Solutions to the Tutorial Problems in
the book "Magnetohydrodynamics of the Sun"
by ER Priest (2014)
CHAPTER 6

PROBLEM 6.1. Two-Dimensional X- and O-points.
Show that any linear null with field components

$$B_X = bX + 2cY, \qquad B_Y = -2aX + dY,$$

can be transformed to

$$B_x = B_0 \frac{y}{L_0}, \qquad B_y = B_0 \bar{\alpha}^2 \frac{x}{L_0}.$$

SOLUTION.
Suppose we expand the two-dimensional field (B_X, B_Y) near a neutral point in a Taylor series and keep only the first-order, linear terms, so that

$$B_X = bX + 2cY, \qquad B_Y = -2aX + dY,$$

where $a, b, c,$ and d are arbitrary constants, and $\nabla \cdot \mathbf{B} = 0$ implies that $d = -b$.

In terms of the *magnetic flux function* (A) the field components are

$$B_X = \frac{\partial A}{\partial Y}, \qquad B_Y = -\frac{\partial A}{\partial X},$$

which implies that the corresponding flux function is

$$A = aX^2 + bXY + cY^2,$$

where we have chosen the constant of integration to make A vanish at the origin.

This expression for A may be simplified by rotating the XY-axes through an angle θ to give new xy-axes, such that $X = x\cos\theta - y\sin\theta, \quad Y = x\sin\theta + y\cos\theta$. If in particular the angle (θ) is chosen such that $\tan(2\theta) = b/(a-c)$, then the xy-term in the resulting expression for A vanishes and we are left with a flux function

$$A = \frac{B_0}{2L_0}(y^2 - \bar{\alpha}^2 x^2), \tag{1}$$

where

$$\frac{B_0}{L_0} = (a+c) - \sqrt{b^2 + (a-c)^2}, \qquad \bar{\alpha}^2 = \frac{\sqrt{b^2+(a-c)^2} + (a+c)}{\sqrt{b^2+(a-c)^2} - (a+c)}$$

and L_0 is the length-scale over which the field is varying. The corresponding field components are

$$B_x = B_0 \frac{y}{L_0}, \qquad B_y = B_0 \bar{\alpha}^2 \frac{x}{L_0},$$

so that B_x vanishes on the x-axis and B_y on the y-axis.

PROBLEM 6.2. Magnetic Relaxation.
Consider the process of magnetic relaxation described in Section 6.3.2.4 of an ideal incompressible plasma satisfying the model equation of motion

$$\rho \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B} - K\mathbf{v}. \tag{2}$$

(i) Prove that the magnetic energy decreases when the Lorentz force does positive work on the plasma. (ii) Prove Equation

$$\frac{d}{dt}(W_m + W_k) = -2KW_k, \tag{3}$$

so that the total energy decreases monotonically.

SOLUTION.

$$\frac{dW_m}{dt} = \frac{1}{\mu} \int_{\mathcal{D}} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}\, dV = \frac{1}{\mu} \int_{\mathcal{D}} \mathbf{B} \cdot \nabla \times (\mathbf{v} \times \mathbf{B})\, dV,$$

which may be transformed using the divergence theorem and the boundary conditions on $\partial\mathcal{D}$ to give

$$\frac{dW_m}{dt} = -\int_{\mathcal{D}} \mathbf{v} \cdot \mathbf{j} \times \mathbf{B}\, dV, \tag{4}$$

so that the magnetic energy decreases when the Lorentz force does positive work on the plasma and vice versa. The kinetic energy $[W_k(t) = \frac{1}{2}\rho \int_{\mathcal{D}} v^2 dV]$ changes at a rate

$$\frac{dW_k}{dt} = \rho \int_{\mathcal{D}} \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t}\, dV = \int_{\mathcal{D}} -\mathbf{v} \cdot \nabla p + \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} - K\mathbf{v} \cdot \mathbf{v}\, dV,$$

which may also be transformed using the divergence theorem, the incompressibility condition and the boundary conditions to

$$\frac{dW_k}{dt} = \int \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} \, dV - 2KW_k, \tag{5}$$

so that the kinetic energy is increased by a (positive) Lorentz force and decreased by dissipation.

Combining (**??**) and (**??**) we find

$$\frac{d}{dt}(W_m + W_k) = -2KW_k. \tag{6}$$

Since $W_k > 0$ and $K > 0$, the total energy decreases monotonically and, being positive, it must tend to a finite limit.

PROBLEM 6.3. Cusp Points in Sheared X-point Fields.
Consider the field near a cusp point in Figure 6.10e of section 6.3.3. Suppose shearing is present only in region I below the X-point, so that in regions II and III to either side $B_z = 0$. In region I seek a self-similar solution of the form $A = r^a f(\xi)$, where $\xi = \theta / r^b$, for which the equilibrium equation becomes $\nabla^2 A = -\epsilon A^{-n}$ and the separatrix is $\xi = 1$. In region II seek a potential field. Show that for total pressure balance across the separatrix $a = 1 + 3b/2$ and $n = (2 + b)/(2 + 3b)$.

SOLUTION.
Consider the simplest case where there is shearing present only in the region (I), say, below the X-point, so that in the regions (II) and (III) to either side $B_z = 0$ and the field is potential with $\nabla^2 A = 0$.

In (I) near the cusp there is a self-similar solution

$$A = r^a f(\xi), \tag{7}$$

where

$$\xi = \frac{\theta}{r^b}, \tag{8}$$

so the separatrix ($A = 0$, say) is $\xi = 1$; in other words, it is not a straight line but a curve $\theta = r^b$, where $b > 0$. Then the field components are

$$\begin{aligned} B_r &= \frac{1}{r}\frac{\partial A}{r \partial \theta} = r^{a-1-b} f'(\xi), \\ B_\theta &= -\frac{\partial A}{\partial r} = -a r^{a-1} f(\xi) + b^{a-1} f'(\xi) \xi. \end{aligned}$$

The equilibrium equation

$$\nabla^2 A = -B_z \frac{dB_z}{dA}$$

must have the right-hand side of the form $-\epsilon A^{-n}$, where substitution of (**??**) gives

$$n = \frac{2b + 2 - a}{a}, \tag{9}$$

and to lowest order the function $f(\xi)$ is given by

$$f'' = -\epsilon f^{-n}. \tag{10}$$

In region II the field is potential and an appropriate form for A is

$$A = B_0 r \sin\theta + B_1 r^{K_1} \sin K_1(\theta - \pi), \tag{11}$$

where B_1 and K_1 are constants. This has $A = 0$ on $\theta = \pi$, the vertical arm of the separatrix, as required. As far as region II is concerned, the curved part of the separatrix is given by

$$\theta = \frac{B_1}{B_0} r^{K_1 - 1} \sin K_1 \pi, \tag{12}$$

so, by comparing with the form $\theta = r^b$ in region I, we see that

$$K_1 = 1 + b. \tag{13}$$

Finally, magnetic pressure balance across the separatrix dividing regions I and II gives

$$B_{z0}^2 + c r^{2(a-1-b)} = B_0^2 + 2K B_1 B_0 r^b \cos K_1 \pi.$$

In order to match the variations in r across the separatrix, we need

$$a = 1 + \frac{3b}{2}. \tag{14}$$

Thus, our cusp solutions have one free parameter (namely, b) whose value can be determined by the global equilibrium solution. The parameters K_1 and a are given in terms of it by (**??**) and (**??**), while the current parameter (n) follows from (**??**) as

$$n = \frac{2 + b}{2 + 3b}.$$

PROBLEM 6.4. Diffusion of a Current Sheet.
Prove the result
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{B^2}{2\mu}dx = -\int\frac{j^2}{\sigma}dx. \tag{15}$$
that for a one-dimensional current sheet obeying the diffusion equation
$$\frac{\partial B}{\partial t} = \eta\,\frac{\partial^2 B}{\partial x^2}, \tag{16}$$
the magnetic energy is converted continuously into heat as it diffuses.

SOLUTION.
The magnetic energy ($\int_{-\infty}^{\infty} B^2/(2\mu)dx$) decreases in time at a rate
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{B^2}{2\mu}dx = \int_{-\infty}^{\infty}\frac{B}{\mu}\frac{\partial B}{\partial t}dx.$$
Substituting for $\partial B/\partial t$ from Eq.(6.21) and integrating by parts, this becomes
$$\int_{-\infty}^{\infty}\frac{B\eta}{\mu}\frac{\partial^2 B}{\partial x^2}dx = \frac{1}{\mu^2\sigma}\left\{\left[B\frac{\partial B}{\partial x}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(\frac{\partial B}{\partial x}\right)^2 dx\right\}.$$
Since $\partial B/\partial x$ remains equal to zero at infinity, the first term vanishes, and, since the electric current is $j = \mu^{-1}\partial B/\partial x$, we finally have
$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{B^2}{2\mu}dx = -\int\frac{j^2}{\sigma}dx.$$
In other words, magnetic energy is converted entirely into heat by ohmic dissipation (j^2/σ per unit volume).

PROBLEM 6.5. Advection of a One-Dimensional Magnetic Field.

Find the effect in the limit of $R_m >> 1$ of a stagnation-point flow [$v_x = -V_0x/a$, $v_y = V_0y/a$] on a field that is initially $\mathbf{B} = B_0\cos(x/a)\hat{\mathbf{y}}$ at $t = 0$ between $x = -\frac{1}{2}\pi a$ and $\frac{1}{2}\pi a$.

SOLUTION.

The equations of the streamlines (namely, xy = constant) are obtained from $dy/dx = v_y/v_x = -y/x$. These are rectangular hyperbolae (Figure 6.12a) with inflow along the x-axis and outflow along the y-axis when $V_0 > 0$.

The velocity field corresponds to a hydrodynamic *stagnation-point flow*. The effect of this flow on the magnetic field is to carry the field lines inwards from the sides and accumulate them near $x = 0$, increasing the field strength there. Since the component (v_x) of the velocity perpendicular to the field lines is constant along a particular field line (x =constant), the field lines are not distorted but remain straight as they come in.

Now, the y-component of the induction equation (6.21) when $R_m \gg 1$ is $\partial B/\partial t = -\partial/\partial x(v_x B)$ or

$$\frac{\partial B}{\partial t} - \frac{V_0 x}{a}\frac{\partial B}{\partial x} = \frac{V_0 B}{a}, \tag{17}$$

and this determines $B(x,t)$. In order to solve such a partial differential equation, we consider characteristic curves in the xt-plane, which are defined to be such that

$$\frac{dx}{dt} = -\frac{V_0 x}{a}, \tag{18}$$

with solution

$$x = x^* e^{-V_0 t/a}, \tag{19}$$

where $x = x^*$, say, at $t = 0$. We wish to determine $B(x,t)$ at every point of the xt-plane and the elegance of considering characteristic curves, $x = x(t)$ given by (**??**) (Figure 6.17Ba), is that on such curves $B(x(t),t)$ has the derivative

$$\frac{dB}{dt} = \frac{\partial B}{\partial t} + \frac{dx}{dt}\frac{\partial B}{\partial x} = \frac{\partial B}{\partial t} - \frac{V_0 x}{a}\frac{\partial B}{\partial x},$$

by (**??**), or, from (**??**), $dB/dt = V_0 B/a$. In other words, on the characteristic curves we have a simple ordinary differential equation to solve in place of (**??**): the solution is B = constant $e^{V_0 t/a}$ or, since $x = x^*$ and $B = B_0 \cos x^*/a$ at $t = 0$, we have

$$B(x,t) = B_0 \cos(x^*/a)\, e^{V_0 t/a}.$$

However, in this solution x^* is a constant which we have introduced for convenience and which was not present in the initial statement of the problem, so we should eliminate it by using (**??**), with the final result

$$B(x,t) = B_0 \cos(\frac{x}{a} e^{V_0 t/a})\, e^{V_0 t/a}.$$

This solution may be plotted against x for several times. It can be shown that the field does indeed, as expected, concentrate near $x = 0$ as time proceeds. The field strength at the origin is $B(0,t) = B_0 e^{V_0 t/a}$, which grows exponentially in time (or decreases if the flow is reversed by taking $V_0 < 0$).

PROBLEM 6.6. Magnetic Annihilation.
Show that the stagnation-point flow

$$v_x = -\frac{V_0\ x}{a}, \quad v_y = \frac{V_0\ y}{a}, \tag{20}$$

acting on a magnetic field $[\mathbf{B} = B(x)\hat{\mathbf{y}}]$ satisfying

$$E - \frac{V_0 x}{a} B = \eta \frac{dB}{dx} \tag{21}$$

also satisfies the equation of motion and so is an exact solution of the MHD equations.

SOLUTION.
For straight magnetic field lines the equation of motion reduces to

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\left(p + \frac{B^2}{2\mu}\right)$$

or

$$\rho[-\mathbf{v}\times(\nabla\times\mathbf{v}) + \nabla(\tfrac{1}{2}v^2)] = -\nabla(p + \tfrac{B^2}{2\mu}).$$

However, the flow (**??**) has zero vorticity ($\nabla\times\mathbf{v} = 0$) and ρ is constant, so the first term in the above equation vanishes while the other terms imply that

$$\nabla\left(p + \frac{B^2}{2\mu} + \tfrac{1}{2}\rho v^2\right) = 0,$$

and thus

$$p = p_s - \tfrac{1}{2}\rho v^2 - \frac{B^2}{2\mu},$$

where p_s is the pressure at the stagnation point situated at the origin.

PROBLEM 6.7. Energetics of the Sweet-Parker Mechanism.

(i) Prove the steady-state *electromagnetic energy equation* from Ohm's law and Maxwell's equations:

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \frac{j^2}{\sigma} + \mathbf{v} \cdot \mathbf{j} \times \mathbf{B}.$$

(ii) Hence prove that half of the inflowing electromagnetic energy in the Sweet-Parker mechanism goes into ohmic heating and half into the work done by $\mathbf{j} \times \mathbf{B}$ when $R_m \gg 1$.

(iii) Use the *mechanical energy equation* to show that the work done by $\mathbf{j} \times \mathbf{B}$ goes into kinetic energy when the plasma is incompressible and that the work done by the $\boldsymbol{\nabla} p$ is negligible.

SOLUTION.
(i) a vector identity for the divergence of a vector product together with $\mathbf{j} = \boldsymbol{\nabla} \times \mathbf{B}/\mu$ and $\nabla \times \mathbf{E} = 0$ (for a steady state) imply that

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{E} \cdot \mathbf{j}.$$

However, the scalar product of Ohm's law in turn with $\mathbf{j}$ gives

$$\mathbf{E} \cdot \mathbf{j} = \frac{j^2}{\sigma} + \mathbf{v} \cdot \mathbf{j} \times \mathbf{B}.$$

Combining these together gives the electromagnetic energy equation.

(ii) Integrate it over the diffusion region with volume V and surface S and use the divergence theorem to give

$$-\int_S \mathbf{E} \times \mathbf{H} \cdot \mathbf{dS} = \int_V \left(\frac{j^2}{\sigma} + \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} \right) dV,$$

in which $E \approx -v_i B_i$ along the inflow part of S. The outflow contribution ($EB_o l/\mu$) is smaller than the inflow contribution ($EB_i L/\mu$) by a factor $l^2/L^2 \approx R_{mi}^{-1}$ so that it may be neglected. Furthermore, the current in the centre of the sheet is roughly $B_i/(\mu l)$ and so we may approximate the mean value of j^2 by roughly $\frac{1}{2}B_i^2/(\mu l)^2$ and the mean value of $\mathbf{v} \cdot \mathbf{j} \times \mathbf{B}$ by $\frac{1}{2}v_o[B_i/(\mu l)]B_o$. Thus, the above equation becomes approximately

$$v_i B_i \frac{B_i}{\mu} 4L = \left[\frac{B_i^2}{2\mu^2 l^2 \sigma} + \frac{v_o B_i B_o}{2\mu l} \right] 4Ll,$$

since the two opposing inflow sides of the current sheet have a total length of $4L$ and the volume is $4Ll$ (per unit length in the z-direction). Replacing v_oB_o by v_iB_i in the last term due to the uniformity of the electric field, we find

$$v_i = \frac{\eta}{2l} + \frac{v_i}{2} \tag{$*$}$$

or $v_i = \eta/l$. In other words, we recover the diffusion result (6.43), which is not surprising since both are essentially a consequence of Ohm's Law (6.41). However, what we can deduce from (**??**) is that half of the inflowing electromagnetic energy goes into ohmic heating and half into the work done by the magnetic force.

(iii) The *equation of mechanical energy* may be derived in the incompressible limit by taking the scalar product of the equation of motion with $\mathbf{v}$ and using $\nabla \cdot \mathbf{v} = 0$, so that

$$\nabla \cdot \left(\tfrac{1}{2}\rho v^2 \mathbf{v}\right) = \mathbf{v} \cdot \mathbf{j} \times \mathbf{B} - \mathbf{v} \cdot \nabla p,$$

which implies that a change of kinetic energy is produced by the work done by $\mathbf{j} \times \mathbf{B}$ and $-\nabla p$. Next, rewriting $\mathbf{v} \cdot \nabla p$ as $\nabla \cdot (p\mathbf{v})$ since $\nabla \cdot \mathbf{v} = 0$ in the present model and integrating over the diffusion region, we find

$$\int_S \left(\tfrac{1}{2}\rho v^2 \mathbf{v} + p\mathbf{v}\right) \cdot \mathbf{dS} = \int_V (\mathbf{v} \cdot \mathbf{j} \times \mathbf{B})\, dV,$$

and so the magnetic force term on the right of this equation is a combination of the change in kinetic energy and the net work done by pressure on the surface. In particular, when the pressure term is negligible, we find that all the work done by $\mathbf{j} \times \mathbf{B}$ goes into kinetic energy.

PROBLEM 6.8. Quasi-Separatrix Layer.
Calculate the norm (N) of the mapping for the sheared X-field $(B_x, B_y, B_z) = (x, -y, l)$ inside a cube.

SOLUTION.
Consider the sheared X-field

$$(B_x, B_y, B_z) = (x, -y, l)$$

inside a cube with $l \ll 1$. The mapping from the base (S_0) to the top and

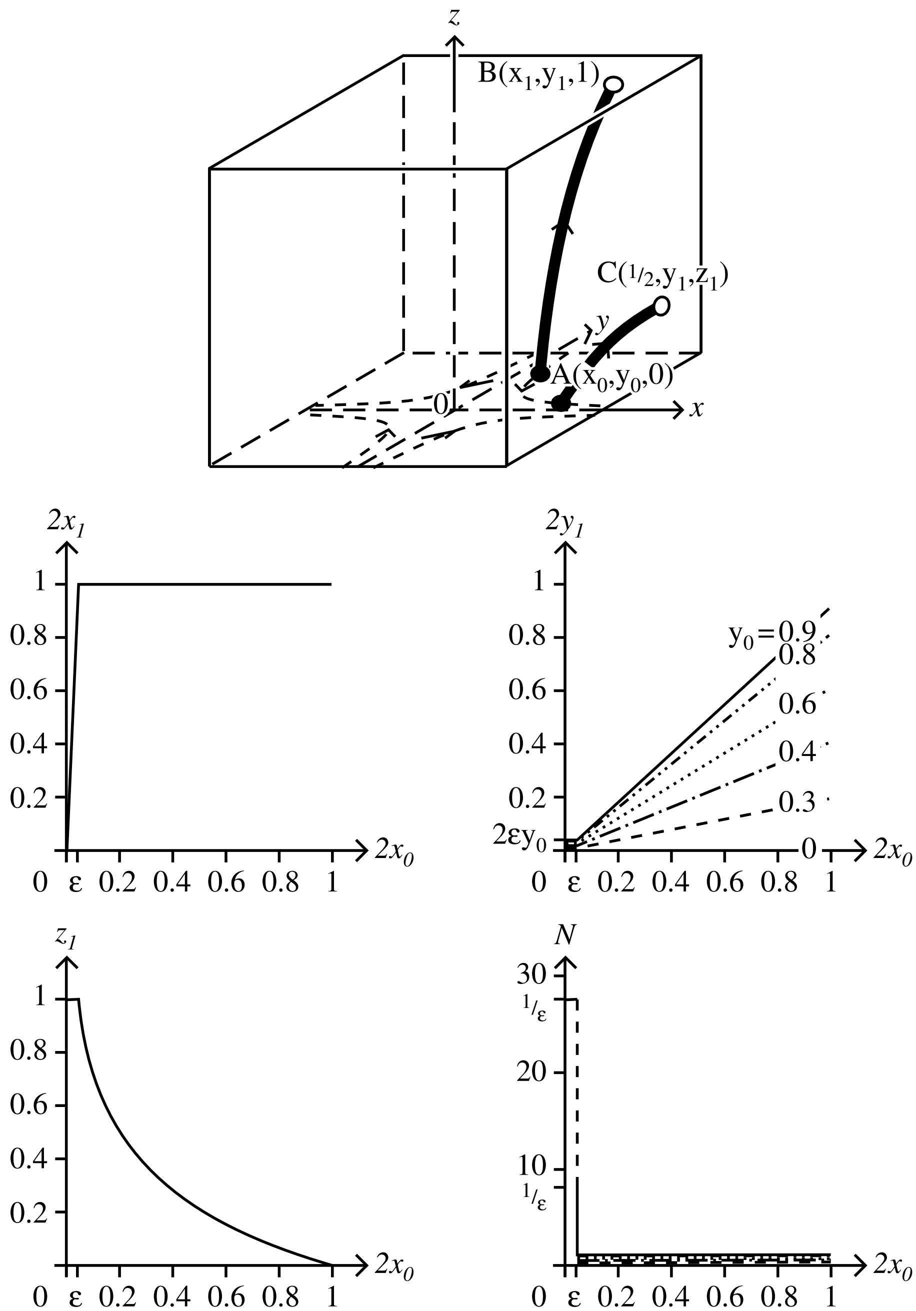

Figure 1: Sheared X-field in a cube together with the variations of the end-point coordinates (x_1, y_1, z_1) on the top and side boundaries and of the norm (N) with the initial footpoint coordinates (x_0 and y_0) on the bottom boundary.

sides (S_1) is given by

$$x_1 = x_0\, e^{z_1/l} \quad , \quad y_1 = y_0\, e^{-z_1/l}.$$

Thus, when the point A$(x_0, y_0, 0)$ on S_0 is so close to the y-axis that $2x_0 < \epsilon$, A maps to a point B on the top ($z_1 = 1$) and

$$\mathcal{F} = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix},$$

while

$$N \approx \frac{1}{\epsilon},$$

where

$$\epsilon = e^{-1/l} \ll 1.$$

On the other hand, when $\epsilon < 2x_0 < 1$, A maps to C on the side ($x_1 = \frac{1}{2}$), while the elements of $\mathcal{F}$ and the value of N are of order unity. The resulting variations of x_1, y_1, z_1, N with x_0 are shown in Figure 8.27, which reveals the quasi-separatrix layer as a very narrow region of width ϵ where $N \gg 1$. When $l = 0.1$ the value of N in the quasi-separatrix layer is 10^4, and even when l is as large as 0.3, N is about 28 in the quasi-separatrix layer. If the cube is replaced by a hemisphere or sphere, similar forms are produced but the functions become continuous and differentiable.

PROBLEM 6.9. Field-Conservation in Ideal MHD.
Use the equation of mass continuity and the ideal induction equation to show that an elemental segment ($\delta\mathbf{l}$) of a line moving with the plasma obeys the same equation as $\mathbf{B}/\rho$.

SOLUTION.
A direct proof of line conservation in ideal MHD is as follows. Assume ideal MHD

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}). \tag{22}$$

and use the mass continuity equation

$$d\rho/dt \equiv \partial\rho/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla}\rho = -\rho \boldsymbol{\nabla} \cdot \mathbf{v}$$

to eliminate $\boldsymbol{\nabla}\cdot\mathbf{v}$. The result is

$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right)=\left(\frac{\mathbf{B}}{\rho}\cdot\boldsymbol{\nabla}\right)\mathbf{v}, \tag{23}$$

where $d/dt\,(=\partial/\partial t+\mathbf{v}\cdot\boldsymbol{\nabla})$ is the *total* or *convective derivative.*

Consider next an elemental segment $\delta\mathbf{l}$ along a line moving with the plasma. If $\mathbf{v}$ is the plasma velocity at one end of the element and $\mathbf{v}+\delta\mathbf{v}$ is the velocity at the other end, then the differential velocity between the two ends is $\delta\mathbf{v}=(\delta\mathbf{l}\cdot\boldsymbol{\nabla})\mathbf{v}$. During the time interval dt, the segment $\delta\mathbf{l}$ therefore changes at the rate

$$\frac{d\delta\mathbf{l}}{dt}=\delta\mathbf{v}=(\delta\mathbf{l}\cdot\boldsymbol{\nabla})\mathbf{v}.$$

Since this equation has exactly the same form as (**??**) for the vector $\mathbf{B}/\rho$, it follows that, if $\delta\mathbf{l}$ and $\mathbf{B}/\rho$ are initially parallel, then they will remain parallel for all time. In other words, any two neighbouring plasma elements on a field line are always on the same field line, with the distance between them proportional to $\mathbf{B}/\rho$ – i.e., the field lines are "frozen" to the plasma.

PROBLEM 6.10. Non-Ideal Field-Line Velocity.
Show that for two-dimensional resistive flow there is no unique definition of field-line velocity.

SOLUTION.
Consider, for example, a two-dimensional flow and magnetic field in the xy-plane, say, satisfying the resistive Ohm's Law

$$\mathbf{E}+\mathbf{v}\times\mathbf{B}=\frac{\mathbf{j}}{\sigma},$$

where $\mathbf{E}$ and $\mathbf{j}$ are aligned in the z-direction. Then the plasma velocity normal to the field is

$$\mathbf{v}_\perp=\frac{(\mathbf{E}-\mathbf{j}/\sigma)\times\mathbf{B}}{B^2}$$

and a possible field-line velocity is

$$\mathbf{w}_\perp=\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{24}$$

so that the slippage velocity is the difference $(\mathbf{v}_\perp-\mathbf{w}_\perp)$.

Now, in general The component ($\mathbf{w}_\perp$) of a field-line velocity may then be defined if and only if Ohm's Law can be transformed into the form

$$\mathbf{E} + \mathbf{w} \times \mathbf{B} = \mathbf{a}, \tag{25}$$

where

$$\nabla \times \mathbf{a} = \lambda_w \mathbf{B}$$

and λ_w is some scalar function.

The form (**??**) therefore works because it implies that $\mathbf{E} + \mathbf{w}_\perp \times \mathbf{B} = 0$, which is of the required form (**??**) with $\mathbf{a} = 0$ and $\lambda_w = 0$. However, another possible field-line velocity is

$$\mathbf{w}'_\perp = \frac{(\mathbf{E} - K^* \,\hat{\mathbf{z}}) \times \mathbf{B}}{B^2},$$

where K^* is a constant. This also works since it implies that

$$\mathbf{E} + \mathbf{w}_\perp \times \mathbf{B} = K^* \hat{\mathbf{z}},$$

which is again of the form (**??**) but now with $\mathbf{a} = K^* \hat{\mathbf{z}}$ and $\lambda_w = 0$. In other words, the field-line velocity is not unique.

PROBLEM 6.11. Euler Potentials.
Use the equations (6.73) for Euler potentials to show that line conservation holds if $\mathbf{B} \times (\nabla \times \mathbf{N}) = \mathbf{0}$ and flux conservation holds if $\nabla \times \mathbf{N} = \mathbf{0}$.

SOLUTION.
We first express the magnetic field as

$$\mathbf{B} = \nabla\alpha \times \nabla\beta \tag{26}$$

in terms of *Euler potentials* (α and β). Then we write Ohm's law ($\mathbf{E}+\mathbf{v}\times\mathbf{B} = \mathbf{N}$) In terms of components parallel to $\nabla\alpha, \nabla\beta$ and ∇s as

$$\frac{d\alpha}{dt} = -\frac{\partial\Psi}{\partial\beta} - N^\beta, \tag{27}$$

$$\frac{d\beta}{dt} = \frac{\partial\Psi}{\partial\alpha} + N^\alpha, \tag{28}$$

$$\frac{\partial\Psi}{\partial s} = -N^s, \tag{29}$$

Line conservation would hold if the right-hand sides of (**??**) and (**??**) were independent of s, i.e.,

$$\frac{\partial}{\partial s}\left(\frac{\partial \Psi}{\partial \beta} + N^{\beta}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \Psi}{\partial \alpha} + N^{\alpha}\right) = 0, \tag{30}$$

so that the behaviour of α and β is independent of distance along a field line. The general solution of these equations (**??**) is

$$N^{\alpha} = \frac{\partial f}{\partial \alpha} + \frac{\partial g}{\partial \beta}\,, \qquad N^{\beta} = \frac{\partial f}{\partial \beta} - \frac{\partial g}{\partial \alpha}\,, \qquad N^{s} = \frac{\partial f}{\partial s},$$

which in turn is equivalent to

$$\mathbf{B} \times (\nabla \times \mathbf{R}) = \mathbf{0}, \tag{31}$$

where $f(\alpha, \beta, s, t,)$ and $g(\alpha, \beta, t)$ are arbitrary functions.

Furthermore, *flux conservation* would still hold if Liouville's theorem holds or, in other words, if α and β are of Hamiltonian form so that

$$\frac{d\alpha}{dt} = -\frac{\partial F}{\partial \beta}, \qquad \frac{d\beta}{dt} = \frac{\partial F}{\partial \alpha}, \tag{32}$$

where $F = F(\alpha, \beta, t)$. Then, if we write the function Ψ in (**??**) - (**??**) in the form $\Psi(\alpha, \beta, s, t) = F(\alpha, \beta, t) - G(\alpha, \beta, s, t)$, the pair (**??**) and (**??**) is the same as (**??**) if and only if

$$N^{\alpha} = \frac{\partial G}{\partial \alpha}, \qquad N^{\beta} = \frac{\partial G}{\partial \beta},$$

or, in other words,

$$\nabla \times \mathbf{N} = \mathbf{0}. \tag{33}$$