Transformation to canonical forms

In Chapter 4, we introduced different canonical forms and made that statement that we can go from one form or another. We show how to do the transformations here. This explanation requires terminology that will not be introduced until Chapter 9. You may defer the reading till then. First, we show that a system which is completely state controllable can be transformed to the observable canonical form. If an *n*-th order system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u$$

is completely state controllable, the controllability matrix (from Chapter 9)

$$\mathbf{P} = [\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\ldots\; \mathbf{A}^{n-1}\mathbf{B}]$$

is of rank n, *i.e.*, it is not singular. In this case, we can show that the transform $\mathbf{x}_{ob} = \mathbf{P}^{-1}\mathbf{x}$ can be used to convert $\mathbf{A}$ into the observable canonical form, $\mathbf{A}_{ob}$:

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{A}_{ob} \tag{A4-1}$$

We multiply both sides with $\mathbf{P}$ to obtain

$$\mathbf{AP} = \mathbf{PA}_{ob} \tag{A4-2}$$

The task is to show that they are really the same. The LHS is easy. It is

$$\mathbf{AP} = [\mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\; \mathbf{A}^3\mathbf{B} \;\ldots\; \mathbf{A}^n\mathbf{B}] \tag{A4-3}$$

Substitution for $\mathbf{A}_{ob}$ from Chapter 9, the RHS of (A4-2) becomes

$$\mathbf{PA}_{ob} = [\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \cdots \;\; \mathbf{A}^{n-1}\mathbf{B}] \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_o \\ 1 & 0 & \ldots & 0 & -a_1 \\ 0 & 1 & & \vdots & -a_2 \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \tag{A4-4}$$

You may recognize that $\mathbf{A}_{ob}$ is the transpose of the controllable canonical matrix in Eq. (4-19). The coefficients a_o, a_1, … a_{n-1} are those of the characteristic equation, $|s\mathbf{I} - \mathbf{A}| = 0$, as in Eq. (4-21).

When we carry out the matrix multiplication in (A4-4), we should remind ourselves that each "entry" in $\mathbf{P}$ (e.g., $\mathbf{A}^2\mathbf{B}$) is an (n x 1) column partition. The product $\mathbf{PA}_{ob}$ is very clean until we get to the last column when we have to add up all the terms. Hence, we have

$$\mathbf{PA}_{ob} = [\mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\ldots\; \mathbf{A}^{n-1}\mathbf{B} \;\; (-a_o\mathbf{B} - a_1\mathbf{AB} - a_2\mathbf{A}^2\mathbf{B} - \ldots - a_{n-1}\mathbf{A}^{n-1}\mathbf{B})]$$

To make sense out of the last messy column, we need the Cayley-Hamilton theorem from linear algebra, which states that each matrix A satisfies it own characteristic equation as in Eq. (4-21):

$$\mathbf{A}^n + a_{n-1}\mathbf{A}^{n-1} + \ldots + a_1\mathbf{A} + a_o\mathbf{I} = 0 \tag{A4-6}$$

With (A4-6), the last column of the matrix in (A4-5) can be factored as

$$(-a_o\mathbf{B} \ldots -a_{n-1}\mathbf{A}^{n-1}\mathbf{B}) = (-a_o\mathbf{I} - a_1\mathbf{A} - a_2\mathbf{A}^2 - \ldots - a_{n-1}\mathbf{A}^{n-1})\mathbf{B} = \mathbf{A}^n\mathbf{B}$$

Finally, we see that RHS of (A4-2) is

$$\mathbf{PA}_{ob} = [\mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\ldots\; \mathbf{A}^{n-1}\mathbf{B} \;\; \mathbf{A}^n\mathbf{B}] \tag{A4-7}$$

which is the same as the LHS (A4-3), and thus (A4-2) is the proper transformation to the observable canonical form.

We can check whether $\mathbf{P}^{-1}\mathbf{B} = \mathbf{B}_{\text{ob}}$ too. In the observable canonical form,

$$\mathbf{B}_{\text{ob}} = [1\ 0\ \ldots\ 0]^{\text{T}} \tag{A4-8}$$

If so, we should recover $\mathbf{B} = \mathbf{P}\mathbf{B}_{\text{ob}}$. Now

$$\mathbf{P}\mathbf{B}_{\text{ob}} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{B}$$

which is exactly what we look for.

Our next agenda is to find the controllable canonical form. We now need a transformation matrix $\mathbf{T} = \mathbf{PM}$, where

$$\mathbf{M} = \begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & 1 \\ a_2 & a_3 & \ldots & 1 & 0 \\ \vdots & \vdots & & 0 & \vdots \\ a_{n-1} & 1 & & \vdots & \\ 1 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{A4-9}$$

To see that

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \mathbf{A}_{\text{ctr}} \tag{A4-10}$$

where $\mathbf{A}_{\text{ctr}}$ is the controllable canonical form in Eq. (4-19), we repeat the same procedure as before. First, we substitute for $\mathbf{T} = \mathbf{PM}$ such that

$$\mathbf{M}^{-1}\mathbf{P}^{-1}\mathbf{APM} = \mathbf{A}_{\text{ctr}}$$

We have just shown that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{A}_{\text{ob}}$. Thus if we multiply both sides by $\mathbf{M}$, we have

$$\mathbf{A}_{\text{ob}}\mathbf{M} = \mathbf{M}\mathbf{A}_{\text{ctr}}$$

Once again, we want to show that both sides are the same. The algebra will be messier this time. It be may more instructive to use the simple $n = 4$ case as an illustration. In this case, the LHS is

$$\mathbf{A}_{\text{ctr}}\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 & 0 \\ a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 1 \\ 0 & a_3 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

And the RHS is

$$\mathbf{M}\mathbf{A}_{\text{ob}} = \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 & 0 \\ a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} = \begin{bmatrix} -a_0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 1 \\ 0 & a_3 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Hence (A4-10) is true, at least for $n = 4$, and we can use $\mathbf{T} = \mathbf{PM}$ to find the controllable canonical form $\mathbf{A}_{\text{ctr}}$.

We can check whether $\mathbf{T}^{-1}\mathbf{B} = \mathbf{B}_{ctr}$, where now

$$\mathbf{B}_{ctr} = [0\ 0\ \ldots\ 1]^T \tag{A4-11}$$

in the controllable canonical form. If so, we should recover $\mathbf{B} = \mathbf{T}\mathbf{B}_{ctr} = \mathbf{PMB}_{ctr}$. Here,

$$\mathbf{PMB}_{ctr} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & 1 \\ a_2 & a_3 & 1 & 0 \\ a_3 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which is reduced to what we look for:

$$\mathbf{PMB}_{ctr} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{B}$$