Solutions to the Tutorial Problems in
the book "Magnetohydrodynamics of the Sun"
by ER Priest (2014)
CHAPTER 7

PROBLEM 7.1. Effect of Steady Flow on the Energy Method.

Consider a steady flow and magnetic field having y- and z-components of the form $\mathbf{v}_0(x)$ and $\mathbf{B}_0(x)$ and show that the linearised equation of motion for perturbations of the form $\boldsymbol{\xi}(x,y,z,t) = \boldsymbol{\xi}(x)\exp i(k_y y + k_z z - \omega t)$ becomes

$$-\tilde{\omega}^2 \rho_0 \;\boldsymbol{\xi}(x) = \mathbf{F}[\boldsymbol{\xi}(x)],$$

in place of

$$-\omega^2 \rho_0 \;\boldsymbol{\xi}(\mathbf{r}_0) = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r}_0)],$$

where

$$-\tilde{\omega} = \omega - \mathbf{k}.\mathbf{v}_0(x).$$

SOLUTION.
With a steady flow $\mathbf{v}_0$, the usual linearised equation of motion,

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \mathbf{F}[\boldsymbol{\xi}(\mathbf{r}_0, t)],$$

is modified to

$$\rho_0 \left(\frac{\partial}{\partial t} + i\mathbf{v}_0 \cdot \boldsymbol{\nabla}\right)^2 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\xi}\rho_0 \mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_0),$$

where $\mathbf{F}$ is the usual linearised total force given by

$$\begin{aligned}\mathbf{F}(\boldsymbol{\xi}) \equiv -\boldsymbol{\nabla} p_1 \;&+\; \rho_1 \mathbf{g} + \mathbf{j}_1 \times \mathbf{B}_0 + \mathbf{j}_0 \times \mathbf{B}_1 \\ = -\boldsymbol{\nabla} p_1 \;&+\; \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\xi})\;\mathbf{g} + (\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0)]) \times \mathbf{B}_0/\mu \\ &+\; (\boldsymbol{\nabla} \times \mathbf{B}_0) \times [\boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \mathbf{B}_0)]/\mu.\end{aligned}$$

For normal modes of the form $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{r})e^{-i\omega t}$, this reduces to

$$-\rho_0(\omega + i\mathbf{v}_0 \cdot \boldsymbol{\nabla})^2 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) + \boldsymbol{\nabla} \cdot (\boldsymbol{\xi}\rho_0 \mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_0).$$

Now, for a flow of the form $\mathbf{v}_0 = \mathbf{v}_0(x)$ having y- and z-components that depend on x alone, $\mathbf{v}_0 \cdot \boldsymbol{\nabla}\mathbf{v}_0 = 0$ and so normal modes of the form

$$\boldsymbol{\xi} = \boldsymbol{\xi}(x)\exp[i(k_y y + k_z z - \omega t)]$$

reduce the above equation to

$$-\tilde{\omega}^2 \rho_0 \, \boldsymbol{\xi}(x) = \mathbf{F}[\boldsymbol{\xi}(x)],$$

as required, in which the usual frequency ω in the case with no flow is replaced by a Doppler-shifted frequency $\tilde{\omega} = \omega - \Omega_0(x)$, with $\Omega_0(x) = \mathbf{k}.\mathbf{v}_0(x)$ and $\mathbf{k} = k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$.

Whereas in the static case, when $\tilde{\omega} = \omega$, transition to instability is through the marginal point ($\omega = 0$), the presence of flow allows such a transition to be through a value $\omega \neq 0$. In this case, overstable modes can appear in the form of propagating waves with exponentially growing amplitude. For more details, see Goedbloed et al's 2010 book on Advanced MHD, from which this example was taken.

PROBLEM 7.2. Rayleigh-Taylor Instability.
Consider two incompressible, inviscid plasmas of uniform densities ($\rho_0^{(-)}$, $\rho_0^{(+)}$), separated by a horizontal boundary, with gravity acting vertically downwards. Show that if the plasma of larger density $\rho_0^{(+)}$ rests on top of the other ($\rho_0^{(+)} > \rho_0^{(-)}$), the system is unstable with perturbations like $e^{i\omega t}$ growing at a rate $|\omega|$ given by

$$\omega^2 = -gk\left(\frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}}\right),$$

SOLUTION.
The incompressible equations are

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla}p + \rho\mathbf{g},$$
$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0,$$
$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0,$$

and the equilibrium satisfies

$$\frac{dp_0}{dz} = -\rho_0 g.$$

The linearised equations are

$$\begin{aligned}
\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= -\boldsymbol{\nabla} p_1 + \rho_1 \mathbf{g}, \\
\frac{\partial \rho_1}{\partial t} &= -v_{1z}\rho_0', \\
\boldsymbol{\nabla} \cdot \mathbf{v}_1 &= 0.
\end{aligned}$$

Next assume perturbations of the form $f(z)e^{i(kx+\omega t)}$, so that

$$i\omega\rho_0 v_{1x} = -ikp_1, \tag{1}$$

$$i\omega\rho_0 v_{1z} = -\frac{dp_1}{dz} - \rho_1 \mathbf{g}, \tag{2}$$

$$i\omega\rho_1 = -\rho_0' v_{1z}, \tag{3}$$

$$ikv_{1x} + v_{1z}' = 0. \tag{4}$$

Eliminate ρ_1 by taking $d(\mathbf{??})/dz - ik(\mathbf{??})$ to give

$$i\omega[(\rho_0 v_{1x})' - ik\rho_0 v_{1z}] = ik\rho_1 g.$$

Then use Eq.(**??**) to substitute for ρ_1 and Eq.(**??**) to eliminate v_{1x}, so that

$$\omega^2 \frac{d}{dz}\left(\rho_0 \frac{dv_{1z}}{dz}\right) - k^2(\rho_0\omega^2 + g\rho_0')v_{1z} = 0. \tag{5}$$

Now, for $z > 0$, $d\rho_0/dz = 0$, and so Eq.(**??**) reduces to

$$\frac{d^2 v_{1z}}{dz} - k^2 v_{1z} = 0,$$

with solution

$$v_{1z} = Ae^{kz} + Be^{-kz},$$

in which the condition $v_{1z} \to 0$ as $z \to \infty$ implies that $A = 0$.

Similarly, in the region $z < 0$ with the condition $v_{1z} \to 0$ as $z \to -\infty$, we find

$$v_{1z} = Ce^{kz}.$$

The next step is to link these two solutions across the interface $z = 0$ with two conditions, the first of which is that v_{1z} be continuous, so that $C = B$ and

$$v_{1z} = \begin{cases} Be^{-kz}, & z > 0, \\ Be^{kz}, & z < 0. \end{cases}$$

For the second condition, integrate Eq.(**??**) across the interface to give

$$\left[\omega^2 \rho_0 \frac{dv_{1z}}{dz} - k^2 g \rho_0 v_{1z}\right]_{z=0^+} = \left[\omega^2 \rho_0 \frac{dv_{1z}}{dz} - k^2 g \rho_0 v_{1z}\right]_{z=0^-},$$

or

$$(-\omega^2 \rho_+ k - k^2 g \rho_+) = (\omega^2 \rho_- k - k^2 g \rho_-),$$

or

$$\omega^2 = -gk\, \frac{\rho_0^{(+)} - \rho_0^{(-)}}{\rho_0^{(+)} + \rho_0^{(-)}},$$

as required.

PROBLEM 7.3. Continuously Stratified Medium.
Consider an equilibrium horizontal magnetic field $\mathbf{B}_0 = [B_0(z), 0, 0]$ in a gravitationally stratified atmosphere, with $p_0 = p_0(z)$ and $\rho_0 = \rho_0(z)$. Linearise the inviscid, ideal, adiabatic MHD equations and assume perturbations of the form $f(x, y, z, t) = f(z)e^{i(ly-\omega t)}$. When the wavenumber l is large ($l \to \infty$), prove that the local dispersion relation is given by

$$\left(c_s^2 + v_A^2\right)\omega^2 = c_s^2 N^2 + g v_A^2 \frac{d}{dz}\left[\log\left(\frac{B_0}{\rho_0}\right)\right],$$

where $v_A^2 = B_0^2/\mu\rho_0$, $c_s^2 = \gamma p_0/\rho_0$ and $N^2 = (g/\gamma)d/dz[\log(p_0/\rho_0^\gamma)]$. Deduce that the plasma can be unstable ($\omega^2 < 0$) even when it is convectively stable ($N^2 > 0$).

SOLUTION.

The inviscid, ideal, adiabatic MHD equations are

$$\begin{aligned}
\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} &= -\boldsymbol{\nabla} p + (\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}/\mu + \rho\mathbf{g},\\
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) &= 0,\\
\frac{\partial \mathbf{B}}{\partial t} &= \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}),\\
\frac{\partial p}{\partial t} &= -\mathbf{v}\cdot\boldsymbol{\nabla} p - \gamma p\boldsymbol{\nabla}\cdot\mathbf{v}.
\end{aligned}$$

If $\mathbf{B}_0 = [B_0(z), 0, 0]$, $p_0 = p_0(z)$ and $\rho_0 = \rho_0(z)$, then the equilibrium satisfies

$$\frac{d}{dz}\left(p_0 + \frac{B_0^2}{2\mu}\right) = -\rho_0 g,$$

and the linearised MHD equations are

$$\begin{aligned}
\rho_0\frac{\partial \mathbf{v}_1}{\partial t} &= -\boldsymbol{\nabla} p_1 + (\boldsymbol{\nabla}\times\mathbf{B}_1)\times\mathbf{B}_0/\mu + (\boldsymbol{\nabla}\times\mathbf{B}_0)\times\mathbf{B}_1/\mu + \rho_1\mathbf{g},\\
\frac{\partial \rho_1}{\partial t} &= -\rho_0\boldsymbol{\nabla}\cdot(\mathbf{v}_1) - v_{1z}\rho_0',\\
\frac{\partial \mathbf{B}_1}{\partial t} &= \boldsymbol{\nabla}\times(\mathbf{v}_1\times\mathbf{B}_0) = -\mathbf{B}_0(\boldsymbol{\nabla}\cdot\mathbf{v}_1) + \mathbf{B}_0\frac{\partial v_{1x}}{\partial x} - v_{1z}\frac{d\mathbf{B}_0}{dz},\\
\frac{\partial p_1}{\partial t} &= -v_{1z}p_0' - \gamma p_0\boldsymbol{\nabla}\cdot\mathbf{v}_1.
\end{aligned}$$

Next assume all perturbations are of the form $f(x,y,z,t) = f(z)e^{i(ly-\omega t)}$. Then the components of the equation of motion become

$$-i\omega\rho_0 v_{1x} = \frac{B_0'}{\mu}B_{1z}, \tag{6}$$

$$-i\omega\rho_0 v_{1y} = -il\left(p_1 + \frac{B_0 B_{1x}}{\mu}\right), \tag{7}$$

$$-i\omega\rho_0 v_{1z} = -\left(p_1 + \frac{B_0 B_{1x}}{\mu}\right)' - \rho_1 g, \tag{8}$$

while the continuity equation becomes

$$-i\omega\rho_1 = -\rho_0\nabla\cdot\mathbf{v}_1 - \rho_0' v_{1z}, \tag{9}$$

and the induction equation reduces to

$$\begin{aligned} -i\omega B_{1x} &= -B_0\nabla\cdot\mathbf{v}_1 - B_0' v_{1z}, \\ -i\omega B_{1y} &= 0, \\ -i\omega B_{1z} &= 0, \end{aligned} \tag{10}$$

Since $B_{1z} = 0$, Eq.(**??**) implies $v_{1x} = 0$ and the adiabatic equation becomes

$$-i\omega p_1 = -\gamma p_0 \nabla\cdot\mathbf{v}_1 - p_0' v_{1z}, \tag{11}$$

while $\boldsymbol{\nabla}\cdot\mathbf{B}_1 = 0$ is

$$0 = ilB_{1y} + B_{1z}'.$$

Now, assume that the wavenumber l is large ($l \to \infty$), so that $\boldsymbol{\nabla}\cdot\mathbf{v}_1 = ilv_{1y} + v_{1z}'$ will remain finite only if $v_{1y} \sim 1/l \to 0$. Then Eq.(**??**) becomes

$$p_1 + \frac{B_0 B_{1x}}{\mu} = 0, \tag{12}$$

and so Eq.(**??**) reduces to

$$-i\omega\rho_0 v_{1z} = -\rho_1 g. \tag{13}$$

The next step is to take $B_0 \times$ (**??**)$/\mu +$ (**??**) to give

$$-i\omega\left(p_1 + \frac{B_0 B_{1x}}{\mu}\right) = -\left(\gamma p_0 + \frac{B_0^2}{\mu}\right)\boldsymbol{\nabla}\cdot\mathbf{v}_1 - \left(p_0' + \frac{B_0 B_0'}{\mu}\right)v_{1z}.$$

However, Eq.(**??**) implies that the left-hand side vanishes and so

$$\left(\gamma p_0 + \frac{B_0^2}{\mu}\right)\boldsymbol{\nabla}\cdot\mathbf{v}_1 = -\left(p_0' + \frac{B_0 B_0'}{\mu}\right)v_{1z}. \tag{14}$$

Then forming $-i\omega \times$ (**??**) and substituting for ρ_1 from Eq.(**??**) gives

$$\omega^2\rho_0 v_{1z} = -i\omega\rho_1 g = -(\rho_0' g v_{1z} + \rho_0 g \boldsymbol{\nabla}\cdot\mathbf{v}_1)$$

After rearranging this and substituting for $\boldsymbol{\nabla}\cdot\mathbf{v}_1$ from Eq.(**??**), we find

$$(\rho_0\omega^2 + \rho_0' g)v_{1z} = -\rho_0 g\boldsymbol{\nabla}\cdot\mathbf{v}_1 = \frac{\rho_0 g}{\gamma p_0 + B_0^2/\mu}\left(p_0' + \frac{B_0 B_0'}{\mu}\right)v_{1z}.$$

This may be rearranged to give

$$(\gamma p_0 + B_0^2/\mu)\omega^2 = \frac{\gamma p_0 g}{\gamma}\left(\frac{p_0'}{p_0} - \frac{\gamma\rho_0'}{\rho_0}\right) + \frac{B_0^2 g}{\mu}\left(\frac{B_0'}{B_0} - \frac{\rho_0'}{\rho_0}\right),$$

or, in other words,

$$\left(c_s^2 + v_A^2\right)\omega^2 = c_s^2 N^2 + g v_A^2 \frac{d}{dz}\left\{\log\left(\frac{B_0}{\rho_0}\right)\right\},$$

as required, where $v_A^2 = B_0^2/\mu\rho_0$, $c_s^2 = \gamma p_0/\rho_0$ and the Brunt-Väisälä frequency

$$N^2 = \frac{g}{\gamma}\frac{d}{dz}\left\{\log\left(\frac{p_0}{\rho_0^\gamma}\right)\right\}.$$

Note that the plasma can be unstable ($\omega^2 < 0$) even when it is convectively stable with $N^2 > 0$.

PROBLEM 7.4. Magnetic Rayleigh-Taylor Instability.
Show that *incompressible* perturbations of the form $\mathbf{v}_1 = [v_x(z), 0, v_z(z)]e^{i(kx+\omega t)}$, $\rho_1 = \rho_1(z)e^{i(kx+\omega t)}$, $\mathbf{B}_1 = [B_{1x}(z), 0, B_{1z}(z)]e^{i(kx+\omega t)}$ to an interface at $z = 0$ between two uniform media (with density ρ_+ and magnetic field $B_+\hat{\mathbf{x}}$ above the interface and ρ_- and $B_-\hat{\mathbf{x}}$ below it) have dispersion relation

$$\omega^2 = \frac{k^2}{\mu}\left(\frac{B_+^2 + B_-^2}{\rho_+ + \rho_-}\right) + kg\left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right).$$

SOLUTION.
The incompressible, inviscid, ideal MHD equations are

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + (\boldsymbol{\nabla}\times\mathbf{B})\times\mathbf{B}/\mu + \rho\mathbf{g},$$

$$\frac{\partial \rho}{\partial t} = -\mathbf{v}\cdot\boldsymbol{\nabla}\rho,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B},$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{v} = 0.$$

The equilibrium variables are $\mathbf{B}_0 = (B_0(z), 0, 0)$, $p_0 = p_0(z)$ and $\rho_0 = \rho_0(z)$, and they satisfy

$$\frac{d}{dz}\left(p_0 + \frac{B_0^2}{2\mu}\right) = -\rho_0 g.$$

The linearised MHD equations are

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\boldsymbol{\nabla} p_1 + (\boldsymbol{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_0/\mu + (\boldsymbol{\nabla} \times \mathbf{B}_0) \times \mathbf{B}_1/\mu + \rho_1 \mathbf{g},$$
$$\frac{\partial \rho_1}{\partial t} = -\rho_0' v_{1z},$$
$$\frac{\partial \mathbf{B}_1}{\partial t} = B_0 \frac{\partial \mathbf{v}_1}{\partial x} - v_{1z} \frac{d\mathbf{B}_0}{dz},$$
$$\boldsymbol{\nabla} \cdot \mathbf{B}_1 = 0, \qquad \boldsymbol{\nabla} \mathbf{v}_1 = 0.$$

Consider now *incompressible* perturbations of the form $\mathbf{v}_1 = [v_{1x}(z), 0, v_{1z}(z)]e^{i(kx+\omega t)}$, $\rho_1 = \rho_1(z)e^{i(kx+\omega t)}$, $\mathbf{B}_1 = [B_{1x}(z), 0, B_{1z}(z)]e^{i(kx+\omega t)}$, for which the linearised equations become

$$\begin{aligned}
i\omega\rho_0 v_{1x} &= -ikp_1 + \frac{B_0'}{\mu} B_{1z}, &(15)\\
i\omega\rho_0 v_{1z} &= -\left(p_1 + \frac{B_0 B_{1x}}{\mu}\right)' + ik\frac{B_0}{\mu} B_{1z} - \rho_1 g, &(16)\\
i\omega\rho_1 &= -\rho_0' v_{1z},\\
i\omega B_{1x} &= ikB_0 v_{1x} - v_{1z} B_0' = -(B_0 v_{1z})',\\
-i\omega B_{1z} &= ikB_0 v_{1z}^2\\
ikv_{1x} &= -v_{1z}', \quad \text{and} \quad ikB_{1x} = -B_{1z}'.
\end{aligned}$$

By taking $d/dz(\textbf{??}) - ik(\textbf{??})$, we can eliminate p_1 to give

$$i\omega(\rho_0 v_{1x})' + \omega k \rho_0 v_{1z} = (B_0' B_{1z}/\mu)' + (B_0 ikB_{1x}/\mu)' + k^2 B_0 B_{1z}/\mu + ik\rho_1 g,$$

and then v_{1x}, B_{1z}, B_{1x} and ρ_1 can be eliminated to give the following equation for v_{1z} alone:

$$\omega(-\rho_0 v_{1z}'/k)' + \omega k \rho_0 v_{1z} = (kB_0' B_0 v_{1z}/\omega)' - (B_0 k/(\omega\mu)(B_0 v_{1z})')' + k^3 B_0^2 v_{1z}/(\omega\mu) - k\rho_0' g v_{1z}/\omega$$

which implies

$$\omega^2 \frac{d}{dz}\left(\rho_0 \frac{dv_{1z}}{dz}\right) - \omega^2 k^2 \rho_0 v_{1z} + \frac{k^4 B_0^2 v_{1z}}{\mu} - k^2 \rho_0' g v_{1z} + \frac{k^2}{\mu}\frac{d}{dz}\left(B_0 B_0' v_{1z} - B_0 B_0' v_{1z} - B_0^2 v_{1z}'\right) = 0$$

or

$$\frac{d}{dz}\left\{\left(\omega^2 \rho_0 - k^2 \frac{B_0^2}{\mu}\right)\frac{dv_{1z}}{dz}\right\} - k^2(\omega^2\rho_0 + \rho_0' g - k^2 \frac{B_0^2}{\mu})v_{1z} = 0. \qquad (17)$$

Next consider the interface at $z = 0$ across which $p_0(z) + B_0^2(z)/2\mu$ is continuous and either side of which

$$\rho_0 = \begin{cases} \rho_+, & z > 0, \\ \rho_-, & z < 0, \end{cases} \qquad B_0 = \begin{cases} B_+, & z > 0, \\ B_-, & z < 0. \end{cases}$$

In the regions $z > 0$ and $z < 0$ either side of the interface, the conditions are uniform and so $\rho_0' = 0$. As with PROBLEM 7.1, the continuous solution for v_{1z} subject to the conditions $v_{1z} \to 0$ as $z \to \infty$ and $z \to -\infty$ is

$$v_{1z} = \begin{cases} Ae^{-kz}, & z > 0, \\ Ae^{kz}, & z < 0. \end{cases}$$

Integrating Eq.(**??**) across the interface gives

$$\left[\left(\omega^2\rho_0 - \frac{kB_0^2}{\mu}\right)\frac{dv_{1z}}{dz} - k^2 g\rho_0 v_{1z}\right]_{z=0^-}^{z=0^+} = 0,$$

or

$$-(\omega^2\rho_+ - k^2B_+^2/\mu)k - k^2 g\rho_+ = (\omega^2\rho_- - k^2B_-^2/\mu)k - k^2 g\rho_-,$$

or

$$\omega^2 = \frac{k^2}{\mu}\left(\frac{B_+^2 + B_-^2}{\rho_+ + \rho_-}\right) + kg\left(\frac{\rho_- - \rho_+}{\rho_- + \rho_+}\right),$$

as required. Hence, there is only an instability if $\rho_+ > \rho_-$ (and small values of k), but the magnetic field is stabilizing, due to the magnetic tension effect introduced by an Alfvén wave.

PROBLEM 7.5. Sausage Instability.
Use the Energy Method to show that a cylindrical tube in equilibrium with pressure $p_0(R)$, uniform current and azimuthal magnetic field $B_{0\phi}(R)$ is unstable to compressible perturbations of the form $\boldsymbol{\xi} = \exp i(kz+\omega t)[\xi_R(R)\hat{\mathbf{R}} + i\xi_z(R)\hat{\mathbf{z}}]$ with $\omega^2 < 0$.

SOLUTION.
Consider an equilibrium tube with pressure $p_0(R)$, azimuthal magnetic field $B_{0\phi}(R)$ and electric current given by

$$\mu j_0(R) \;=\; \frac{1}{R}\frac{d}{dR}(RB_0) \;=\; \frac{B_0}{R} + B_0'$$

and

$$\mu p_0' = -\mu j_0 B_0 = B_0\left(\frac{B_0}{R} + B_0'\right).$$

Suppose there is a perturbation of the form

$$\boldsymbol{\xi} = \exp i(kz + \omega t)[\xi_R(R)\hat{\mathbf{R}} + i\xi_z(R)\hat{\mathbf{z}}],$$

in which the tube takes on a periodic sausage-like shape of wavelength $2\pi/k$ along its length.

Now, Eq.(7.28), namely,

$$\omega^2 \int \tfrac{1}{2}\rho_0\, \xi^2\, dV = \delta W,$$

implies that we have instability if $\delta W < 0$, so that ω is imaginary and the perturbation grows exponentially in time.

In turn, Eq.(7.30) determines the change in energy as

$$\delta W = \frac{1}{2}\int \frac{B_1^2}{\mu} - \mathbf{j}_0\cdot(\mathbf{B}_1\times\boldsymbol{\xi}) + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)(\boldsymbol{\nabla}\cdot\boldsymbol{\xi}) + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2\, dV,$$

where

$$\mathbf{B}_1 = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\mathbf{B}_0) = -(\mathbf{B}_0 D + \xi_R\Delta)\hat{\boldsymbol{\phi}},$$

in terms of $D = \boldsymbol{\nabla}\cdot\boldsymbol{\xi}$ and $\Delta = B_0' - B_0/R$. In other words,

$$\delta W = \frac{1}{2}\int \frac{1}{\mu}(B_0 D + \xi_R\Delta)^2 - j_0\xi_R(B_0 D + \xi_R\Delta) + \xi_R p_0' D + \gamma p_0 D^2\, dV,$$

After rearranging and substituting for Δ, p_0' and j_0, we find

$$\delta W = \frac{1}{2\mu}\int a^2 D^2 + 2bD + c\, dV,$$

where $a^2 = \gamma\mu p_0 + B_0^2$, $b = -2B_0^2\xi_R/R$ and $c = -2\xi_R^2 B_0(B_0/R)'$. Thus, we may complete the square in the integral to give

$$\delta W = \frac{1}{2\mu}\int \left(aD + \frac{b}{a}\right)^2 + c - \frac{b^2}{a^2}\, dV.$$

Then we choose a perturbation whose z-component satisfies $D = -b/a^2$, so that the integral reduces to

$$\delta W = \frac{1}{2\mu}\int c - \frac{b^2}{a^2}\, dV$$

or

$$\delta W = -\frac{1}{\mu}\int \xi_R^2 \left[B_0 \left(\frac{B_0}{R}\right)' + \frac{2B_0^4/R^2}{\gamma\mu p_0 + B_0^2}\right] dV.$$

The second term in the square brackets is destabilising since it produces a negative δW, whereas the first term tends to be stabilising since for typical profiles $(B_0/R)' < 0$.

Now, for a uniform-current flux tube $B_0(R) = B_e R/R_0$ inside a tube of radius R_0, say, where B_e is the field at $R = R_0$, whereas outside the tube the field is $B_0(R) = B_e R_0^2/R$. In this case, the destabilising term is present for all R, whereas the stabilising term vanishes inside the tube. Thus, if we pick a ξ_R that is nonzero inside the tube, but vanishes outside the tube, then δW will definitely be negative and we have instability, as required.

PROBLEM 7.6. Kelvin-Helmholtz Instability.
Consider two homogeneous plasma layers with field and flow $\mathbf{v}_0^+$ and $\mathbf{B}_0^+$ in $x > 0$ and $\mathbf{v}_0^-$ and $\mathbf{B}_0^-$ in $x < 0$, having y- and z-components parallel to the interface. Show that, under an incompressible displacement of the form $\boldsymbol{\xi} \sim \exp[i(k_y + k_z z - \omega t)]$, the interface is unstable when

$$[\mathbf{k}\cdot(\mathbf{v}_0^+ - \mathbf{v}_0^-)]^2 > \frac{(\mathbf{k}\cdot\mathbf{B}_0^+)^2 + (\mathbf{k}\cdot\mathbf{B}_0^-)^2}{\mu\rho^+}\frac{\rho^+ + \rho^-}{\rho^-},$$

where $\mathbf{k} = k_y\hat{\mathbf{y}} + k_z\hat{\mathbf{z}}$.

SOLUTION.
The details of this solution can be found in chapter 13 of the excellent book on Advanced MHD by Goedbloed et al (2010). The wave equation for an ideal incompressible MHD displacement of the form $\boldsymbol{\xi} = \xi(x)\exp[i(k_y + k_z z - \omega t)]\hat{\mathbf{x}}$, in which the disturbance is Fourier analysed in the ignorable y- and z-directions, is

$$\frac{d}{dx}\left[\rho_0(\tilde{\omega}^2 - \omega_A^2)\frac{d\xi}{dx}\right] - k^2[\rho_0(\tilde{\omega}^2 - \omega_A^2)]\xi = 0,$$

where $\tilde{\omega} = \omega - \mathbf{k}\cdot\mathbf{v}_0$ is the Doppler-shifted frequency, $\omega_A = \mathbf{k}\cdot\mathbf{B}_0/\sqrt{(\mu\rho_0)}$ is the Alfvén frequency and $k^2 = k_y^2 + k_z^2$.

Suppose we have an eigenvalue problem with boundary conditions ξ vanishing at $x = a$ and $x = -b$, say. Then above the interface we have a uniform medium, for which the above equation reduces to

$$\frac{d^2\xi}{dx^2} - k^2\xi = 0$$

with solution subject to $\xi(a) = 0$

$$\xi^+ = c^+ \frac{\sinh[k(a-x)]}{\sinh(ka)}.$$

Similarly, below the interface we have the same differential equation and so the solution subject to $\xi(-b) = 0$ is

$$\xi^- = c^- \frac{\sinh[k(b+x)]}{\sinh(kb)}.$$

Thus, the eigenfunctions have the usual cusp-shaped form characteristic of surface modes. The first boundary condition at the interface is continuity of normal velocity or displacement $\xi^+(0) = \xi^-(0)$, which implies $c^+ = c^-$. The second condition is essentially pressure balance and may be obtained by integrating the above full differential equation across the interface to give

$$\left[\rho(\tilde{\omega}^2 - \omega_A^2)\frac{d\xi}{dx}\right]_-^+ = 0$$

or, after substituting for ξ^+ and ξ^-, we find the dispersion relation

$$-\rho^+[(\omega - \Omega_0^+)^2 - (\omega_A^+)^2]\coth(ka) = \rho^-[(\omega - \Omega_0^-)^2 - (\omega_A^-)^2]\coth(kb).$$

In the limit of small wavelength disturbances, the boundaries are essentially at infinity and so both coth functions become unity. The solutions then reduce to

$$\omega = \frac{\rho^+\Omega_0^+ + \rho^-\Omega_0^-}{\rho^+ + \rho^-} \pm \sqrt{\left[-\frac{\rho^+\rho^-(\Omega_0^+ - \Omega_0^-)^2}{(\rho^+ + \rho^-)^2} + \frac{\rho^+{\omega_A^+}^2 + \rho^-{\omega_A^-}^2}{\rho^+ + \rho^-}\right]}.$$

The Kelvin-Helmholotz instability occurs when the expression under the square root is negative, namely, when

$$[\mathbf{k}\cdot(\mathbf{v}_0^+ - \mathbf{v}_0^-)]^2 > \frac{(\mathbf{k}\cdot\mathbf{B}_0^+)^2 + (\mathbf{k}\cdot\mathbf{B}_0^-)^2}{\mu\rho^+}\frac{\rho^+ + \rho^-}{\rho^-},$$

as required, so that the frequency is complex and the positive square root represents a solution that is growing exponentially in time.

Note that when this does not hold, the expression under the square root is positive and we have two waves of real frequency.

Also, if there is no shear so that $\mathbf{B}_0^+$ and $\mathbf{B}_0^-$ have the same direction, then we may choose $\mathbf{k}$ such that $\mathbf{k} \cdot \mathbf{B}_0 = 0$, for which the above instability condition is always satisfied for arbitrarily small velocity differences $\mathbf{v}_0^+ - \mathbf{v}_0^-$. However, when the field is sheared it is stabilising, since the right-hand side of the above instability criterion is always then positive and so instability only occurs when the velocity difference $\mathbf{v}_0^+ - \mathbf{v}_0^-$ is large enough.

For treatment of a continuous rather than a discrete transition between to media, the analysis is very much more complex (see Goedbloed's book, section 13.2).