Problems and Exercises

Vectors

1. *Useful vector relations*:
Show that

$$|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}|, \qquad |\mathbf{a}+\mathbf{b}| \le |\mathbf{a}| + |\mathbf{b}|,$$

and

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a},\\ (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}),\\ |\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2.\end{aligned}$$

Note: it is often helpful to use component representations.
Recall $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

2. *Vector differentiation*:
Show that

$$\frac{\partial(\mathbf{a}\cdot\mathbf{b})}{\partial\mathbf{a}} = \mathbf{b}, \qquad \frac{\partial|\mathbf{a}|}{\partial\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}.$$

3. *Differential operations*: for vector $\mathbf{v}$ and scalar ψ
Show that

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) &= \mathbf{0},\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\psi) &= 0,\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{v}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{v}) - \boldsymbol{\nabla}^2\mathbf{v},\\ \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{v}) &= \tfrac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}.\end{aligned}$$

PartI

Strain

1. A unit cube is deformed homogeneously by the mapping $\boldsymbol{\xi} \to \mathbf{x} = \mathsf{F}\boldsymbol{\xi}$ with respect to coordinates along the edge of the cube. Obtain formulae for the lengths of the corresponding edges of the resulting parallelepiped, their included angles, and the distances between pairs of opposite faces.

Hence find the specific values of these quantities for a rhombohedron obtained by doubling the length of the long diagonal of the cube from the origin to $(1, 1, 1)$, without changing the directions in transverse planes.

Verify that a congruent rhombohedron could be generated by a homogeneous deformation that maps the cube edges $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ onto $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$ respectively and find the required matrix F.

2. A plane deformation is specified by

$$x_1 = \lambda_1\xi_1 + \gamma\lambda_2\xi_2, \quad x_2 = \lambda_2\xi_2$$

which is a pure strain with principal stretches λ_1, λ_2 followed by a simple shear of amount γ. For $\lambda_1 \neq \lambda_2$ and small γ, verify that the overall principal stretches are $\lambda_1(1 + \eta + \dots)$ and $\lambda_2(1 - \eta + \dots)$ where

$$(\lambda_1^2/\lambda_2^2 - 1)\eta = \tfrac{1}{2}\gamma^2.$$

Show that after the shear γ the Lagrangian diad has an inclination to the reference axes of $\lambda_1\lambda_2\gamma/(\lambda_1^2 - \lambda_2^2)$ to first order. Find the corresponding inclination of the Eulerian diad.

3. A simple shear $\mathbf{I} + \gamma\mathbf{m}\mathbf{n}^{\mathsf{T}}$ is followed by a deformation F. Show that the final configuration could alternatively be produced by F followed by a simple shear. Determine its amount, direction and basal plane.

4. Consider the displacement $\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$ in a homogeneous deformation. Define the displacement gradient A such that $\mathbf{u} = \mathsf{A}\boldsymbol{\xi}$. Show that $\mathsf{A} = \mathsf{F} - \mathbf{I}$ in terms of the deformation gradient F. Using the decomposition theorem demonstrate that:

$$\mathsf{A} = \mathsf{R}\mathsf{U} - \mathbf{I} = (\mathsf{R} - \mathbf{I}) + (\mathsf{U} - \mathbf{I}) + (\mathsf{R} - \mathbf{I})(\mathsf{U} - \mathbf{I}),$$

and express the Green strain tensor E in terms of A. What simplifications occur when the displacement and rotation are small?

Find explicit forms for the displacement gradient for simple shear and pure shear.

5. For a uniform dilatation $\mathbf{x} = \kappa\boldsymbol{\xi}$ show that the Green strain tensor is $E_{ij} = \frac{1}{2}(\kappa^2 - 1)\delta_{ij}$, and the Cauchy strain tensor is $e_{ij} = \frac{1}{2}(1 - 1/\kappa^2)\delta_{ij}$.
When $\kappa = 1 + \epsilon$ with $\epsilon \ll 1$ demonstrate that the two definitions of the strain tensor agree to first order in ϵ.

6. A body undergoes the deformation specified by

$$x_1 = \xi_1(1 + a^2t^2), \quad x_2 = \xi_2, \quad x_3 = \xi_3.$$

Find the displacement and velocity in both the material and spatial descriptions.

Stress

1. Verify that any stress of the type $\sigma_{ij} = r^{n-2}[(n+2)r^2\delta_{ij} - x_i x_j]$, where $r = |\mathbf{x}| \neq 0$ is self-equilibrated. Find the traction vector at any point on the surface of a hemispherical shell with internal and external radii a and b centred on the origin. Check your tractions are in overall balance by resolving in the direction of the axis of symmetry.

2. Define the mean stress in a volume V by

$$V\bar{\sigma}_{ij} = \int_V \sigma_{ij} dV$$

Show that for a volume with surface tractions τ, body force $\mathbf{g}$ and mass-acceleration $\mathbf{f}$ that

$$V\bar{\sigma}_{ij} = \int_S x_i \tau_j dS + \int_V x_i(g_j - f_j) dm$$

An isolated spherical planet, with radius a and uniform density ρ, has a constant spin ω against the stellar background. Show that the mean stress in the planet is a hydrostatic pressure $(1/5)\rho g a$ combined with a biaxial tension $(1/5)\rho a^2\omega^2$ perpendicular to the polar axis, where g is the acceleration at the surface of the planet due to its own gravitation.

Constitutive Relations

1. What modification to the constitutive law within a body would be needed to describe the behaviour when the temperature is raised at fixed strain?

2. Consider a material in which the stress depends non-linearly on the strain and strain rate through power law dependencies.

Formulate the description of three-dimensional deformation in such a material. How wouuld the behaviour of the material be expected to differ from the linear viscoelastic case? What simplifications might you wish to impose on this system – are such assumptions appropriate to Earth materials?

Elasticity and linear viscoelasticity

1. Show that the displacement field for elastic waves in an isotropic medium at angular frequency ω can be written in the form

$$\mathbf{u} = \{A_P[p, 0, \pm q_\alpha]\, e^{\pm i\omega q_\alpha x_3} \\ +B_V[q_\beta, 0, \mp p]\, e^{\pm i\omega q_\beta x_3} \\ + B_H[0, 1, 0]\, e^{\pm i\omega q_\beta x_3}\}\; e^{i\omega(px-t)}$$

and find expressions for q_α, q_β. Demonstrate that the $\pm$ signs correspond to phase fronts progressing in the $\pm x_3$ direction respectively. What happens if $p > 1/\alpha$?

If the plane $x_3 = 0$ is free from traction, find the reflected waves due to an incident wavefield from $x_3 > 0$. Show that the SH components are decoupled from the P and SV waves.

Discuss how the behaviour would be modified for a linear visco-elastic medium.

2. For an isotropic elastic medium consider two-dimensional deformation (independent of the 3-coordinate). Using greek indices α, β, γ =1,2 show that
(a) the displacement componet $u_3(x_1, x_2, t)$ satisfies

$$\mu\partial_{\beta\beta}u_3 + \rho f_3 = \rho\partial_{tt}u_3.$$

(b) show in the plane strain case where all field variables are independent of x_3

$$\mu\partial_{\beta\beta}u_\alpha + (\lambda + \mu)\partial_{\beta\alpha}u_\beta\rho f_\alpha = \rho\partial_{tt}u_\alpha.$$

and derive the stress components in terms of the horizontal components and displacement.
(c) for plane stress when $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ show that

$$\sigma_{\alpha\beta} = \frac{2\mu\lambda}{\lambda + 2\mu}\partial_\gamma u_\gamma\delta_{\alpha\beta} + \mu(\partial_\alpha u_\beta + \partial_\beta u_\alpha),$$

and find the equations of motion for displacement. [*This situation is often used as an approximate representation of the motions of a thin sheet*]

Continua under Pressure

1. For pressures upto 10 GPa the equation of state for water is well approximated by

$$\frac{p + B}{p_0 + B} = \left(\frac{\rho}{\rho_0}\right)^n,$$

with $B = 300$ MPa, $p_0 = 100$ kPa and $\rho_0 = 1000\ \mathrm{kg\,m^{-3}}$.

(a) calculate the bulk modulus K for water, (b) calculate the density and pressure distribution in the ocean, (c) what is the pressure and the relative compression of the water and the deepest point in the ocean (10.9 km deep)?

2. The planet Mercury is sufficiently small that it is a reasonable approximation to neglect compression and treat the mantle and core as having constant densities, similar to those on Earth.

If the mean density of Mercury is estimated to be 5300 kg m^{-3}, what is the ratio of the the core radius c to the total radius a.

Calculate the moment of inertia $\int_0^a 4\pi r^2 \rho r^2 dr$ and compare with the value for the case where density is uniform.

3. For an adiabatic radial gradient in hydrostatic equilibrium

$$\frac{dp}{dr} + \rho g(r) = 0,$$

where g is the gravitational acceleration. Derive the Adams-Williamson equation for the radial density gradient

$$-\frac{d\rho}{dr} = \frac{\rho g(r)}{\Phi_s},$$

where Φ_s is the adiabatic seismic parameter $(= K_s/\rho)$.

If the radial temperature gradient differs from adiabatic by an amount τ, show that the radial temperature gradient is modified to

$$-\frac{d\rho}{dr} = \frac{\rho g(r)}{\Phi_s} + \alpha_{th}\rho\tau.$$

For a homogeneous adiabatic layer show that

$$\left(1 - \frac{1}{g}\frac{d\Phi_s}{dr}\right) = \left(1 + \rho\frac{d\Phi_s}{dp}\right) = \frac{dK_s}{dp}.$$

Deviations from this relation were used by Bullen to infer departures from homogeneity.

Fluids

1. Formulate the linear stability problem for the onset of convection in a layer of fluid heated from within. Assume that the boundaries are stress-free. Take the upper boundary to be isothermal and the lower boundary to be insulating. Carry the formulation to the point where the solution of the problem depends only on the integration of an ordinary differential equation for the stream function, subject to appropriate boundary conditions.

2. Verify that the solution (originally derived by Stokes) for a spherical ball of radius a moving at constant speed U in a fluid with viscosity η can be written in spherical polar coordinates (r, θ, ϕ), with the polar axis aligned in the asymptotic

flow direction as:

$$v_r = \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right) U \cos\theta,$$
$$v_\theta = -\left(1 - \frac{3a}{4r} + \frac{a^3}{4r^3}\right) U \sin\theta,$$
$$v_\phi = 0,$$

with associated pressure

$$p = -\eta \frac{3a}{2r^2} U \cos\theta.$$

Show that the radial stress

$$\sigma_{rr} = \eta \frac{3a}{2r^2}\left(3 - 2\frac{a^2}{r^2}\right) U \cos\theta,$$

and the only non-vanishing shear stress is

$$\sigma_{\theta r} = \eta \frac{1}{r}\left(2 - 3\frac{a}{r} - \frac{a^3}{2r^3}\right) U \sin\theta.$$

Hence show that the traction on the surface of the sphere is of constant magnitude and points in the direction of the asymptotic flow.

How far away from the sphere can the influence of the ball be largely ignored (e.g. reduced by a factor of 10).

3. Consider the scenario of a packet of fluid of density ρ in a rotating Earth with angular velocity $\boldsymbol{\omega}_E$. The Coriolis force is then $2\rho\mathbf{v}\times\boldsymbol{\omega}_E$. Take a right handed coordinate system with z upwards and show that the equations of motion for steady laminar flow in a horizontal direction, for viscosity η, are:

$$\eta \frac{\partial^2 v_x}{\partial z^2} + (2\rho\omega_E sin\vartheta) v_y = 0,$$
$$\eta \frac{\partial^2 v_y}{\partial z^2} + (2\rho\omega_E sin\vartheta) v_x = 0,$$

where ϑ is the latitude (measured in a northerly direction). Show that that v_x satisfies the fourth-order differential equation

$$\eta^2 \frac{\partial^4 v_x}{\partial z^4} + (2\rho\omega_E sin\vartheta)^2 v_x = 0,$$

and hence that there are solutions for which the velocity attenuates by a factor of e^{-1} over a depth ϵ which should be evaualted. This thickness ϵ represents the Ekman layer.

Electromagnetic disturbances

1. Consider the penetration of electromagnetic disturbances generated in the outer core (conductivity 5×10^5 S m^{-1}) into the mantle and core for periods of a day or longer.

If the ratio of conductivities between mantle and core is 10^3 to 1, how much of the field would be lost in passage to the Earth's surface and how does this vary with period?

In contrast what would be expected for the inner core where the conductivities are approximately equal?

Heat

1. Assuming a thermal conductivity of 2.5 W m^{-1} K^{-1} and a specific heat of 10^3 J kg^{-1} K^{-1}, estimate the depth to which temperatures in the Earth will be influenced by ice ages.

2. Volcanic flood basalts such as the Deccan traps rapidly accumulate large thickness of material and cover extensive areas. Consider a 2-km thick basalt lava flow erupted at 1500 K. If the solidus temperature is 1170 K estimate the time required for the flow to solidify. How far away from the flow would a significant thermal anomaly penetrate and what might be the consequences for the underlying rocks?

3. Consider the temperature distribution in a spherical planet of radius a, with heat production h and surface temperature T_0.

Find the temperature distributive in conductive equilibrium as a function of radius r for:
(a) constant conductivity k,
(b) radiative conductivity $k = CT^3$,
and hence deduce the central temperature.

For constant conductivity k and a power law variation in heat production ($h = h_0[r/a]^l$, with l a constant), find the central temperature as a function of a when h_0 varies with a such that the average $\langle h \rangle$ is independent of a.

Part II

These questions are somewhat open ended and frequently require reference to more than one chapter of the book

1. What are the likely circumstances that control the switch from dislocation-controlled to diffusion-controlled creep in the Earth (*consider both microscopic and macroscopic behaviour*)?
What are then the consequences for the style of deformation?

2. Discuss the development of shear zones in the Earth. Under what circumstances might such zones be regarded as equivalent to faults?

3. Much of the seismological analysis of earthquake mechanisms is based on the simplifying assumption of a "point source". Up to what magnitude is such a representation likely to be able to represent the main features of the behaviour of the radiated seismic waves.

How might the source description be modified to better capture the processes involved in the earthquake?

4. The discussion of the normal modes of the Earth is based on a spherical non-rotating model. For a rotating system the Coriolis force needs to be included in the equation of motion, and this has a small but important effect on the low-frequency modes.

Give a qualitative description of the consequence of including rotation with attention to the degeneracy of the modes and inter-mode coupling.

5. What are the forces imposed on the base of the lithosphere by the presence of a mountain chain? (Recall the likely effects of isostatic compensation and gravitational gradients).

How might asthenospheric flow be modified by the presence of topography at the base of the lithosphere - is this at all likely to lead to such stress as to induce volcanism?

6. For the general creep rheology of equation (9.3.3), with grain size and temperature dependence, find those conditions than result in strain-rate weakening or self-lubrication (cf. Section 14.5.2).

7. Consider the presence of loads within the Earth in conditions of dynamic compensation. Working in tterms of a spherical harmonic expansion show that in the presence of interfaces with mean radius R_i and density jumps $[\Delta\rho]_i$, the coefficients in the expansion of the gravitational potential φ take the form

$$C_l^m \approx \frac{4\pi G}{(2l+1)r_e^{l+1}} \left(-\int_0^{r_e} dr\, r^{l+2} \rho_l^m(r) - \sum_i [\Delta\rho]_i (h_i)_l^m r_i^{l+2} \right),$$

where ρ_l^m are the coefficients of the speherical harmonic expansion of the density distribution and $(h_i)_l^m$ the coefficients of the expansion of the interface topography.

Hence deduce that
(a) density anomalies close to the surface obey the isostatic rule

$$\rho_C H_l^m + \int_0^{r_e} dr\, \rho_l^m(r) \approx 0,$$

for crustal density ρ_C.
(b) the presence of an internal mass of degree l induces an interface topography that reaches an equilibrium shape with a time constant of order $2\eta l/(\rho_M g_0 r_e)$ for a viscosity η, surface gravity G_0 and mantle density ρ_M. [*cf. glacial rebound*].
(c) the gravity contribution has generally the sign opposite to a deep density anomaly, except when depp masses are close to a signficant viscosity increase.

8. Estimate the forces acting on a subducted plate associated with passage through the major phase transitions and the related strains?

To what extent would the boundary layers at the surfaces of the plate lubricate or impede the downward motion of the material.

9. What differences are to be expected in the nature of mantle circulation when the mantle viscosity increases linearly with depth rather than having an increase in vicosity of a factor of 30 at around 800 km depth?
How might the configuration of subducting slabs penetrating into the lower mantle differ in the two caes, and would this have any effect on thermal equilibration with the rest of the mantle?
Consider scenarios with and without an endothermic phase transition at 650 km depth.

10. Consider Earth-like planets with a metallic core approximately half the radius of the surface when
(a) most of the initial radioactivity is segregated in the core, and
(b) most of the radioactivity remains in the mantle.
What differences would be expected in the thermal evolution of the mantle and core in the two cases?
[*You may find it helpful to consider scaling via the Rayleigh and Nusselt numbers.*]

11. Estimate the relative contributions of the thermal driver from heat loss to the mantle and compositional convection from solidification of the inner core to the energetics of the core as a function of time since the formation of the Earth.

12. Aspects of core behaviour can be illustrated by critical stability analysis of a plane-layer, rotating fluid, subject to an externally imposed magnetic magnetic field. This yields the following non-dimensional equations, (with a similar scaling

to that employed in Section 15.2.5):

$$\frac{\partial v}{\partial t} + 2\mathrm{Ek}^{-1}\hat{\mathbf{e}}_z \times \mathbf{v} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla}^2 \mathbf{v} + \mathrm{Pr}^{-1}\mathrm{Ra}\,\theta \hat{\mathbf{e}}_z + \mathrm{Ha}^2 \mathrm{Pm}^{-1}(\boldsymbol{\nabla} \times b) \times \hat{\mathbf{e}}_z,$$

$$\frac{\partial \Theta}{\partial t} - \mathbf{v} \cdot \hat{\mathbf{e}}_z = \mathrm{Pr}^{-1}\boldsymbol{\nabla}^2 \Theta,$$

$$\frac{\partial \mathbf{b}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \hat{\mathbf{e}}_z) = \mathrm{Pm}^{-1}\boldsymbol{\nabla}^2 \mathbf{b},$$

where we have employed the Rayleigh number Ra, the Ekman number Ek, the regular Prandtl number Pr, the magnetic Prandtl number $\mathrm{Pm} = \eta/\lambda$ and the Hartmann number $\mathrm{Ha} = B_0 d/\sqrt{\mu_0 \rho \eta \lambda}$ where d is the layer thickness and B_0 the magnitude of the imposed magnetic field.

For a steady disturbance with the functional form $\exp(\sigma t + ik_x x + ik_y y)$ we find that

$$\begin{aligned}
\sigma\,\Theta &= v_z + \mathrm{Pr}^{-1}\boldsymbol{\nabla}^2\Theta \\
\sigma\,\boldsymbol{\nabla} v_z &= -2\mathrm{Ek}^{-1}\omega_z' + \boldsymbol{\nabla}^4 v_z - \mathrm{Pr}^{-1}\mathrm{Ra}\;k^2\theta + \mathrm{Ha}^2\mathrm{Pm}^{-1}\boldsymbol{\nabla}^2\,b_z' \\
\sigma\,\omega_z &= 2\mathrm{Ek}^{-1}v_z' + \boldsymbol{\nabla}^2\omega_z + \mathrm{Ha}^2\mathrm{Pm}^{-1}j_z' \\
\sigma\,b_z &= v_z' + \mathrm{Pm}^{-1}\boldsymbol{\nabla}^2 b_z \\
\sigma\,j_z &= \omega_z' + +\mathrm{Pm}^{-1}\boldsymbol{\nabla}^2 j_z
\end{aligned}$$

where v_z and b_z are the z-components of $\mathbf{v}$ and $\mathbf{b}$, ω_z and j_z the z-components of $\boldsymbol{\nabla} \times \mathbf{v}$, $'$ indicates differentiation with respect to z.

In the absence of rotation and magnetism the critical Rayleigh number for the onset of convection will be associated with the critical wavelength k_c = 3.12. Explain, how the critical Rayleigh number will change in the presence of rotation or magnetism alone. [*Use Figure 7.1 as a guide to behaviour*].

When magnetism is suppressed, which special case is recovered for frictionless flows?

How might the critical Rayleigh number change, when both rotation and magnetism are allowed to be present?

Consider your arguments in the context of weak- and strong-field dynamos.