7 Chapter 7 Solution Set

Problems

7.1 Prove that a square integrable free field (one that satisfies the homogeneous Helmholtz equation over all of space) has finite multipole moments.

A free field admits the multipole expansion

$$U(\mathbf{r}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m j_l(k_0 r) Y_l^m(\hat{\mathbf{r}})$$

from which we find that

$$\int_{\tau_0} d^3r \, |U(\mathbf{r})|^2 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} |a_l^m|^2 \mu_l^2(k_0 R_0)$$

where

$$\mu_l^2(k_0 R_0) = \int_0^{R_0} r^2 dr |j_l(k_0 r)|^2$$

and where τ_0 is a sphere of radius R_0 centered at the origin. The parameters $\mu_l^2(k_0 R_0)$ are exponentially small (but finite) for $l > k_0 R_0$ so that we find that

$$\int_{\tau_0} d^3r \, |U(\mathbf{r})|^2 \approx \sum_{l=0}^{k_0 R_0} \sum_{m=-l}^{l} |a_l^m|^2 \mu_l^2(k_0 R_0)$$

so long as all the multipole moments from $l = k_0 R_0$ to $l = \infty$ are finite. If the free field is square integrable the r.h.s. of the above equation must be finite for any choice of R_0 which then requires that the multipole moments all be finite.

7.2 Derive Eq.(7.5b).

The total field in the l.h.s. $z < 0$ in the presence of a Dirichlet plane at $z = 0$ must consist of an incoming free field to the plane as well as a reflected (scattered) free field. Ignoring evanescent plane waves we then conclude that

$$U(\mathbf{r}) = \int_{K_\rho < k_0} d^2K_\rho \, A(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{i\gamma z} + \int_{K_\rho < k_0} d^2K_\rho \, A_s(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{-i\gamma z}$$

where the second term represents the reflected (scattered) wavefield that propagates into the l.h.s. The sum must be zero on the Dirichlet plane which then yields

$$A_s(\mathbf{K}_\rho) = -A(\mathbf{K}_\rho)$$

and establishes Eq.(7.5b).

7.3 Use the angular spectrum expansions developed in Section 7.2.1 to compute the outgoing wave Green functions in the half-space $z < 0$ that satisfy homogeneous Dirichlet and Neumann conditions on the plane $z = 0$.

We must consider the two cases where $-\infty < z < z' < 0$ and $z' < z < 0$ corresponding the field points $\mathbf{r} = (x, y, z < 0)$ lying to the left of the source point $\mathbf{r}' = (x', y', z' < 0)$ and field points $\mathbf{r}$ lying to the right of the source point $\mathbf{r}'$ but with both points still in the l.h.s. In both cases the field scattered from the plane at $z = 0$ can be represented by a plane wave expansion of the form

$$U^{(s)}(\mathbf{r}, \mathbf{r}') = \int_{K_\rho < k_0} d^2K_\rho\, A_s(\mathbf{K}_\rho, \mathbf{r}')e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}e^{-i\gamma z}$$

where we have made the usual assumption of ignoring evanescent plane waves. The field generated by a delta function source at $\mathbf{r}'$ is the free space Green function $G_+(\mathbf{r} - \mathbf{r}')$ so that in the region $z' < z < 0$ the field incident to the plane at $z = 0$ propagates in the positive z direction and thus admits the Weyl expansion

$$U^{(in)}(\mathbf{r}) = G_+(\mathbf{r} - \mathbf{r}') = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}e^{i\gamma(z-z')}.$$

The total Green function in the region $z' < z < 0$ is then given by the plane wave expansion

$$G(\mathbf{r}, \mathbf{r}') = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}e^{i\gamma(z-z')} + \int_{K_\rho < k_0} d^2K_\rho\, A_s(\mathbf{K}_\rho, \mathbf{r}')e^{i\mathbf{K}_\rho\cdot\boldsymbol{\rho}}e^{-i\gamma z}.$$

Considering for the moment a Dirichlet plane at $z = 0$ we then require that $G = G_D$ vanish at $z = 0$ which yields

$$A_s(\mathbf{K}_\rho, \mathbf{r}') = \frac{i}{8\pi^2}\frac{1}{\gamma}\, e^{-i\mathbf{K}_\rho\cdot\boldsymbol{\rho}'}e^{-i\gamma z'}$$

so that

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{-i}{8\pi^2}\int_{-\infty}^{\infty}\frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}e^{i\gamma(z-z')} + \frac{i}{8\pi^2}\int_{K_\rho < k_0}\frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}e^{-i\gamma(z+z')}.$$

On making use of the Weyl expansion we see that the above plane wave expansion of the Dirichlet Green function can be expressed in the form (see Problems 2.15 and 4.9)

$$G_D(\mathbf{r}, \mathbf{r}') = G_+(\mathbf{r} - \mathbf{r}') - G_+(\tilde{\mathbf{r}} - \mathbf{r}') \tag{7.1}$$

where $\tilde{\mathbf{r}} = (x, y, -z)$.

In the region $z < z' < 0$ the Green function component scattered from the plane at $z = 0$ remains that same but the component generated by the source point at $z = z'$ has the Weyl expansion

$$G_+(\mathbf{r} - \mathbf{r}') = \frac{-i}{8\pi^2} \int_{-\infty}^{\infty} \frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\gamma(z-z')}.$$

The total Dirichlet Green function in this region is then given by

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{-i}{8\pi^2} \int_{-\infty}^{\infty} \frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\gamma(z-z')} + \frac{i}{8\pi^2} \int_{K_\rho < k_0} \frac{d^2K_\rho}{\gamma}\, e^{i\mathbf{K}_\rho \cdot (\boldsymbol{\rho} - \boldsymbol{\rho}')} e^{-i\gamma(z+z')},$$

which again reduces to the expression given in Eq.(7.1). A Neumann plane is done in completely parallel fashion.

7.4 Generalize the angular spectrum expansions developed in Section 7.2.1 to the case of an incident wave radiated by a source in the right half-space $z > z_0$ and reflecting off of the plane $z = z_0$ where it satisfies homogeneous Dirichlet or Neumann conditions.

The total field in the r.h.s. $z > 0$ in the presence of a Dirichlet plane at $z = 0$ must consist of an incoming free field to the plane as well as a reflected (scattered) free field. Ignoring evanescent plane waves we then conclude that

$$U(\mathbf{r}) = \int_{K_\rho < k_0} d^2K_\rho\, A(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{-i\gamma z} + \int_{K_\rho < k_0} d^2K_\rho\, A_s(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{i\gamma z}$$

where the first term represents the incident wave that propagates in the $-z$ direction and second term represents the reflected (scattered) wavefield that propagates into the $+z$ direction. The sum must be zero on a Dirichlet plane located at $z = 0$ which then yields

$$A_s(\mathbf{K}_\rho) = -A(\mathbf{K}_\rho)$$

and yields

$$U(\mathbf{r}) = \int_{K_\rho < k_0} d^2K_\rho\, A(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{-i\gamma z} - \int_{K_\rho < k_0} d^2K_\rho\, A(\mathbf{K}_\rho) e^{i\mathbf{K}_\rho \cdot \boldsymbol{\rho}} e^{i\gamma z}.$$

A Neumann plane is done in completely parallel fashion.

7.5 Derive the 2D versions of Eqs.(7.25).

The scattered field from a Dirichlet surface within the P.O. approximation is given by Eq.(7.21b). The 2D version of this equation is obtained by taking $\mathbf{r} = (x, y) = (r, \phi)$ and

$$G_{0+}(\mathbf{r} - \mathbf{r}') = -\frac{i}{4} H_0^+(k_0|\mathbf{r} - \mathbf{r}'|) \sim -\frac{i}{4}\sqrt{\frac{2}{\pi k_0}} e^{-i\frac{\pi}{4}} e^{-ik_0 \mathbf{s} \cdot \mathbf{r}'} \frac{e^{ik_0 r}}{\sqrt{r}}.$$

The scattered field from a 2D Dirichlet surface is then found to be

$$U^{(s)}_{P.O.}(\mathbf{r}) = 2 \int_{\partial\tau_0} dS'\, \overbrace{-\frac{i}{4} H_0^+(k_0|\mathbf{r} - \mathbf{r}'|)}^{G_{0+}(\mathbf{r}-\mathbf{r}')} \frac{\partial}{\partial n'} U_<^{(in)}(\mathbf{r}')$$

$$\sim -\frac{i}{2}\sqrt{\frac{2}{\pi k_0}} e^{-i\frac{\pi}{4}} \int_{\partial\tau_0} dS'\, e^{-ik_0 \mathbf{s} \cdot \mathbf{r}'} \frac{\partial}{\partial n'} U_<^{(in)}(\mathbf{r}') \frac{e^{ik_0 r}}{\sqrt{r}}$$

which, for plane wave incidence, yields

$$U^{(s)}_{P.O.}(\mathbf{r}) \sim -\frac{i}{2}\sqrt{\frac{2}{\pi k_0}}e^{-i\frac{\pi}{4}}\int_{\partial\tau_{0l}} dS'\, ik_0\mathbf{s}_0\cdot\hat{\mathbf{n}}' e^{ik_0\mathbf{s}_0\cdot\mathbf{r}'}e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{e^{ik_0 r}}{\sqrt{r}}.$$

The scattering amplitude is then found to be

$$f_{PO}(\mathbf{s},\mathbf{s}_0) = \sqrt{\frac{k_0}{2\pi}}e^{-i\frac{\pi}{4}}\int_{\partial\tau_{0l}} dS'\,\mathbf{s}_0\cdot\hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'},$$

where τ_{0l} is the lit portion of the surface. A completely parallel development is used for the Neumann surface.

7.6 Derive Eq.(7.26a).

This is a straightforward calculation that follows the steps given in the book.

7.7 Derive the generalized scattering amplitudes for Dirichlet and Neumann surfaces within the P.O. approximation

1. From the definition of the P.O. scattered fields given in Eqs.(7.21),
   We find from Eqs.(7.21) that the scattered field within the P.O. approximation for a Dirichlet surface scatterer is given by

$$U^{(s)}_{P.O.}(\mathbf{r}) = 2\int_{\partial\tau_0} dS'\, G_{0+}(\mathbf{r}-\mathbf{r}')\frac{\partial}{\partial n'}U^{(in)}_{<}(\mathbf{r}').$$

   If we now let $r\to\infty$ we obtain

$$U^{(s)}_{P.O.}(\mathbf{r}) \sim 2\int_{\partial\tau_0} dS'\,[-\frac{1}{4\pi}e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{e^{ik_0 r}}{r}]\frac{\partial}{\partial n'}U^{(in)}_{<}(\mathbf{r}').$$

   from which we find that

$$f_{P.O.}(\mathbf{s},\nu) = -\frac{1}{2\pi}\int_{\partial\tau_0} dS'\, e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{\partial}{\partial n'}U^{(in)}_{<}(\mathbf{r}').$$

   A Neumann plane is done in a parallel fashion.

2. From their plane wave scattering amplitudes and the relationship Eq.(6.35b),
   We have from Eq.(6.35b)

$$f(\mathbf{s},\nu) = \int_{4\pi} d\Omega_{s_0}\, A(\mathbf{s}_0,\nu) f(\mathbf{s},\mathbf{s}_0)$$

   where $A(\mathbf{s}_0,\nu)$ is the plane wave amplitude of the incident wave to the scatterer and $f(\mathbf{s},\mathbf{s}_0)$ is the plane wave scattering amplitude. If we then express the incoming component of an incident wave to a Dirichlet or Neumann surface scatterer in the plane wave expansion

$$U^{(in)}_{<}(\mathbf{r}) = \int d\Omega_{s_0} A^{(in)}_{<}(\mathbf{s}_0,\nu)e^{ik_0\mathbf{s}_0\cdot\mathbf{r}}$$

   we conclude that

$$f_{P.O.}(\mathbf{s},\nu) = \int_{4\pi} d\Omega_{s_0}\, A^{(in)}_{<}(\mathbf{s}_0\nu) f_{P.O.}(\mathbf{s},\mathbf{s}_0).$$

Again restricting our attention to a Dirichlet surface scatterer we have that

$$f_{P.O.}(\mathbf{s},\mathbf{s}_0) = -\frac{ik_0}{2\pi}\int_{\partial\tau_{0l}(\mathbf{s}_0)} dS'\,\mathbf{s}_0\cdot\hat{\mathbf{r}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}$$

yielding

$$f_{P.O.}(\mathbf{s},\nu) = \int_{4\pi} d\Omega_{s_0}\, A_<^{(in)}(\mathbf{s}_0\nu)\overbrace{-\frac{ik_0}{2\pi}\int_{\partial\tau_{0l}(\mathbf{s}_0)} dS'\,\mathbf{s}_0\cdot\hat{\mathbf{r}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}}^{f_{P.O.}(\mathbf{s},\mathbf{s}_0)}$$
$$= -\frac{1}{2\pi}\int_{\partial\tau_{0l}(\mathbf{s}_0)} dS'\,\{ik_0\int_{4\pi} d\Omega_{s_0}\, A_<^{(in)}(\mathbf{s}_0\nu)\mathbf{s}_0\cdot\hat{\mathbf{r}}' e^{ik_0\mathbf{s}_0\cdot\mathbf{r}'}\}e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}$$
$$= -\frac{1}{2\pi}\int_{\partial\tau_0} dS'\, e^{-ik_0\mathbf{s}\cdot\mathbf{r}'}\frac{\partial}{\partial n'}U_<^{(in)}(\mathbf{r}').$$

A completely parallel development is used for Neumann scatterers.

3. Verify that they are the same and reduce to the plane wave scattering amplitudes for plane wave incidence.

   This is easily verified.

7.8 Compute the scattered field and scattering amplitude for plane wave incidence with the P.O. approximation for a Dirichlet sphere.

The scattering amplitudes for a surface scatterer within the P.O. approximation are given in Eqs.(7.25) which for a Dirichlet scatterer yields

$$f_{P.O.}(\mathbf{s},\mathbf{s}_0) = -\frac{ik_0}{2\pi}\int_{\partial\tau_{0l}(\mathbf{s}_0)} dS'\,\mathbf{s}_0\cdot\hat{\mathbf{r}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}.$$

If we now take the surface to be a sphere with radius a_0 and take the polar axis of the $\mathbf{r}'$ spherical coordinate system to be the negative of the unit wave vector $\mathbf{s}_0$ of the incident plane wave we obtain

$$f_{P.O.}(\mathbf{s},\mathbf{s}_0) = \frac{ik_0}{2\pi}a_0^2\int_{-\pi}^{\pi} d\phi'\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta' e^{-ik_0a_0[\cos(\theta_s-\theta')+\cos\theta']}$$
$$ik_0a_0^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta' e^{-ik_0a_0[\cos(\theta_s-\theta')+\cos\theta']},$$

where θ_s is the angle that $\mathbf{s}_0$ makes with the polar axis $-\mathbf{s}_0$ and we have used the fact that $\mathbf{s}_0\cdot\mathbf{r}' = -\cos\theta'$ and $\mathbf{s}\cdot\mathbf{r}' = \cos(\theta_s-\theta')$. The final integral cannot be evaluated in closed form and would have to be implemented numerically to evaluate.

7.9 Use the scattering amplitude for a Dirichlet cylinder within the P.O. approximation given in Section 7.3 to verify the 2D version of Eq.(7.31).

It is first necessary to derive the 2D version of Eq.(7.31). The 2D scattering amplitude for a Dirichlet scatterer was found in Problem 7.5 to be

$$f_{PO}(\mathbf{s},\mathbf{s}_0) = \sqrt{\frac{k_0}{2\pi}}e^{-i\frac{\pi}{4}}\int_{\partial\tau_{0l}} dS'\,\mathbf{s}_0\cdot\hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}$$

It then follows that

$$F(k_0\mathbf{s}_0) = e^{i\frac{\pi}{4}} f_{PO}(-\mathbf{s}_0, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(\mathbf{s}_0, -\mathbf{s}_0)$$

$$= \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_{0l}} dS'\, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{2ik_0\mathbf{s}_0\cdot\mathbf{r}'} + \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_{0l}^{\perp}} dS'\, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{2ik_0\mathbf{s}_0\cdot\mathbf{r}'}$$

$$= \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_0} dS'\, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{2ik_0\mathbf{s}_0\cdot\mathbf{r}'} = \sqrt{\frac{k_0}{2\pi}} \int_{\tau_0} d^2r'\, \nabla_{r'} \cdot \mathbf{s}_0 e^{2ik_0\mathbf{s}_0\cdot\mathbf{r}'} = 2ik_0\sqrt{\frac{k_0}{2\pi}} \int_{\tau_0} d^2r'\, e^{2ik_0\mathbf{s}_0\cdot\mathbf{r}'},$$

thus yielding

$$F(k_0\mathbf{s}_0) = 2ik_0\sqrt{\frac{k_0}{2\pi}}\tilde{\Gamma}(-2k_0\mathbf{s}_0), \tag{7.2}$$

where $\Gamma(\mathbf{r})$ is the 2D characteristic function of the scatterer. Eq.(7.2) is the 2D version of the Bojarski indentity given in Eq.(7.31).

Now that we have derived the 2D Bojarski identity we can apply it to a cylinder and show that the P.O. approximation given in Section 7.3 leads to Eq.(7.2). We first compute $F(k_0\mathbf{s}_0)$ directly from Eq.(7.2) for a cylinder. We find that

$$\tilde{\Gamma}(-K) = \int_0^{a_0} rdr \int_0^{2\pi} d\phi\, e^{iKr\cos\phi} = 2\pi \int_0^{a_0} rdr\, J_0(Kr) = 2\pi a_0 \frac{J_1(Ka_0)}{K}$$

where we have used the identity

$$\frac{d}{dx} xJ_1(x) = xJ_0(x).$$

On making use of the above result we then find that

$$F(k_0\mathbf{s}_0) = 2ik_0\sqrt{\frac{k_0}{2\pi}} 2\pi a_0 \frac{J_1(2k_0a_0)}{2k_0} = i\sqrt{2\pi k_0 a_0^2} J_1(2k_0a_0). \tag{7.3}$$

We must now show that the same result is obtained from the 2D definition of F:

$$F(k_0\mathbf{s}_0) = e^{i\frac{\pi}{4}} f_{PO}(-\mathbf{s}_0, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(\mathbf{s}_0, -\mathbf{s}_0)$$

using the P.O. approximation to the cylinder given in Section 7.3. The scattering amplitude was found in that section to be given by

$$f_{P.O.}(\phi) = \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{-i\frac{\pi}{4}} [C_0 J_0(k_0a_0) + 2\sum_{n=1}^{\infty} (-i)^n C_n J_n(k_0a_0)\cos n\phi],$$

where the incident plane wave is assumed to be propagating in the positive x direction and ϕ is the scattering angle relative to the positive x axis. For the following development it is preferable to rewrite the above expansion in the alternative form

$$f_{P.O.}(\phi) = \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{-i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} (-i)^n C_n J_n(k_0a_0) e^{in\phi},$$

which follows using the fact that $C_{-n} = C_n$. Using this expression we find that

$$f_{PO}(-\mathbf{s}_0, \mathbf{s}_0) = f_{PO}(\mathbf{s}_0, -\mathbf{s}_0) = f_{P.O.}(\phi = \pi) = \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{-i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} i^n C_n J_n(k_0 a_0),$$

where we have made use of the fact that for a cylinder $f(\mathbf{s}_0, -\mathbf{s}_0) = f(-\mathbf{s}_0, \mathbf{s}_0)$. We then conclude that

$$\begin{aligned} F(k_0\mathbf{s}_0) &= e^{i\frac{\pi}{4}} f_{PO}(-\mathbf{s}_0, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(\mathbf{s}_0, -\mathbf{s}_0) \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} [\sum_{n=-\infty}^{\infty} i^n C_n J_n(k_0 a_0) - \sum_{n=-\infty}^{\infty} (-i)^n C_n^* J_n(k_0 a_0)] \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \{\sum_{n=-\infty}^{\infty} i^n (C_n - (-1)^n C_n^*) J_n(k_0 a_0)\}. \end{aligned} \tag{7.4}$$

To proceed we need to compute $C_n - (-1)^n C_n^*$. We find from the definition of C_n given in Section 7.3 that

$$C_n^* = \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\phi' \cos\phi' e^{-ik_0 a_0 \cos\phi'} e^{in\phi'} = -(-1)^n \int_{\frac{3\pi}{2}}^{\frac{5}{2}\pi} d\phi'' \cos\phi'' e^{ik_0 a_0 \cos\phi''} e^{in\phi''}$$

where we have made the transformation $\phi'' = \phi' + \pi$. Since $C_n = C_{-n}$ we can also express this result in the form

$$C_n^* = -(-1)^n \int_{\frac{3\pi}{2}}^{\frac{5}{2}\pi} d\phi'' \cos\phi'' e^{ik_0 a_0 \cos\phi''} e^{-in\phi''}.$$

On using this result we find that

$$\begin{aligned} C_n - (-1)^n C_n^* &= \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-in\phi'} + \int_{\frac{3\pi}{2}}^{\frac{5}{2}\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-in\phi'} \\ &= \int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-in\phi'}. \end{aligned}$$

Using this result in Eq.(7.4) then yields

$$\begin{aligned} F(k_0\mathbf{s}_0) &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \sum_{n=-\infty}^{\infty} i^n \overbrace{\int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-in\phi'}}^{C_n - (-1)^n C_n^*} J_n(k_0 a_0) \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} \overbrace{\sum_{n=-\infty}^{\infty} i^n e^{-in\phi'} J_n(k_0 a_0)}^{e^{ik_0 a_0 \cos\phi'}} \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \int_0^{2\pi} d\phi' \cos\phi' e^{2ik_0 a_0 \cos\phi'} = \sqrt{2\pi k_0 a_0^2}\, i J_1(2k_0 a_0) \end{aligned}$$

which is the same result we found earlier in Eq.(7.3). In deriving the above result we have made use of the expansion

$$J_1(x) = 2\pi i \int_0^{2\pi} d\phi \, e^{i\phi} e^{ix\cos\phi}.$$

7.10 Use the scattering amplitude for a Dirichlet cylinder within the P.O. approximation given in Section 7.3 to verify the 2D version of Eq.(7.35).

This problem is done in a parallel manner as was used in the previous problem. It is first necessary to derive the 2D version of Eq.(7.35). The 2D scattering amplitude for a 2D Dirichlet scatterer was found in Problem 7.5 to be

$$f_{PO}(\mathbf{s}, \mathbf{s}_0) = \sqrt{\frac{k_0}{2\pi}} e^{-i\frac{\pi}{4}} \int_{\partial\tau_{0l}} dS' \, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}$$

It then follows that

$$F_g(\mathbf{s}_0, \mathbf{s}) = e^{i\frac{\pi}{4}} f_{PO}(\mathbf{s}, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(-\mathbf{s}, -\mathbf{s}_0)$$

$$= \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_{0l}} dS' \, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'} + \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_{0l}^{\perp}} dS' \, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}$$

$$= \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_0} dS' \, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'} = \sqrt{\frac{k_0}{2\pi}} \int_{\tau_0} d^2r' \, \nabla_{r'} \cdot \mathbf{s}_0 e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'},$$

yielding the result

$$F_g(\mathbf{s}_0, \mathbf{s}) = e^{i\frac{\pi}{4}} f_{PO}(\mathbf{s}, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(-\mathbf{s}, -\mathbf{s}_0) = ik_0\sqrt{\frac{k_0}{2\pi}}(1-\mathbf{s}_0\cdot\mathbf{s})\tilde{\Gamma}[k_0(\mathbf{s}-\mathbf{s}_0)].$$

We will also need later the form for F_g given by the surface integral on the l.h.s. in the third line of the above development:

$$F_g(\mathbf{s}_0, \mathbf{s}) = \sqrt{\frac{k_0}{2\pi}} \int_{\partial\tau_0} dS' \, \mathbf{s}_0 \cdot \hat{\mathbf{n}}' e^{-ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}'}. \qquad (7.5)$$

Note that putting $\mathbf{s} = -\mathbf{s}_0$ we obtain

$$F_g(\mathbf{s}_0, -\mathbf{s}_0) = e^{i\frac{\pi}{4}} f_{PO}(-\mathbf{s}_0, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(\mathbf{s}_0, -\mathbf{s}_0) = 2ik_0\sqrt{\frac{k_0}{2\pi}}\tilde{\Gamma}(-2k_0\mathbf{s}_0),$$

which was the result we found for $F(k_0\mathbf{s}_0)$ in the previous problem.

We showed in the previous problem that

$$\tilde{\Gamma}(-K) = \int_0^{a_0} rdr \int_0^{2\pi} d\phi \, e^{iKr\cos\phi} = 2\pi \int_0^{a_0} rdr \, J_0(Kr) = 2\pi a_0 \frac{J_1(Ka_0)}{K}$$

from which we conclude that

$$F_g(\mathbf{s}_0, \mathbf{s}) = ik_0\sqrt{2\pi k_0 a_0^2}(1 - \mathbf{s}_0 \cdot \mathbf{s})\frac{J_1(k_0 a_0|\mathbf{s} - \mathbf{s}_0|)}{k_0|\mathbf{s} - \mathbf{s}_0|}. \qquad (7.6)$$

Again putting $\mathbf{s} = -\mathbf{s}_0$ we obtain the solution obtained for $F(k_0\mathbf{s}_0)$ in Eq.(7.3) in the previous problem.

We must now show that the same result is obtained from the 2D definition of $F_g(\mathbf{s}_0, \mathbf{s})$:

$$F_g(\mathbf{s}_0, \mathbf{s}) = e^{i\frac{\pi}{4}} f_{PO}(\mathbf{s}, \mathbf{s}_0) - e^{-i\frac{\pi}{4}} f^*_{PO}(-\mathbf{s}, -\mathbf{s}_0)$$

using the P.O. approximation to the cylinder given in Section 7.3. The scattering amplitude was found in that section to be given by

$$\begin{aligned} f_{PO}(\mathbf{s}, \mathbf{s}_0) &= \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{-i\frac{\pi}{4}} [C_0 J_0(k_0 a_0) + 2 \sum_{n=1}^{\infty} (-i)^n C_n J_n(k_0 a_0) \cos n\phi] \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{-i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} (-i)^n C_n J_n(k_0 a_0) e^{in\phi}. \end{aligned}$$

where the incident plane wave is assumed to be propagating in the positive x direction and ϕ is the scattering angle relative to the positive x axis; i.e., is the angle made between the wave vectors $\mathbf{s}$ and the positive x axis. Using this expression we find that

$$\begin{aligned} f^*_{PO}(-\mathbf{s}, -\mathbf{s}_0) = \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} i^n C^*_n J_n(k_0 a_0) e^{-in\phi} \\ \sqrt{\frac{k_0 a_0^2}{2\pi}} e^{i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} i^n C^*_n J_n(k_0 a_0) e^{in\phi} \end{aligned}$$

where we have made use of the fact that for a cylinder $f(-\mathbf{s}, -\mathbf{s}_0) = f(\mathbf{s}, \mathbf{s}_0)$ and that the scattering amplitude is an even function of the scattering angle ϕ. We then find in analogy to Eq.(7.4) in the solution of the previous problem that

$$F_g(\mathbf{s}_0, \mathbf{s}) = \sqrt{\frac{k_0 a_0^2}{2\pi}} \sum_{n=-\infty}^{\infty} (-i)^n (C_n - (-1)^n C^*_n) J_n(k_0 a_0) e^{in\phi},$$

with

$$C_n - (-1)^n C^*_n = \int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-in\phi'},$$

found in that problem solution. On using the above expression we find that the expression for F_g given above reduces to

$$\begin{aligned} F_g(\mathbf{s}_0, \mathbf{s}) &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} \sum_{n=-\infty}^{\infty} (-i)^n e^{-in(\phi'-\phi)} J_n(k_0 a_0) \\ &= \sqrt{\frac{k_0 a_0^2}{2\pi}} \int_0^{2\pi} d\phi' \cos\phi' e^{ik_0 a_0 \cos\phi'} e^{-ik_0 a_0 \cos(\phi-\phi')} \end{aligned}$$

where we have made use of the Jacobi-Anger expansion

$$e^{-ik_0 a_0 \cos\theta} = \sum_{n=-\infty}^{\infty} (-i)^n e^{-in\theta} J_n(k_0 a_0)$$

with $\theta = \phi' - \phi$.

Our goal is to show that the above expression for $F_g(\mathbf{s}_0, \mathbf{s})$ obtained above is the same as that obtained in Eqs.(7.5) and (7.6). It is far easier to compare the above result with the expression given in Eq.(7.5) than that given in Eq.(7.6) which is what we will do here[1]. If we specialize Eq.(7.5) to the case of a cylinder it becomes

$$F_g(\mathbf{s}_0, \mathbf{s}) = \sqrt{\frac{k_0 a_0^2}{2\pi}} \int_0^{2\pi} d\phi' \cos\phi' e^{-ik_0 a_0 \cos(\phi - \phi')} e^{ik_0 a_0 \cos\phi'},$$

which is seen to be identical to the expression for F_g obtained above and thus proves the equivalence of the two expressions.

7.11 Derive the expression for the Kirchoff diffraction pattern given in Eq.(7.46b).

We begin with Eq.(7.41b) where we let r tend to infinity to obtain

$$U^{(d)}(\mathbf{r}, \nu) \sim \frac{-ik_0}{2\pi} \frac{z}{r} \int_{z=z_0} d^2\rho' \, \mathcal{T}(\boldsymbol{\rho}') U^{(in)}(\mathbf{r}_0, \nu) e^{-ik_0 \mathbf{s}\cdot\mathbf{r}_0'} \frac{e^{ik_0 r}}{r},$$

where $\mathbf{s} = \mathbf{r}/r$ is the unit vector along the position vector $\mathbf{r}$. If we then neglect the so-called "obliquity factor" z/r and take the aperture to be located at $z' = 0$ we obtain the expression given in Eq.(7.46b) for the Kirchoff diffraction pattern $f_d(\mathbf{s}, \nu)$ as a function of $\mathbf{s}$ over a hemisphere in the r.h.s. $z >> 0$.

7.12 Compute the Kirchoff diffraction pattern of a circular disk as a function of the transverse coordinates in an observation plane located at distance z from the center of the disk.

Here we use the expression for the diffraction pattern given in Eq.(7.46a) which, for a circular disk and normally incident plane wave reduces to

$$f_d(\mathbf{u}, \nu) = -\frac{ik_0}{2\pi} \int_0^{a_0} \rho' d\rho' \int_0^{2\pi} d\phi' \, e^{-i|\mathbf{u}|\rho' \cos\phi'}$$

where ϕ' is the polar angle of $\boldsymbol{\rho}'$ and $\mathbf{u} = k_0\boldsymbol{\rho}/z$. The above integrals are easily performed and we obtain

$$f_d(\mathbf{u}, \nu) = -\frac{ik_0}{2\pi} \int_0^{a_0} \rho' d\rho' 2\pi J_0(|\mathbf{u}|\rho') = -ik_0 a_0 \frac{J_1(|\mathbf{u}|a_0)}{|\mathbf{u}|}.$$

7.13 Compute the diffraction pattern of the disk in problem 7.12 in the far field as a function of the unit vector $\mathbf{s}$.

Here we use Eq.(7.46b) for the diffraction pattern which for a circular disk and normally incident plane wave reduces to

$$f_d(\mathbf{s}, \nu) = -\frac{ik_0}{2\pi} \int_0^{a_0} \rho' d\rho' \int_0^{2\pi} d\phi' \, e^{-ik_0 \mathbf{s}_T \cdot \boldsymbol{\rho}'}$$

[1] Any reader who can provide a direct comparison with Eq.(7.6) should be commended. I would also be grateful to receive a copy of that solution since I have not been able to provide such a direct comparison!

where $\mathbf{s}_T = (s_x, s_y) = \boldsymbol{\rho}/r$ is the component of the unit vector $\mathbf{s} = \mathbf{r}/r$ lying in the (x, y) plane. We can simplify the above expression to obtain

$$f_d(\mathbf{s}, \nu) = -\frac{ik_0}{2\pi}\int_0^{a_0} \rho' d\rho' \int_0^{2\pi} d\phi'\, e^{-ik_0|\mathbf{s}_T|\rho'\cos\phi'}$$
$$= -ik_0\int_0^{a_0} \rho' d\rho' J_0(k_0|\mathbf{s}_T|\rho') = -ik_0 a_0 \frac{J_1(k_0|\mathbf{s}_T|a_0)}{k_0|\mathbf{s}_T|} = -ik_0 a_0 \frac{J_1(k_0 a_0\rho/r)}{k_0\rho/r}.$$

7.14 Give an argument why the field diffracted by a circular aperture which subtends the solid angle Ω_0 from a source located in the l.h.s. is approximately given by

$$U^{(d)}(\mathbf{r}) \approx \int_{\Omega_0} d\Omega_s A^{(in)}(\mathbf{s}) e^{ik_0\mathbf{s}\cdot\mathbf{r}} \tag{7.7}$$

where $A(\mathbf{s})$ is the angular spectrum of the incident wave to the aperture. State the requirements for the approximation to be accurate.

We can represent the field radiated by a source located in the l.h.s. of a diffracting aperture in the angular spectrum expansion

$$U_+(\mathbf{r}) = \int_{-\pi}^{\pi} d\beta \int_{C_+} d\alpha \sin\alpha A^{(in)}(\mathbf{s}) e^{ik_0\mathbf{s}\cdot\mathbf{r}}$$

where C_+ is the α contour of integration that runs from $\alpha = 0$ to $\alpha = \pi/2 - i\infty$ and $A^{(in)}(\mathbf{s})$ is the angular spectrum of the field with α and β being the polar and azimuthal angles of $\mathbf{s}$, respectively. The incident wave field to the aperture then consists of a superposition of homogeneous plane waves propagating in various real directions $\mathbf{s}$ with $s_z > 0$ and evanescent plane waves that propagate perpendicular to the z axis and that decay exponentially fast with z.

If the diffracting aperture is large compared with the wavelength and located at a distance much larger than a wavelength from the source then we can ignore the contribution of the evanescent plane waves to the diffracted field. Moreover, the action of the diffracting aperture to the incident wave can be regarded to be a filter that allows those plane waves who's propagation vector intersects the aperture to pass through but blocks the plane waves who's propagation vectors lie outside the aperture. This then leads to the field representation given by Eq.(7.7).

7.15 Derive Eq.(7.58) from Eq.(7.56).

We begin by writing Eq.(7.56) in the form

$$I(\mathbf{r}) = \mp\frac{k_0^2}{8\pi^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\, \sqrt{1-(\mathbf{s}\cdot\mathbf{s}_0)^2}\,\overbrace{[f(\mathbf{s},\mathbf{s}_0) + f^*(-\mathbf{s},-\mathbf{s}_0)]}^{F_g(\mathbf{s},\mathbf{s}_0)}\, e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$
$$= \mp\frac{k_0^2}{8\pi^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\, \sqrt{1-(\mathbf{s}\cdot\mathbf{s}_0)^2} f(\mathbf{s},\mathbf{s}_0) e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$
$$\mp\frac{k_0^2}{8\pi^2}\int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\, \sqrt{1-(\mathbf{s}\cdot\mathbf{s}_0)^2} f^*(-\mathbf{s},-\mathbf{s}_0) e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}} \tag{7.8}$$

where we have substituted for F_g from Eq.(7.55).

According to fig. 8.3 in the Chapter 8 α_0 is the angle formed between the positive y axis and the unit vector $\mathbf{s}_0$ and α is the angle formed between $\mathbf{s}$ and $\mathbf{s}_0$. Moreover, the Cartesian (ξ, η) coordinate system is defined to have its positive η axis aligned along $\mathbf{s}_0$ so that

$$\mathbf{s} \cdot \mathbf{s}_0 = \cos\alpha, \quad \mathbf{s}_0 = \hat{\boldsymbol{\eta}}, \quad \mathbf{s} = \sin\alpha\hat{\boldsymbol{\xi}} + \cos\alpha\hat{\boldsymbol{\eta}}, \quad \mathbf{r} = \xi\hat{\boldsymbol{\xi}} + \eta\hat{\boldsymbol{\eta}}$$

from which we conclude that

$$\mathbf{s} - \mathbf{s}_0 = \sin\alpha\ \hat{\boldsymbol{\xi}} + (\cos\alpha - 1)\ \hat{\boldsymbol{\eta}}, \quad (\mathbf{s} - \mathbf{s}_0) \cdot \mathbf{r} = \xi \sin\alpha + \eta(\cos\alpha - 1)$$

where ξ and η are the components of $\mathbf{r}$ in the rotated (ξ, η) coordinate system. We then find that

$$\int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \sqrt{1 - (\mathbf{s} \cdot \mathbf{s}_0)^2} f(\mathbf{s}, \mathbf{s}_0) e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$
$$= \int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha\, |\sin\alpha| f(\alpha, \alpha_0) e^{ik_0[\xi \sin\alpha + \eta(\cos\alpha - 1)]},$$

where $f(\alpha, \alpha_0)$ is $f(\mathbf{s}, \mathbf{s}_0)$ expressed in terms of α and α_0. We also find that

$$\int_{-\pi}^{\pi} d\alpha_0 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha \sqrt{1 - (\mathbf{s} \cdot \mathbf{s}_0)^2} f^*(-\mathbf{s}, -\mathbf{s}_0) e^{ik_0(\mathbf{s}-\mathbf{s}_0)\cdot\mathbf{r}}$$
$$= \int_{0}^{2\pi} d\alpha_0' \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\alpha'\, |\sin\alpha'| f^*(\alpha', \alpha_0') e^{-ik_0[\xi \sin\alpha' + \eta(\cos\alpha' - 1)]}$$

where $\alpha_0' = \alpha_0 + \pi$ and α' is the angle formed between $-\mathbf{s}$ and $-\mathbf{s}_0$. Substituting these two expressions into Eq.(7.8) then yields Eq.(7.58).

7.16 Generalize the formulation developed in Example 7.1 to the case of a sphere.

This problem is easily done using the eigen functions appropriate to a spherical coordinate system as opposed to a cylindrical system.