Mechanical and Electromechanical Models

Many control systems contain mechanical and electromechanical components. Their models are based on introductory physics. We provide three simple examples. The first one is a model on translational motion, the second is rotational, and the last one is electrocmechanical. To go beyond these examples, refer to the *Web Support* Reference for recommended resources.

Simple Mass and Spring System

An automobile suspension can be modeled by translational (also called rectilinear) mechanical systems. We will not be that complicated. In place, we use the simple mass and spring with a damper as our introductory illustration.[1]

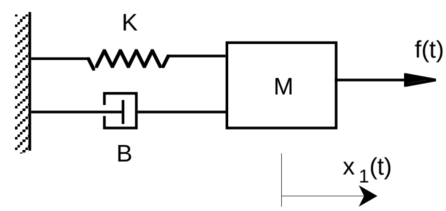

Figure 1. A simple mass and spring with damper system. The displacement (deviation) is denoted by x_1.

As shown in Fig. 1, we apply a force f(t) on a mass M. The motion is opposed by a spring and viscous friction.

The model begins with Newton's law of motion, *F* = *ma*. We'll do the derivation differently because the Hook's law is based on displacement. The law of motion, using notations in Fig. 1, is

$$F = M\frac{d^2x}{dt^2} \tag{1}$$

We next rewrite it in terms of the deviation variables,

$$x_1 = x - x_s \ , \quad \text{and} \quad F_1 = F - F_s$$

For all practical purpose, the so-called deviation x_1 is the *displacement* from some initial position x_s, which is maintained by some force F_s.[2] Equation (1) hence becomes

$$F_1 = M\frac{d^2x_1}{dt^2} \tag{2}$$

There are three forces acting on the mass: the applied force, and opposing it forces from the spring and viscous friction due to a fluid (liquid, air):

$$F_1 = (f - f_s) - K(x - x_s) - B\frac{d}{dt}(x - x_s) \tag{3}$$

The first term on the right is the deviation of applied force, f = f(t). The force f_s can be considered as static friction that prevents motion at the beginning. The second term is Hook's law with K denoting the spring constant. The last term describes viscous friction, and B is the viscous damping coefficient. Without being explicit, it is understood that f = f(t), and K and B are constants.

Substitution of Eq. (3) in (2), and making use of deviation variables would lead to

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[1] FYI. This same mass and spring system in Fig. 1 can be used to model a visco-elastic polymer chain.

[2] We use subscript 1, and not an apostrophe, to denote the displacement because it conveniently becomes the first state variable later when we do the state space representation. And for consistency, we apply the same notation to F and f too.

$$M\frac{d^2x_1}{dt^2} = f_1 - Kx_1 - B\frac{d\,x_1}{dt} \tag{4}$$

or as

$$M\frac{d^2x_1}{dt^2} + B\frac{d\,x_1}{dt} + Kx_1 = f_1 \tag{4a}$$

This equation should have the zero initial condition, $x_1(0) = 0$. This is the result that you find in control texts, which typically skip over our additional steps from (1) to (3).

The Laplace transform of (4a) is

$$\frac{x_1(s)}{f_1(s)} = \frac{1}{Ms^2 + Bs + K} \tag{5}$$

The RHS is, of course, the transfer function of this model.

If we define the natural frequency $\omega = (K/M)^{1/2}$, Eq. (5) can be rewritten as

$$\frac{x_1(s)}{f_1(s)} = \frac{1\,/\,M}{s^2 + 2\zeta\omega s + \omega^2} \tag{5a}$$

where the damping ratio is

$$\zeta = \frac{B}{2}\sqrt{\frac{1}{MK}} \tag{6}$$

If there is no viscous damping, B = 0, and the transfer function becomes

$$\frac{x_1(s)}{f_1(s)} = \frac{1\,/\,M}{s^2 + \omega^2} \tag{7}$$

To reformulate the model in state space (Chapter 4), we use x_1 as the first state variable and define the second state variable as $x_2 = dx_1/dt$. With them, we can easily write Eq. (4) as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{-K}{M} & \dfrac{-B}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \end{bmatrix} f_1 \tag{8}$$

You should double check that the model matrix here has the same characteristic equation as the characteristic polynomial in Eq. (4a).

Simple Torsion Disk

There are systems involving rotational motion. A computer disk drive is one example. There are also systems involving both translational and rotational motions (robotics or crane systems), but they are definitely beyond the scope of what we can do here.

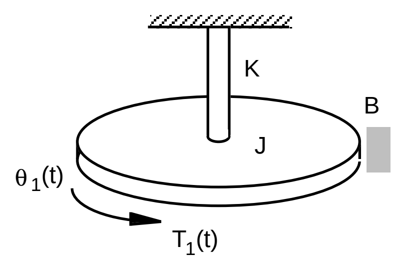

Figure 2. A simple torsion disk.

The simple model that we will study is a torsion disk (Fig. 2). It is a disk suspended by a thin shaft. The disk can rotate if we apply a torque, but the motion is opposed by the spring constant of the shaft and viscous friction.

The derivation and end result is very similar to the last example. We just have to work with the inertia of the disk and the angular acceleration in the law of motion. The equivalent step to Eq. (2) is

$$\tau_1 = J\frac{d^2\theta_1}{dt^2} \tag{9}$$

where now τ_1 is the deviation in total torque, J is the moment of inertia of the disk, and θ_1 is the angular *displacement*, $\theta_1 = \theta - \theta_s$. The total torque is made up of three terms as in (3):

$$\tau_1 = (T - T_s) - K(\theta - \theta_s) - B\frac{d}{dt}(\theta - \theta_s) \tag{10}$$

where K is the spring constant of the shaft, and B is the viscous friction coefficient.

Again, if we define $T_1 = T - T_s$, and substitute (10) into (9), we can arrive at the equivalent of Eq. (4a):

$$J\frac{d^2\theta_1}{dt^2} + B\frac{d\,\theta_1}{dt} + K\theta_1 = T_1 \tag{11}$$

with $\theta_1(0) = 0$.

The Laplace transform is

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{Js^2 + Bs + K} \tag{12}$$

We will skip the state space representation since it is so similar to Eq. (8). Instead, we should point out the simplified cases. When there is no viscous friction, we have the undamped model as analogous to Eq. (7).

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1/J}{s^2 + K/J} \tag{13}$$

where the natural frequency here is $\omega = (J/M)^{1/2}$.

When the torsion spring constant is negligible, (12) becomes

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{s(Js + B)} \tag{14}$$

And when both B and K are negligible, we have a free rotation body (very much like a satellite):

$$\frac{\theta_1(s)}{T_1(s)} = \frac{1}{Js^2} \tag{15}$$

which is a double-integrator plant.

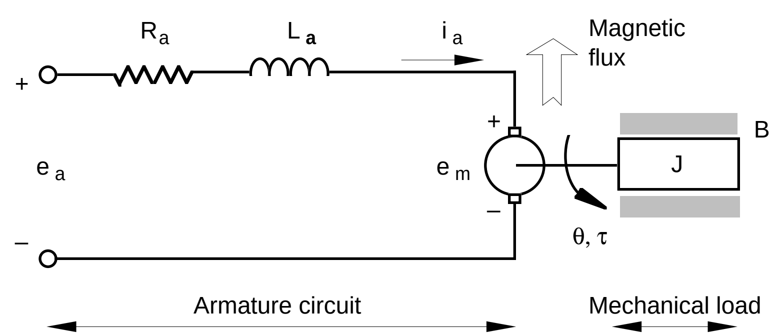

Figure 3. Schematic diagram of a dc motor.

DC Motor

The dc motor is one of the most common actuator in electromechanical systems. We derive here the model equations for a permanent magnet dc motor. The main physical component is an armature rotor, which is placed in the center of a permanent magnet. We apply a voltage to the armature, which is modeled as a circuit with resistance in series with inductance and a back-emf when the motor rotates as current flows through the armature under the magnetic field. This idea is illustrated in Fig. 3. The torque generated by the motor drives a mechanical load and the rotational motion is opposed by viscous friction.

In the following derivation, we will follow the typical texts in mechanical engineering. With the understanding that the angular motion θ is really a displacement, we skip the formality of deviation variables.

First, we apply Kirchhoff's voltage law to the motor armature circuit. The applied (input) voltage e_a is equal to the voltage drops in the field coil resistance and inductance, and the induced back-emf e_m in the armature:

$$e_a = L_a \frac{di_a}{dt} + R_a i_a + e_m \tag{16}$$

where i_a is the armature current, R_a is the armature resistance, and L_a is the armature inductance.

The induced back-emf e_m is related to the magnetic field flux ϕ and the armature rotation rate as

$$e_m = K\phi \frac{d\theta}{dt} = K_m \omega \tag{17}$$

where θ is the angular *displacement*, and $\omega = d\theta/dt$ is the angular velocity. The generated voltage is affected by the physical structure of the generator and we use the factor K to reflect that. More often, we simply lump K and ϕ together to make K_m, the back-emf constant.

If we now substitute for e_m in (16) using (17), we have a differential equation for the change in the armature current:

$$\frac{di_a}{dt} = \frac{1}{L_a} [e_a - R_a i_a - K_m \omega] \tag{18}$$

Next, we need to do a force balance on the mechanical load. The developed torque is

$$\tau = K_i \, i_a \tag{19}$$

where K_i is the torque constant, and analogous to K_m, it includes some factor that depends on the motor physical design. The force balance, as analogous to Eq. (11), but without the shaft spring constant, is

$$J\frac{d^2\theta}{dt^2} = \tau - B\frac{d\theta}{dt} \tag{20}$$

Substitution for τ using (19) will lead us to

$$\frac{d^2\theta}{dt^2} = \frac{1}{J}[K_i \, i_a - B\frac{d\theta}{dt}] \tag{21}$$

Eqs. (18) and (21) constitute a third order model for the dc motor. Let us first write the state space representation since we are almost there. We define the state variables:

$$x_1 = \theta \,, \quad x_2 = d\theta/dt = \omega \,, \quad \text{and} \quad x_3 = i_a \tag{22}$$

and Eqs. (21) and (18) can be put in matrix form:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & K_i/J \\ 0 & -K_m/L_a & -R_a/L_a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} e_a \tag{23}$$

To find the transfer function, the Laplace transform of (21) is

$$Js^2\theta(s) = K_i\, i_a(s) - Bs\theta(s) \tag{24}$$

and that of (18) without the ω substitution for $d\theta/dt$ is

$$L_a s i_a(s) = e_a(s) - R_a i_a(s) - K_m s\theta(s)$$

which can be rearranged to

$$i_a(s) = \frac{e_a(s) - K_m s\theta(s)}{L_a s + R_a} \tag{25}$$

Now we substitute for i_a in (24) using (25), and after algebraic rearrangement, we can arrive at the transfer function:

$$\frac{\theta(s)}{e_a(s)} = \frac{K_i}{L_a J s^3 + (R_a J + BL_a)\, s^2 + (K_i K_m + R_a B)\, s} \tag{26}$$

Note that the model has a pole at the origin. We can calculate the coefficients based on the specs of the manufacturer. Often, we make the approximation to ignore the armature inductance. The model becomes second order as in

$$\frac{\theta(s)}{e_a(s)} = \frac{K_i}{s\,(R_a J s + K_i K_m + R_a B)} \tag{27}$$

or

$$\frac{\theta(s)}{e_a(s)} = \frac{K_i/R_a J}{s\left(s + \dfrac{K_i K_m + R_a B}{R_a J}\right)} \tag{27a}$$

When we design a control system, we can purposely choose a dc motor such that its time constant is much smaller than that of the process (plant) itself. Thus if we can further neglect the time constant of the dc motor in the system, it becomes simply an integrating element.

Under such circumstances, we must be careful in making use of open-loop data. Let's presume that the plant is a first order function. Together with a very powerful and fast dc servomotor, an open-loop test may reveal only a second order model because the motor time constant is masked by the much slower plant time constant. Now with a second order plant and a proportional controller, the system should always be stable. But we actually implement the system, we may find that it can become unstable because we really have a higher order system. So we need to be careful or conservative when we select the controller gain.