Root Locus Plotting Criteria

Most classical books on process control derive guidelines for sketching the root loci by inspection of the transfer function. Using MATLAB, we do not need the details, but we should know the origin of these guidelines. Even though MATLAB does an excellent job for us in making root locus plots, it is always risky to take things totally for granted. Here are some ideas behind root locus construction. As an illustration, we consider a closed-loop characteristic equation in pole-zero form:

$$1 + K\,\frac{(s - z_1)}{(s - p_2)\,(s - p_3)} = 0 \ ,$$

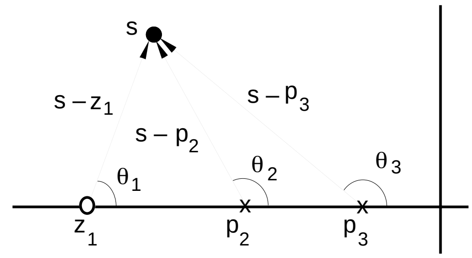

Figure A1.

where K > 0. This equation is better conceived as

$$K\,\frac{(s - z_1)}{(s - p_2)\,(s - p_3)} = -1 \ . \qquad \text{(A-1)}$$

In general, *s* is a complex number and we take the magnitude of the equation to give

$$K = \frac{|s - p_2|\,|s - p_3|}{|s - z_1|} \ , \qquad \text{(A-2)}$$

which becomes the so-called **magnitude criterion** of the root locus. What this equation implies is that for a chosen point *s*, we can calculate the gain K by the magnitude of the individual terms $(s - z_1)$, $(s - p_2)$, and $(s - p_3)$. In vector terms, for example, $(s - z_1)$ is the one pointing from z_1 to *s* (Fig. A1). Thus one can use a ruler (that is exactly what one did in the stone age!) to measure the lengths in Eq. (A-2) to calculate K that corresponds with the particular chosen point *s*.

Now, we could have picked any point on the complex plane and application of Eq. (A-2) would still return a number! We must have another constraint to assure that the point *s* indeed lies on a root locus. This is where the argument of a complex number comes in. Take note that the argument of K is zero and of –1 is 180°. If we take the argument of (A-1), we would have

$$\angle(s - z_1) - \angle(s - p_2) - \angle(s - p_3) = \theta_1 - \theta_2 - \theta_3 = n180^\circ \ , \ \text{ where } \ n = \pm 1, \ \pm 3, ... \qquad \text{(A-3)}$$

This the **angle criterion** of the root locus. Points on a root locus must satisfy this requirement.

If the root locus (the closed-loop pole) happens to be on the real axis, a real open-loop pole on its left contributes 0°, while a real open-loop pole to its right contributes 180°. (We should clarify that the right side of a closed-loop pole is the side closer to the origin.) Two such real poles on the right contributes a total of 360° and would not satisfy the angle criterion. A zero and a pole together cancels each others' contribution to the total angle. This is a very informal observation that we expect to find a root locus on the real axis *only* to the left of an odd number of real open-loop poles and zeros.

In addition, if we pick any point *s* and substitute it into (A-1), we should expect to find K a complex number; it would only be a real number if *s* lies exactly on the root locus. On the real axis, we should see that if we pick a point which does not have an odd number of open-loop poles and zeros on its right, the resulting value of K is negative. In other words, points there cannot satisfy the closed-loop equation and the root locus cannot exist there. These properties are important because they help us locate where the root locus (the closed-loop pole) can possibly lie on the real axis and decide if a system may have complex poles with virtually no work. [1]

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[1] Do not be alarmed when we do a root locus plot with the Padé approximation. When we use [–1 1] as the numerator polynomial in MATLAB, we should find that the result is exactly opposite to our discussion here. That is because if we factor the –1 out such that this equation takes the

In more general terms, we rewrite the characteristic polynomial as

$$1 + KG(s) = 0 \qquad \text{or} \qquad KG(s) = -1 \tag{A-4}$$

where K is the parameter (a real positive number) and G(s) is "everything" else. If s is a solution to the characteristic polynomial, it must satisfy the equation in whichever way we manipulate it, including in polar coordinates. In other words, the solution s (or the locus) must lie on a point (path) which satisfies both the magnitude and angle requirements:

$$|KG(s)| = 1 \qquad \text{or} \qquad K = \frac{1}{|G(s)|} \quad \text{with } K > 0 \tag{A-5}$$

and

$$\angle[KG(s)] = \angle(-1) = n180° \qquad \text{where } n = \pm1, \pm3, \pm5, \text{etc.}$$

With $K > 0$, $\angle K = 0°$, and since $\angle[KG(s)] = \angle K + \angle G(s)$, the angle criterion is

$$\angle G(s) = n180°, \tag{A-6}$$

which is called the 180° locus.

When $K < 0$, $\angle K = 180°$, and $\angle[KG(s)] = 180° + \angle G(s) = n180°$, where $n = \pm1, \pm2$, etc., we have

$$\angle G(s) = n360°, \tag{A-7}$$

which is the 0° or complementary locus.

The angle criterion determines the shape of the root locus, but we will definitely skip these details. What we can do easily, however, is to identify the angle of the locus asymptote if it approaches infinity.

To find the angle of an asymptote, we write the closed-loop equation in Eq. (A-4) as

$$KG(s) = K\,\frac{b_m s^m + b_{m-1} s^{m-1} + \ldots}{a_n s^n + a_{n-1} s^{n-1} + \ldots} = -1 \tag{A-8}$$

Here, we presume that the function G(s) can be expressed as a ratio of two polynomials. The polynomial in the denominator has a higher order such that $n > m$. (If $n = m$, all the loci will go from an open-loop pole to an open-loop zero and there will be no asymptotes.)

A locus asymptote approaches infinity when K and thus s becomes very large. As s approaches infinity, we can approximate the polynomials with only their leading terms:

$$\lim_{s \to \infty} KG(s) \approx K\,\frac{b_m s^m}{a_n s^n} = K\,\frac{b_m}{a_n s^\alpha}$$

where $\alpha = n - m$. Substituting this relation in Eq. (A-8), an asymptote can be approximated by the equation

$$s^\alpha = -K\,\frac{b_m}{a_n} \tag{A-9}$$

or in polar coordinates for the RHS,

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form of (A-1), we really are working with the complementary locus with a negative gain in (A-7). This is confusing in MATLAB because its functions `rlocus()` and `rlocfind()` group ($-K_c$) together and report the gain as a positive number.

Most texts do not bother to do a root locus plot with the Padé approximation because time delay can be handled exactly and easily with frequency response analysis.

$$s^{\alpha} = \left| K \frac{b_m}{a_n} \right| \angle r180^\circ , \quad r = \pm 1, \pm 3, \ldots \tag{A-9}$$

Hence, the angles of the asymptotes *s* are the values of

$$\angle s = \theta = \frac{r180^\circ}{\alpha} , \quad r = \pm 1, \pm 3, \ldots \tag{A-10}$$

For a second order system with two open-loop poles and no zeros, $\alpha = 2$, and the angles of the asymptotes are ±90°, a familiar result to us dated back to Example 7.5. The results for the most probable cases are listed below:

α	Asymptotic Angle
1	180°
2	±90°
3	±60°, 180°
4	±45°, ±135°