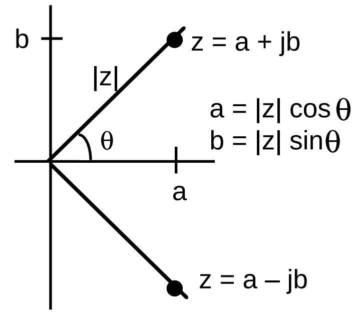
❑ Complex numbers: A review of definitions

A complex number

$$z = a + jb \qquad \text{where} \qquad j = \sqrt{-1}$$



can be taken as an ordered pair $z = (a, b)$. It has a real part $a = \mathrm{Re}(z)$ and an imaginary part $b = \mathrm{Im}(z)$. In mathematics, a more common notation for $\sqrt{-1}$ is i. The complex conjugate of z is

$$\bar{z} = a - jb.$$

Another acceptable notation for the conjugate is z^*. We routinely use a two-dimensional plane (the complex plane) to represent the two components. We also use polar coordinates to represent z:

$$|z| = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$$

Here, $|z|$ is the *magnitude* (or *modulus*) and θ is the *argument* of z, also written as $\angle z$. We can write the complex number as

$$z = (a,b) = |z|\,(\cos\theta + j\sin\theta) = |z|\,e^{j\theta}$$

The complex conjugate is

$$\bar{z} = |z|\,e^{-j\theta}, \quad \text{i.e., } |z| = |\bar{z}| \ \text{ and } \ \arg(z) = -\arg(\bar{z})$$

Some simple arithmetic:

$$z + \bar{z} = 2a\ ; \quad z - \bar{z} = j\,(2b)$$

$$z\,\bar{z} = (a + jb)(a - jb) = a^2 + b^2 = |z|^2$$

$$\frac{z}{\bar{z}} = \frac{(a + jb)}{(a - jb)} = \frac{a^2 - b^2 + j\,(2ab)}{a^2 + b^2}$$

$$z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) = (a_1 a_2 - b_1 b_2) + j(a_1 b_2 + a_2 b_1)$$

or

$$z_1 z_2 = |z_1|\,e^{j\theta_1}\,|z_2|\,e^{j\theta_2} = |z_1|\,|z_2|\,e^{j(\theta_1 + \theta_2)}$$

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\,e^{j(\theta_1 - \theta_2)} \quad \text{or} \quad \frac{(a_1 + jb_1)}{(a_2 + jb_2)}\,\frac{(a_2 - jb_2)}{(a_2 - jb_2)} = \frac{(a_1 a_2 + b_1 b_2) + j\,(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$$

and finally a handy result:

$$(a_1 + jb_1)\,(a_2 - jb_2) + (a_1 - jb_1)\,(a_2 + jb_2) = 2(a_1 a_2 + b_1 b_2)$$

and

$$(a_1 + jb_1)\,(a_2 + jb_2) + (a_1 - jb_1)\,(a_2 - jb_2) = 2(a_1 a_2 - b_1 b_2)$$

We need to know **Euler's identity**:

$$\cos\theta = \frac{1}{2}\,(e^{j\theta} + e^{-j\theta}) \quad \text{and} \quad \sin\theta = \frac{1}{2j}\,(e^{j\theta} - e^{-j\theta})$$

which commonly is presented (by adding the two together) as

$$e^{j\theta} = \cos\,\theta + j\,\sin\,\theta$$

Note that

$$\left| e^{j\theta} \right| = \cos^2\theta + \sin^2\theta = 1$$

We may note that **de Moivre's theorem** is more general:

$$(e^{j\theta})^n = (\cos\theta + j\,\sin\theta)^n = \cos\,n\theta + j\,\sin\,n\theta$$

The following identity is very important in frequency response analysis:

$$a_1\cos\theta + a_2\sin\theta = A\,\sin(\theta + \phi)$$

where

$$A = \sqrt{a_1^2 + a_2^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{a_1}{a_2}\right)$$

We end with a few trigonometry relations, in the second quadrant, $\pi/2 \le \theta \le \pi$

$$\sin(\pi - \theta) = \sin\theta \quad ; \quad \cos(\pi - \theta) = -\cos\theta \quad \text{and} \quad \tan(\pi - \theta) = -\tan\theta$$

In the fourth quadrant, $3\pi/2 \le \theta \le 2\pi$ or $-\pi/2 \le \theta \le 0$

$$\sin(-\theta) = -\sin\theta \quad ; \quad \cos(-\theta) = \cos\theta \quad \text{and} \quad \tan(-\theta) = -\tan\theta$$

$$e^{-j\theta} = \cos\theta - j\,\sin\theta$$

• *Question*: Plot $\tan\theta$ vs θ for $-3\pi/2 \le \theta \le 3\pi/2$. You need to know this in Chapter 8.