Laplace and Fourier Transforms

This brief note is intended for those who know a bit about Fourier transform and now wonder if Laplace and Fourier transforms are related. In a way, they are.

To begin, we first state the definition of the Fourier transform

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x}\, dx \tag{1}$$

and its inverse

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega x}\, d\omega \tag{2}$$

There are two notable features. One, with Laplace transform (see Eqs. (5) below), the lower integration limit is zero, not negative infinity. Second, Laplace transform uses a transform variable in the complex plane. The transform variable in Fouier transform is a pure imaginary number, restricted to $s = j\omega$.

Generally, Laplace transform is for functions that are semi-infinite or piecewise continuous, as in the step or rectangular pulse functions. We also impose the condition that the function is zero at negative times:

$$f(t) = 0, \quad t < 0$$

More formally, we say the function must be of exponential order as t approaches infinity so that the transform integral converges.

A function is of exponential order if there exists (real) constants K, c and T such that

$$|f(t)| < K e^{ct} \quad \text{for} \quad t > T$$

or in other words, the quantity $e^{-ct}|f(t)|$ is bounded. If c is chosen sufficiently large, the so-called *abscissa of convergence*, then $e^{-ct}|f(t)|$ should approach zero as t approaches infinity. In terms of the Laplace transform integral $\int_0^\infty f(t) e^{-st}\, dt$, it means that the real part of s must be larger than the real part of all the poles of $f(t)$ in order the integral to converge. Otherwise, we can force a function to be transformable with $e^{-\gamma t} f(t)$ if we can choose $\gamma > c$ such that $K e^{-(\gamma - c)t}$ approaches zero as t goes to infinity.

We now do a quick two-step to see how the definition of Laplace transform may arise from that of Fourier transform. First, we write the inverse transform of the Fourier transform of the function $e^{-\gamma t} f(t)$, which of course, should recover the function itself:

$$e^{-\gamma t} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t}\, d\omega \int_0^{\infty} e^{-\gamma \tau} f(\tau)\, e^{-j\omega \tau}\, d\tau$$

where we have changed the lower integration limit of the Fourier transfrom from $-\infty$ to 0 because $f(t) = 0$ when $t < 0$. Next, we move the exponential function $e^{-\gamma t}$ to the RHS to go with the inverse integral and then combine the two exponential functions in the transform integral to give

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\gamma + j\omega)t}\, d\omega \int_0^{\infty} f(\tau)\, e^{-(\gamma + j\omega)\tau}\, d\tau \tag{3}$$

Now we define

$$s = \gamma + j\omega$$

and Eqs. (3) will appear as

$$f(t) = \frac{1}{2\pi j} \int_{\gamma - j\infty}^{\gamma + j\infty} e^{st}\, ds \int_0^{\infty} f(\tau)\, e^{-s\tau}\, d\tau \tag{4}$$

From this form, we can extract the definitions of Laplace transform and its inverse:

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{5}$$

and

$$f(t) = \frac{1}{2\pi j}\int_{\gamma-j\infty}^{\gamma+j\infty} F(s)e^{st}\,ds \tag{6}$$

where again, γ must be chosen to be larger than the real parts of all the poles of $F(s)$. Thus the path of integration of the inverse is the imaginary axis shift by the quantity γ to the right.