{v} \times \mathbf{B}$) for $\mathbf{E}$ reduces the above equation to

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{v} \times \mathbf{B} = \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}).$$

Note that one of the unpleasant side effects of this gauge is that in a steady state $\mathbf{A}$ is proportional to t. When the magnetic Reynolds number is infinite, the above equation still holds but with $\mathbf{v}$ replaced by a flux velocity $\mathbf{w}$ provided it exists (which is not always the case – see Sec. 6.12.3).

If a small displacement $\boldsymbol{\xi}$ is produced by a small velocity $\mathbf{v} = \partial \boldsymbol{\xi}/\partial t$ acting on an initial field $\mathbf{B}_0$, linearising the above equation shows that the resulting change $\delta \mathbf{A}$ in $\mathbf{A}$ is given by

$$\delta \mathbf{A} = \boldsymbol{\xi} \times \mathbf{B}_0.$$

PROBLEM 3.2. Setting Up Euler Potentials.
If two coordinates (f, g) label field lines, show that in general they may be redefined to make $\mathbf{B} = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g$.

SOLUTION. Since f and g are constant along magnetic field lines, $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$ are perpendicular to $\mathbf{B}$. Therefore, following Roberts(1967), there exists a scalar function (α) of position such that

$$\mathbf{B} = \alpha \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g.$$

Taking the divergence and using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ gives

$$\boldsymbol{\nabla} \alpha \cdot \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g = 0.$$

This is equivalent to

$$\frac{\partial(\alpha, f, g)}{\partial(x, y, z)} = 0,$$

which implies that

$$\alpha = \alpha(f, g),$$

so that α is constant on field lines.

Therefore, we may redfine either f or g to absorb α. For example, if we introduce

$$\tilde{g}(f, g) = \int^g \alpha(f, g')dg' + \beta(f),$$

where β is an arbitrary function of f, then

$$\alpha \boldsymbol{\nabla} g = \boldsymbol{\nabla} \tilde{g} - \left(\int^g \frac{\partial \alpha}{\partial f} dg' + \frac{d\beta}{df} \right) \boldsymbol{\nabla} f.$$

Thus, substituting for $\alpha \boldsymbol{\nabla} g$ into the expression for $\mathbf{B}$ gives $\mathbf{B} = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} \tilde{g}$, or, after dropping the tilde,

$$\mathbf{B} = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g,$$

as required. Note that this representation is far from unique.

PROBLEM 3.3. **Vector Potential and Euler Potentials.**
How can the vector potential $\mathbf{A}$ be written in terms of Euler potentials? (i) Choose the gauge to make $\mathbf{B} \cdot \mathbf{A} = 0$. (ii) Choose the gauge instead to make a change $\delta \mathbf{A}$ produced by a displacement $\boldsymbol{\xi}$ normal to $\mathbf{B}$, and comment on the consequences.

SOLUTION. We again here follow Roberts (1967). We have by definition

$$\mathbf{B} = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g,$$

but this can be rewritten

$$\mathbf{B} = \boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g).$$

Thus, if $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, then in general

$$\mathbf{A} = f \boldsymbol{\nabla} g + \boldsymbol{\nabla} h,$$

say. Often f, g and h (sometimes called *Clebsch variables*) are more convenient variables than A_x, A_y, and A_z.

(i) There are many choices of gauge, the most convenient depending on the problem. For instance, if $h \equiv 0$, we have

$$\mathbf{A} = f\boldsymbol{\nabla} g,$$

and so

$$\mathbf{B} \cdot \mathbf{A} = \boldsymbol{\nabla} f \times \boldsymbol{\nabla} g \cdot f\boldsymbol{\nabla} g = 0.$$

(ii) As another example, if $\mathbf{B} + \delta\mathbf{B}$ is a neighbouring field with corresponding vector potential $\mathbf{A} + \delta\mathbf{A}$, then we have, after substituting in terms of f, g and h,

$$\delta\mathbf{A} = \delta f\boldsymbol{\nabla} g + f\boldsymbol{\nabla}\delta g + \boldsymbol{\nabla}\delta h$$

or

$$\delta\mathbf{A} = \delta f\boldsymbol{\nabla} g - \delta g\boldsymbol{\nabla} f + \boldsymbol{\nabla}(\delta h + f\delta g).$$

Thus, if we choose $\delta h = -f\delta g$,

$$\delta\mathbf{A} = \delta f\boldsymbol{\nabla} g - \delta g\boldsymbol{\nabla} f,$$

and so $\delta\mathbf{A}$ is normal to $\mathbf{B}$. Furthermore, it can be written as

$$\delta\mathbf{A} = \boldsymbol{\xi} \times \mathbf{B} = \boldsymbol{\xi} \times (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g),$$

or

$$\delta\mathbf{A} = -(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} f)\boldsymbol{\nabla} g + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla} g)\boldsymbol{\nabla} f.$$

This in turn implies that

$$\delta f + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} f = 0$$

and

$$\delta g + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} g = 0,$$

so that the surfaces $f = \textit{constant}$ and $g = \textit{constant}$ (and therefore their intersections, namely, the field lines) follow the motion.

PROBLEM 3.4. Nonlinear Force-Free Fields.
Show that

$$\boldsymbol{\nabla} \times \mathbf{B} = \alpha\mathbf{B} \tag{1}$$

implies that

$$(\nabla^2 + \alpha^2)\mathbf{B} = \mathbf{B} \times \boldsymbol{\nabla}\alpha \tag{2}$$

but that Eq.(**??**) does not necessarily imply Eq.(**??**).

SOLUTION. Take the curl of Eq.(**??**) to give

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla} \times (\alpha \mathbf{B})$$

or, expanding both sides

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \alpha \boldsymbol{\nabla} \times \mathbf{B} + (\boldsymbol{\nabla}\alpha) \times \mathbf{B}.$$

After using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and Eq.(**??**) for $\boldsymbol{\nabla} \times \mathbf{B}$ this becomes

$$-\nabla^2 \mathbf{B} = \alpha^2 \mathbf{B} + (\boldsymbol{\nabla}\alpha) \times \mathbf{B},$$

or

$$(\nabla^2 + \alpha^2)\mathbf{B} = \mathbf{B} \times \boldsymbol{\nabla}\alpha,$$

as required.

However, this is of higher order than the original Eq.(**??**), so in principle we would not expect all solutions of Eq.(**??**) to satisfy Eq.(**??**). This may be shown explicitly as follows. First of all, rearrange the terms in Eq.(**??**) to give

$$\nabla^2 \mathbf{B} + (\boldsymbol{\nabla}\alpha) \times \mathbf{B} = -\alpha^2 \mathbf{B}.$$

Then, after adding and subtracting $\alpha \boldsymbol{\nabla} \times \mathbf{B}$, it becomes

$$\nabla^2 \mathbf{B} + \alpha \boldsymbol{\nabla} \times \mathbf{B} + (\boldsymbol{\nabla}\alpha) \times \mathbf{B} = \alpha \boldsymbol{\nabla} \times \mathbf{B} - \alpha^2 \mathbf{B},$$

or

$$\nabla^2 \mathbf{B} + \boldsymbol{\nabla} \times (\alpha \mathbf{B}) = \alpha(\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B}).$$

Then, using a triple vector product identity and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, this implies

$$-\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \boldsymbol{\nabla} \times (\alpha \mathbf{B}) = \alpha(\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B})$$

or

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B}) + \alpha(\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B}) = \mathbf{0}.$$

In other words,

$$(\boldsymbol{\nabla} \times +\alpha)(\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B}) = \mathbf{0}.$$

Now this implies that

$$\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B} = \mathbf{C},$$

where

$$\boldsymbol{\nabla} \times \mathbf{C} + \alpha \mathbf{C} = \mathbf{0}.$$

Therefore one possible solution is $\mathbf{C} \equiv \mathbf{0}$, namely,

$$\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B} = \mathbf{0},$$

but there are many other solutions, namely, all those with $\boldsymbol{\nabla} \times \mathbf{B} - \alpha \mathbf{B} = \mathbf{C}$ and $\mathbf{C} \neq \mathbf{0}$, and so we have proved that Eq.(**??**) does not necessarily imply Eq.(**??**), as required.

PROBLEM 3.5. **Force-Free Fields in Euler Potentials.**
What are the equations for a linear force-free field in Euler potentials? Comment on the result.

SOLUTION. We have

$$\mathbf{B} = \boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g),$$

for which the electric current density is

$$\mu \mathbf{j} = \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (f \boldsymbol{\nabla} g)].$$

Expanding out the triple vector product gives

$$\mu \mathbf{j} = \boldsymbol{\nabla}[\boldsymbol{\nabla} \cdot (f \boldsymbol{\nabla} g)] - \nabla^2 (f \boldsymbol{\nabla} g)$$

or

$$\mu \mathbf{j} = \boldsymbol{\nabla}(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) + \boldsymbol{\nabla} f \nabla^2 g - \nabla^2 f \boldsymbol{\nabla} g.$$

For a linear force-free field, we have

$$\mu \mathbf{j} = \alpha \mathbf{B},$$

where α is a given constant, and so, after substituting for $\mathbf{j}$ and $\mathbf{B}$, we find from the x- and y-components the following two equations for $f(x, y, z)$ and $g(x, y, z)$:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) + \frac{\partial f}{\partial x}\nabla^2 g - \frac{\partial g}{\partial x}\nabla^2 f = \alpha\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right),$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right) + \frac{\partial f}{\partial y}\nabla^2 g - \frac{\partial g}{\partial y}\nabla^2 f = \alpha\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right).$$

Thus, the equations determining the Euler potentials are highly complex and nonlinear, even though the magnetic field itself is linear.

PROBLEM 3.6. Properties of Magnetostatic Fields.
Prove that for a magnetostatic field (i) $\boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} p) = 0$ and (ii) $(\mathbf{j} \cdot \boldsymbol{\nabla})\mathbf{B} = (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{j}$

SOLUTION. We have

$$\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} p,$$

where

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

and

$$\mathbf{j} = \boldsymbol{\nabla} \times \mathbf{B}/\mu.$$

(i) The expression

$$\boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} p)$$

may be expanded by operating first on $\mathbf{B}$ and then on $\boldsymbol{\nabla} p$ to give

$$\boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} p) = \boldsymbol{\nabla} p \cdot \boldsymbol{\nabla} \times \mathbf{B} + \mathbf{B} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} p,$$

or

$$\boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} p) = \mu \boldsymbol{\nabla} p \cdot \mathbf{j},$$

since the second term vanishes identically. Furthermore, $\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} p$ implies that $\boldsymbol{\nabla} p$ is perpendicular to $\mathbf{j}$ and so

$$\boldsymbol{\nabla} \cdot (\mathbf{B} \times \boldsymbol{\nabla} p) = 0,$$

as required.

(ii) Taking the curl of $\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} p$ implies

$$\boldsymbol{\nabla} \times (\mathbf{j} \times \mathbf{B}) = \mathbf{0},$$

or, after expanding the triple vector product and using $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$

$$(\mathbf{j} \cdot \boldsymbol{\nabla})\mathbf{B} - (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{j} = \mathbf{0},$$

as required.

PROBLEM 3.7. **General Solution for Magnetostatic Fields.**
Prove that the magnetostatic equations are satisfied by

$$\mathbf{j} = (\alpha/\mu)\mathbf{B} + \frac{\mathbf{B} \times \boldsymbol{\nabla} p}{B^2}, \tag{3}$$

where α is given by an integral along a field line of the form

$$\alpha = \alpha_0 - 2\mu \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{B} \cdot (\boldsymbol{\nabla} B) \times (\boldsymbol{\nabla} p)/B^4 ds$$

from a value α_0 at a reference point $\mathbf{x}_0$, say.

SOLUTION. Substitute Eq.(**??**) into the magnetostatic equation

$$\mathbf{j} \times \mathbf{B} = \boldsymbol{\nabla} p, \tag{4}$$

to give

$$\frac{\mathbf{B} \times \boldsymbol{\nabla} p}{B^2} \times \mathbf{B} = \boldsymbol{\nabla} p.$$

Expanding the triple vector product, we have

$$\boldsymbol{\nabla} p - \mathbf{B}\frac{\boldsymbol{\nabla} p \cdot \mathbf{B}}{B^2} = \boldsymbol{\nabla} p.$$

But Eq.(**??**) implies that $\boldsymbol{\nabla} p \cdot \mathbf{B} = 0$ and so this is satisfied identically. In other words (**??**) is satisfied by (**??**), as required.

The value of α follows from taking the divergence of Eq.(**??**) and using $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$, to give

$$0 = \boldsymbol{\nabla} \cdot (\alpha \mathbf{B}/\mu) + \boldsymbol{\nabla} \cdot \left(\frac{\mathbf{B} \times \boldsymbol{\nabla} p}{B^2}\right).$$

Expanding out both expressions and using $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ gives

$$(\mathbf{B} \cdot \boldsymbol{\nabla})\alpha/\mu = -\mathbf{B} \cdot \boldsymbol{\nabla} p \times \boldsymbol{\nabla}\left(\frac{1}{B^2}\right) - \frac{\boldsymbol{\nabla} p}{B^2} \cdot \boldsymbol{\nabla} \times \mathbf{B} - \frac{\mathbf{B}}{B^2} \cdot \boldsymbol{\nabla} \times \boldsymbol{\nabla} p.$$

But the second and third terms on the right vanish since Eq.(**??**) implies that $\boldsymbol{\nabla} p \cdot \mathbf{j} = 0$ and $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p$ is identically zero. Thus, after putting $\boldsymbol{\nabla}(1/B^2) = -2(\boldsymbol{\nabla}\mathbf{B})/B^3$ this reduces to

$$B\frac{d\alpha}{ds} = -\frac{2\mu}{B^3}\mathbf{B} \cdot \boldsymbol{\nabla} p \times \boldsymbol{\nabla} B$$

or, integrating along a field line from $\mathbf{x}_0$ to $\mathbf{x}$,

$$\alpha = \alpha_0 - 2\mu \int_{\mathbf{x}_0}^{\mathbf{x}} \frac{\mathbf{B} \cdot (\boldsymbol{\nabla} B) \times (\boldsymbol{\nabla} p)}{B^4} ds,$$

as required.